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The anti-reflexive extremal rank solutions of the matrix equation $AX = B$

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Abstract. An $n \times n$ complex matrix P is said to be a generalized reflection matrix if $P^* = P$ and $P^2 = I$. An $n \times n$ complex matrix A is said to be a anti-reflexive matrix with respect to the generalized reflection matrix P if $A = -PAP$. We in this paper mainly investigate the anti-reflexive maximal and minimal rank solutions to the matrix equation $AX = B$. We present necessary and sufficient conditions for the existence of the maximal and minimal rank solutions with anti-reflexive to the matrix equation $AX = B$. The expressions of such solutions to this system are also given when the solvability conditions are satisfied. In addition, in corresponding the minimal rank solution set to the matrix equation $AX = B$, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm has been provided.

1. Introduction

Throughout this paper, let $C^{m\times n}$ be the set of all $m \times n$ complex matrices, $UC^{n\times n}$ be the set of all $n \times n$ unitary complex matrices. Denote by I_n the identity matrix with order n. For matrix A, A^*, $\|A\|_F$ and $r(A)$ represent its conjugate transpose, Frobenius norm and rank, respectively. On $C^{m\times n}$ we define inner product, $(A, B) = trace(B^T A)$ for all $A, B \in C^{m\times n}$, then $C^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generated by this inner product is Frobenius norm.

Recall that a reflexive (anti-reflexive) matrix was defined in [1,2]: a complex matrix A is reflexive (anti-reflexive) if $A = PAP(A = -PAP)$, where P is a generalized reflection matrix. By a generalized reflection matrix, say P, one mean that P satisfies the following two conditions: $P^* = P$ and $P^2 = I$. In other words, a generalized reflection matrix is a Hermitian involution matrix.

The reflexive (anti-reflexive) matrix with respect to a generalized reflection matrix P has many special properties and widely used in engineering and scientific computations(see, e.g. [1-4]).

In matrix theory and applications, many problems are closely related to the ranks of some matrix expressions with variable entries, and so it is necessary to explicitly characterize the possible ranks of the matrix expressions concerned. The study on the possible ranks of matrix equations can be traced back

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to the late 1970s (see, e.g. [5-9]). Recently, the extremal ranks, i.e. maximal and minimal ranks, of some matrix expressions have found many applications in control theory [10,11], statistics, and economics (see, e.g. [12-14]).

In this paper, we consider the anti-reflexive extremal rank solutions of the matrix equation

$$AX = B, \tag{1}$$

where A and B are given matrices in $C^{m\times m}$. In 1987, Uhlig [8] gave the maximal and minimal ranks of solutions to system (1). By applying the matrix rank method, recently, Tian [15] obtained the minimal rank of solutions to the matrix equation $A = BX + YC$. Xiao et al. [16] in 2009 considered the symmetric minimal rank solution to system (1). The reflexive and anti-reflexive matrices with respect to the generalized reflection matrix P are two important classes of matrices and have engineering and scientific applications. The anti-reflexive maximal and minimal rank solutions of the matrix equation (1), however, have not been considered yet. In this paper, we will discuss this problem.

We also consider the matrix nearness problem

$$\min_{X\in S_m} \left\|X - \tilde{X}\right\|_F, \tag{2}$$

where $\tilde{X}$ is a given matrix in $C^{n\times m}$ and S_m is the minimal rank solution set of Eq. (1).

The matrix nearness problem (2) is so-called the optimal approximation problem, which has important application in practice, and has been discussed far and wide (see, e.g., [17-22] and the references therein).

We organize this paper as follows. In Section 2, we first establish a representation for the generalized reflection matrix P. Then we give necessary and sufficient conditions for the existence of anti-reflexive solution to (1). We also give the expressions of such solutions when the solvability conditions are satisfied. We in Section 3 establish formulas of maximal and minimal ranks of anti-reflexive solutions to (1), and present the anti-reflexive extremal rank solutions to (1). We in Section 4 present the expression of the optimal approximation solution to the set of the minimal rank solution.

