Functional Analysis,
Approximation and
Computation
4:2 (2012), 23–31

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

# EP matrices in indefinite inner product spaces

**Sachindranath Jayaraman**[a]

[a] *School of Mathematics, Indian Institute of Science Education and Research - Thiruvananthapuram
CET Campus, Engineering College P.O.
Thiruvananthapuram - 695 016, Kerala, India.*

**Abstract.** The aim of this article is to introduce the notion of $J$-EP matrices as a generalization of EP matrices, in the setting of indefinite inner product spaces with respect to the indefinite matrix product. Connections between $J$-EP matrices and EP matrices, an interesting characterization of $J$-EP matrices similar to EP matrices and the reverse order law for the Moore-Penrose inverse with respect to the indefinite matrix product (including the triple product) are brought out. A generalization of a result on polynomials in two variables satisfied by two related matrices is also presented in the setting of indefinite inner product spaces with respect to the indefinite matrix product.

## 1. Introduction

An indefinite inner product in $\mathbb{C}^n$ is a conjugate symmetric sesquilinear form $[x, y]$ together with the regularity condition that $[x, y] = 0$, $\forall\, y \in \mathbb{C}^n$ only when $x = 0$. Associated with any indefinite inner product is a unique invertible Hermitian matrix $J$ (called a weight) with complex entries such that $[x, y] = \langle x, Jy \rangle$, where $\langle ., . \rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$ and vice versa. Motivated by the notion of Minkowski space (as studied by physicists), we also make an additional assumption on $J$, namely, $J^2 = I$. It can be shown that this assumption on $J$ is not really restrictive as the results presented in this manuscript can also be deduced without this assumption on $J$, with appropriate modifications. It should be remarked that this assumption also allows us to compare our results with the Euclidean case, apart from allowing us to present the results with much algebraic ease.

Investigations of linear maps on indefinite inner product spaces employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors (See for instance [4, 11] and the references cited therein.) This causes a problem as there are two different values for dot product of vectors. To overcome this difficulty, Kamaraj, Ramanathan and Sivakumar introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [11]. More precisely, the indefinite matrix product of two matrices $A$ and $B$ of sizes$m \times n$ and $n \times l$ complex matrices, respectively, is defined to be the matrix $A \circ B = AJ_nB$. The adjoint of $A$, denoted by $A^{[*]}$ is defined to be the matrix $J_nA^*J_m$, where $J_m$ and $J_n$ are weights in the appropriate spaces.

2010 *Mathematics Subject Classification*. Primary : 46C20, 15A09.
*Keywords*. Indefinite matrix product; indefinite inner product spaces; EP matrices; $J$-EP matrices; reverse order law.
Received: November 1, 2011; Accepted: May 3, 2012
Communicated by Dragan S. Djordjević
*Email address:* sachindranathj@iisertvm.ac.in, sachindranathj@gmail.com (Sachindranath Jayaraman)

Many properties of this product are similar to that of the usual matrix product (refer [11]). Moreover, it not only rectifies the difficulty indicated earlier, but also enables us to recover some interesting results in indefinite inner product spaces in a manner analagous to that of the Euclidean case. Kamaraj, Ramanathen and Sivakumar also established in [11] that in the setting of indefinite inner product spaces, the indefinite matrix product is more appropriate that the usual matrix product. Recall that the Moore-Penrose inverse exists if and only if $rank(AA^*) = rank(A^*A) = rank(A)$. If we take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $AA^{[*]}$ and $A^{[*]}A$ are both the zero matrix and so $rank(AA^{[*]}) < rank(A)$, thereby proving that the Moore-Penrose inverse doesn't exist with respect to the usual matrix product. However, it can be easily verified that with respect to the indefinite matrix product, $rank(A \circ A^{[*]}) = rank(A^{[*]} \circ A) = rank(A)$. Thus, the Moore-Penrose of a matrix with real or complex entries exists over an indefinite inner product space with respect to the indefinite matrix product, whereas a similar result is false with respect to the usual matrix multiplication. It is therefore really pertinant to extend the study of generalized inverses to the setting of indefinite inner product spaces, with respect to the indefinite matrix product.

The objective of this manuscript is to generalize the notion of EP matrices to indefinite inner product spaces and investigate its relationship with the usual notion of EP-ness in the Euclidean setting. The paper is organized as follows. We recall the basic definitions and facts in the next section. In particular, we recall the definition of an indefinite product of two matrices / vectors (Definition 2.1) and the adjoint with respect to this multiplication in Definition 2.2. The definons of the range and the null space, denoted by $Ra(.)$ and $Nu(.)$, respectively, invertibility, the Moore-Penrose inverse, and the group inverse, all with respect to this indefinite product, and its properties are given next in Definitions 2.3, 2.4, 2.5 and 2.6. The definition of $J$-EP matrices is introduced first (Definition 3.1). The main results are presented in Section 3. Theorems 3.5 and 3.7 (a) - (c) give the basic relationship between $J$-EP matrices and EP matrices. In particular, Theorem 3.7 (c) is a nice generalization of a characterization of EP matrices, namely, that a matrix $A$ is EP if and only if $R(A^2) = R(A^*)$. A result concerning the sum of $J$-EP matrices being $J$-EP is presented next (Theorem 3.12). A few remarks on range additivity results are also brought out (Remarks 3.13). We then move on to reverse order laws for the Moore-Penrose inverse with respect to the indefinite matrix product (Theorems 3.14, 3.17 and Theorems 3.20, 3.21). A generalization to indefinite inner product spaces of a result of Baksalary, Hauke and Johnson on characterizations of EP matrices in terms of polynomials in two variables evaluated at $A$ and $A^\dagger$ is also brought out (Lemma 3.24 and Theorem 3.25). Wherever possible, we give examples to bring out the importance of the assumptions made. We end with a few concluding remarks on the possibility of extending this study to various matrix partial orders and its connection with reverse odrer laws.