2. Anti-reflexive solution to $AX = B$

In this section we first introduce some structure properties of the generalized reflection matrix P and establish the representations of anti-reflexive matrix. Then we give the necessary and sufficient conditions for the existence of and the expressions for anti-reflexive solution of Eq. (1).

Lemma 2.1. [18] *Assume P is a generalized reflection matrix of size n, and let*

$$P_1 = \frac{I_n + P}{2}, \ \ P_2 = \frac{I_n - P}{2}. \tag{3}$$

Then P_1 and P_2 are orthogonal projection matrices, and satisfied $P_1 + P_2 = I_n$, $P_1P_2 = 0$.

Remark 2.2. *Assume P_1 and P_2 are defined as* (3) *and $r(P_1) = r$, then $r(P_2) = n-r$, and there exist column orthogonal matrices $U_{11} \in C^{n\times r}$, $U_{22} \in C^{n\times(n-r)}$, such that*

$$P_1 = U_{11}U_{11}^*, \ \ P_2 = U_{22}U_{22}^*, \ \ P = U_1U_1^T - U_2U_2^T, \ \ U_1^TU_2 = 0.$$

Let $U = [U_{11}, U_{22}]$, From the Remark 2.2, it is easy to verify that U is an unitary matrix and the generalized reflection matrix P can be expressed as

$$P = U\begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix}U^*, \tag{4}$$

where the symbols "0" stand for null matrices with associated orders (in the sequel, we always mark them like this).

Lemma 2.3. [18] *Let $A \in C^{n\times n}$ and a generalized reflection matrix P with the form of* (4), *then A is the anti-reflexive matrix if and only if*

$$A = U\begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}U^*, \tag{5}$$

where $M \in C^{r\times(n-r)}$, $N \in C^{(n-r)\times r}$ are arbitrary and U is the same as in (4).

Lemma 2.4. *(GSVD)*[23] *Given matrices $A_1 \in C^{m\times n}$, $B_1 \in C^{m\times p}$, there exist unitary matrices $U_1 \in UC^{n\times n}$, $V_1 \in UC^{p\times p}$ and nonsingular matrix $M_1 \in C^{m\times m}$ such that*

$$A_1 = M_1\Sigma_{A_1}U_1,\ \ B_1 = M_1\Sigma_{B_1}V_1 \tag{6}$$

where

$$\Sigma_{A_1} = \begin{array}{c} \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} r_1-s_1 \\ s_1 \\ k_1-r_1 \\ m-k_1 \end{array},\quad \Sigma_{B_1} = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_1} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} r_1-s_1 \\ s_1 \\ k_1-r_1 \\ m-k_1 \end{array},$$

$k_1 = r([A_1, B_1])$, $r_1 = r(A_1)$, $s_1 = r(A_1) + r(B_1) - r([A_1, B_1])$, $S_{A_1} = diag(\alpha_1, \ldots, \alpha_{s_1})$, $S_{B_1} = diag(\beta_1, \ldots, \beta_{s_1})$, $0 < \alpha_{s_1} \le \cdots \le \alpha_1 < 1$, $0 < \beta_1 \le \cdots \le \beta_{s_1} < 1$, $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, \ldots, s_1$.

Lemma 2.5. *Given matrices $A_1 \in C^{m\times n}$, $B_1 \in C^{m\times p}$, the generalized singular value decomposition of the matrix pair $[A_1, B_1]$ is given by* (6), *then matrix equation $A_1X = B_1$ is consistent, if and only if*

$$r([A_1, B_1]) = r(A_1), \tag{7}$$

and the expression of its general solution is

$$X = U_1^*\begin{bmatrix} 0 & 0 \\ 0 & S_{A_1}^{-1}S_{B_1} \\ Y_{31} & Y_{32} \end{bmatrix}V_1, \tag{8}$$

where $Y_{31} \in C^{(n-r_1)\times(p-s_1)}$, $Y_{32} \in C^{(n-r_1)\times s_1}$ are arbitrary.

Proof. With (6) we have

$$r(B_1 - A_1X) = r(M_1\Sigma_{B_1}V_1 - M_1\Sigma_{A_1}U_1X) = r(\Sigma_{B_1} - \Sigma_{A_1}U_1XV_1^*).$$

Let $Y = U_1XV_1^*$ and let Y be partitioned as $Y = (Y_{ij})_{3\times 3}$, then

$$\Sigma_{B_1} - \Sigma_{A_1}Y = \begin{bmatrix} -Y_{11} & -Y_{12} & -Y_{13} \\ -S_{A_1}Y_{21} & S_{B_1} - S_{A_1}Y_{22} & -S_{A_1}Y_{23} \\ 0 & 0 & I_{B_1} \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} r_1-s_1 \\ s_1 \\ k_1-r_1 \\ m-k_1 \end{array}. \tag{9}$$

Noting that $Y_{ij} (i = 1, 2, j = 1, 2, 3)$ are arbitrary, then

$$\min r(B_1 - A_1X) = \min r(\Sigma_{B_1} - \Sigma_{A_1}Y) = k_1 - r_1 = r(A_1, B_1) - r(A_1).$$

$A_1X = B_1$ is solvable in $C^{n\times p}$ if and only if $\min r(B_1 - A_1X) = 0$. Then matrix equation $A_1X = B_1$ is consistent, if and only if (7) holds. In this case, from (9) and $Y = U_1XV_1^*$, its general solution can be expressed as (8). The proof is completed. □

Assume the given generalized reflection matrix P with the form of (4). Let

$$AU = [A_2, A_3], \quad BU = [B_2, B_3], \tag{10}$$

where $A_2 \in C^{m\times r}$, $A_3 \in C^{m\times(n-r)}$, $B_2 \in C^{m\times r}$, $B_3 \in C^{m\times(n-r)}$, and the generalized singular value decomposition of the matrix pairs $[A_2, B_3]$, $[A_3, B_2]$ are, respectively,

$$A_2 = M_2\Sigma_{A_2}U_2, \quad B_3 = M_2\Sigma_{B_3}V_2, \tag{11}$$

$$A_3 = M_3\Sigma_{A_3}U_3, \quad B_2 = M_3\Sigma_{B_2}V_3, \tag{12}$$

where $U_2 \in UC^{r\times r}$, $V_2 \in UC^{(n-r)\times(n-r)}$, $U_3 \in UC^{(n-r)\times(n-r)}$, $V_3 \in UC^{r\times r}$, nonsingular matrices $M_2, M_3 \in C^{m\times m}$, $k_2 = r([A_2, B_3])$, $r_2 = r(A_2)$, $s_2 = r(A_2) + r(B_3) - r([A_2, B_3])$, and $k_3 = r([A_3, B_2])$, $r_3 = r(A_3)$, $s_3 = r(A_3) + r(B_2) - r([A_3, B_2])$,

$$\Sigma_{A_2} = \begin{array}{c} \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} r_2 - s_2 \\ s_2 \\ k_2 - r_2 \\ m - k_2 \end{array}, \qquad \Sigma_{B_3} = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_3} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} r_2 - s_2 \\ s_2 \\ k_2 - r_2 \\ m - k_2 \end{array},$$

$$\Sigma_{A_3} = \begin{array}{c} \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} r_3 - s_3 \\ s_3 \\ k_3 - r_3 \\ m - k_3 \end{array}, \qquad \Sigma_{B_2} = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_2} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} r_3 - s_3 \\ s_3 \\ k_3 - r_3 \\ m - k_3 \end{array}.$$

Now we can establish the existence theorems as follows.

Theorem 2.6. *Let $A, B \in C^{m\times n}$ and the generalized reflection matrix P of size n be known. Suppose the generalized reflection matrix P with the form of* (4)*, AU, BU have the partition forms of* (10)*, and the generalized singular value decompositions of the matrix pairs $[A_2, B_3]$ and $[A_3, B_2]$ are given by* (11) *and* (12)*, respectively. Then the equation* (1) *has an anti-reflexive solution X if and only if*

$$r([A_2, B_3]) = r(A_2), \quad r([A_3, B_2]) = r(A_3), \tag{13}$$

and its general solution can be expressed as

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} U^*, \tag{14}$$

where $Z_{31} \in C^{(r-r_2)\times(n-r-s_2)}$, $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{31} \in C^{(n-r-r_3)\times(r-s_3)}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary.

Proof. Suppose the matrix equation (1) has a solution X which is anti-reflexive, then it follows from Lemma 2.3 that there exist $M \in C^{r\times(n-r)}$, $N \in C^{(n-r)\times r}$ satisfying

$$X = U \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} U^* \quad and \quad AX = B \tag{15}$$

By (10), that is

$$[A_2 \;\; A_3] \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} = [B_2 \;\; B_3], \tag{16}$$

i.e.

$$A_2M = B_3, \quad A_3N = B_2. \tag{17}$$

Therefore by Lemma 2.5, (13) hold, and

$$M = U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} V_2, \ \ N = U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} V_3, \tag{18}$$

where $Z_{31} \in C^{(r-r_2)\times(n-r-s_2)}$, $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{31} \in C^{(n-r-r_3)\times(r-s_3)}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary. Substituting (18) into (15) yields that the anti-reflexive solution X of the matrix equation (1) can be represented by (14). The proof is completed. □

3. Anti-reflexive extremal rank solutions to $AX = B$

In this section, we first derive the formulas of the maximal and minimal ranks of anti-reflexive solutions of (1), then present the expressions of anti-reflexive maximal and minimal rank solutions to (1).

Theorem 3.1. *Suppose that the matrix equation* (1) *has an anti-reflexive solution X and Ω is the set of all anti-reflexive solutions of* (1). *Then the extreme ranks of X are as follows:*

(1) *The maximal rank of X is*

$$\max_{X\in\Omega} r(X) = \min\{n-r, r-r(A_2)+r(B_3)\} + \min\{r, n-r-r(A_3)+r(B_2)\}. \tag{19}$$

The general expression of X satisfying (19) *is*

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} U^*, \tag{20}$$

where $Z_{31} \in C^{(r-r_2)\times(n-r-s_2)}$, $W_{31} \in C^{(n-r-r_3)\times(r-s_3)}$ are chosen such that $r(Z_{31}) = \min(r-r_2, n-r-s_2)$, $r(W_{31}) = \min(n-r-r_3, r-s_3)$, $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary.

(2) *The minimal rank of X is*

$$\min_{X\in\Omega} r(X) = r(B_2) + r(B_3). \tag{21}$$

The general expression of X satisfying (20) *is*

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} U^*, \tag{22}$$

where $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary.

Proof. (1) By (14),

$$\max_{X\in\Omega} r(X) = \max_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} + \max_{W_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & W_{32} \end{bmatrix}, \tag{23}$$

$$\max_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} = s_2 + \min\{r-r_2, n-r-s_2\} \tag{24}$$

$$= \min\{n-r, r-r_2+s_2\} = \min\{n-r, r-r(A_2)+r(B_3)\},$$

and

$$\max_{W_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} = s_3 + \min\{n-r-r_3, r-s_3\} \tag{25}$$

$$= \min\{r, n-r-r_3+s_3\} = \min\{r, n-r-r(A_3)+r(B_2)\}.$$

Taking (24) and (25) into (23) yields (19).

According to the general expression of the solution in theorem 2.6, it is easy to verify the rest of part in (1).

(2) By (14),

$$\min_{X\in\Omega} r(X) = \min_{Z_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} + \min_{W_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix}, \tag{26}$$

$$\min_{Z_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} = s_2 = r(B_3) \tag{27}$$

and

$$\min_{W_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} = s_3 = r(B_2). \tag{28}$$

Taking (27) and (28) into (26) yields (21).

According to the general expression of the solution in theorem 2.6, it is easy to verify the rest of part in (2). The proof is completed. □

4. The expression of the optimal approximation solution to the set of the minimal rank solution

From (22), When the solution set $S_m = \{X \mid AX = B, X = -PXP, r(X) = \min_{Y\in\Omega} r(Y)\}$ is nonempty, it is easy to verify that S_m is a closed convex set, therefore there exists a unique solution $\hat{X}$ to the matrix nearness Problem (2).

Theorem 4.1. *Given matrix $\tilde{X}$, and the other given notations and conditions are the same as in Theorem 2.6. Let*

$$U^*\tilde{X}U = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{bmatrix}, \ \tilde{X}_{12} \in C^{r\times(n-r)}, \ \tilde{X}_{21} \in C^{(n-r)\times r}, \tag{29}$$

and we denote

$$U_2\tilde{X}_{12}V_2^* = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \\ \tilde{Z}_{31} & \tilde{Z}_{32} \end{bmatrix}, \ U_3\tilde{X}_{21}V_3^* = \begin{bmatrix} \tilde{W}_{11} & \tilde{W}_{12} \\ \tilde{W}_{21} & \tilde{W}_{22} \\ \tilde{W}_{31} & \tilde{W}_{32} \end{bmatrix}. \tag{30}$$

If S_m is nonempty, then Problem (2) *has a unique $\hat{X}$ which can be represented as*

$$\hat{X} = U\begin{bmatrix} 0 & U_2^*\begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ 0 & \tilde{Z}_{32} \end{bmatrix}V_2 \\ U_3^*\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ 0 & \tilde{W}_{32} \end{bmatrix}V_3 & 0 \end{bmatrix}U^*, \tag{31}$$

where $\tilde{Z}_{32}, \tilde{W}_{32}$ *are the same as in* (4.2).

Proof. When S_m is nonempty, it is easy to verify from (22) that S_m is a closed convex set. Since $C^{n\times n}$ is a uniformly convex Banach space under Frobenius norm, there exists a unique solution for Problem (2). By theorem 3.1, for any $X \in S_m$, X can be expressed as

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} U^*, \tag{32}$$

where $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary.

Using the invariance of the Frobenius norm under unitary transformations, we have

$$\begin{aligned} \|X - \tilde{X}\|_F^2 &= \left\| \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} - U^* \tilde{X} U \right\|_F^2 \\ &= \left\|Z_{32} - \tilde{Z}_{32}\right\|_F^2 + \left\|W_{32} - \tilde{W}_{32}\right\|_F^2 + \left\|S_{A_2}^{-1} S_{B_3} - \tilde{Z}_{22}\right\|_F^2 + \left\|S_{A_3}^{-1} S_{B_2} - \tilde{W}_{22}\right\|_F^2 \\ &+ \left\|\tilde{X}_{11}\right\|_F^2 + \left\|\tilde{X}_{22}\right\|_F^2 + \left\|\tilde{Z}_{11}\right\|_F^2 + \left\|\tilde{Z}_{12}\right\|_F^2 + \left\|\tilde{Z}_{21}\right\|_F^2 + \left\|\tilde{Z}_{31}\right\|_F^2 \\ &+ \left\|\tilde{W}_{11}\right\|_F^2 + \left\|\tilde{W}_{12}\right\|_F^2 + \left\|\tilde{W}_{21}\right\|_F^2 + \left\|\tilde{W}_{31}\right\|_F^2 . \end{aligned}$$

Therefore, $\|X - \tilde{X}\|_F$ reaches its minimum if and only if

$$Z_{32} = \tilde{Z}_{32}, \quad W_{32} = \tilde{W}_{32}. \tag{33}$$

Substituting (33) into (32) yields (31). The proof is completed. □

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