## 2. Notations, Definitions and Preliminaries

We first recall the notion of an indefinite multiplication of matrices. We refer the reader to [11], wherein various properties and also advantages of this product have been discussed in detail.

**Definition 2.1.** *Let $A$ and $B$ be $m \times n$ and $n \times l$ complex matrices, respectively. Let $J_n$ be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n^* = J_n^{-1}$. The indefinite matrix product of $A$ and $B$ (relative to $J_n$) is defined by $A \circ B = AJ_nB$.*

Note that there is only one value for the indefinite product of vectors / matrices. When $J_n = I_n$, the above product becomes the usual product of matrices.

**Definition 2.2.** *Let $A$ be an $m \times n$ complex matrix. The adjoint $A^{[*]}$ of $A$ (relative to $J_n, J_m$) is defined by $A^{[*]} = J_nA^*J_m$.*

When the dimensions are equal, the subscripts $n, m$ will be dropped. $A^{[*]}$ satisfies the following identities : $[Ax, y] = [x, A^{[*]}y]$ and $[A \circ x, y] = [x, (I \circ A \circ I)^{[*]} \circ y]$.

**Definition 2.3.** *Let $A$ be an $m \times n$ complex matrix. Then the range space $Ra(A)$ is defined by $Ra(A) = \{y = A \circ x \in \mathbb{C}^m : x \in \mathbb{C}^n\}$ and the null space $Nu(A)$ of $A$ is defined by $Nu(A) = \{x \in \mathbb{C}^n : A \circ x = 0\}$.*

**Definition 2.4.** *Let $A \in \mathbb{C}^{n\times n}$. A is said to be J-invertible if there exists $X \in \mathbb{C}^{n\times n}$ such that $A \circ X = X \circ A = J$.*

It follows that $A$ is $J$-invertible if and only if $A$ is invertible and in this case the $J$-inverse is given by $A^{[-1]} = JA^{-1}J$. We now pass on to the notion of the Moore-Penrose inverse in indefinite inner product spaces.

**Definition 2.5.** *For $A \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{n\times m}$ is called the Moore-Penrose inverse if it satisfies the following equations : $A \circ X \circ A = A, X \circ A \circ X = X, (A \circ X)^{[*]} = A \circ X, (X \circ A)^{[*]} = X \circ A$.*

Such an $X$ will be denoted by $A^{[\dagger]}$. It can be shown that $A^{[\dagger]}$ exists if and only if $rank(A) = rank(A \circ A^{[*]}) = rank(A^{[*]} \circ A)$ [11]. The Moore-Penrose has the representation $A^{[\dagger]} = J_n A^{\dagger} J_m$. We also have, $Ra(A \circ A^{[\dagger]}) = Ra(A)$ and $Ra(A^{[\dagger]} \circ A) = Ra(A^{[*]})$ (see for instance Lemma 2.1(v), [10].) One can similarly define the notion of the group inverse in indefinite inner product spaces.

**Definition 2.6.** *For $A \in \mathbb{C}^{n\times n}, X \in \mathbb{C}^{n\times n}$ is called the group inverse of A if it satisfies the equations : $A \circ X \circ A = A, X \circ A \circ X = X, A \circ X = X \circ A$.*

As in the Euclidean setting, it can be proved that the group inverse exists in the indefinite setting if and only if $rank(A) = rank(A^{[2]})$ and is denoted by $A^{[\#]}$. In particular, if $A = A^{[*]}$, then $A^{[\#]}$ exists. However, an analagous formula for the group inverse similar to that of the Moore-Penrose inverse does not hold in the indefinite setting. However, if $A = B \circ C$ is a rank factorization, then the group inverse of $A$ exists if and only if $C \circ B$ is invertible and in this case, the group inverse is given by $A^{[\#]} = B \circ (C \circ B)^{[-2]} \circ C$.

## 3. Main Results

We present our main results in this section. We first start by a notion of EP matrices in the setting of indefinite inner product spaces.

**Definition 3.1.** *$A \in \mathbb{C}^{n\times n}$ is said to be J-EP if $A \circ A^{[\dagger]} = A^{[\dagger]} \circ A$.*

**Remark 3.2.** *Notice that when $J = I$, the above definition coincides with the usual definition of EP matrices. Note also that A is J-EP if and only if AJ is EP. It is a well known result that a square matrix A is EP if and only if $A^{\dagger}$ is a polynomial in A (see, for instance, Corollary 3 on page 178, Chapter 4, [3]). Noting that $(AJ)^{\dagger} = JA^{\dagger}$, we see that A is J-EP if and only if $A^{[\dagger]}$ is a polynomial in A. It is also interesting to note that there are matrices that are EP but not J-EP and J-EP but not EP. The following two examples illustrate this.*

**Example 3.3.** *Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Clearly, A is EP. One then computes $A^{[\dagger]}$ to be the matrix $A^{[\dagger]} = (1/4)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. It is then not hard to see that $A \circ A^{[\dagger]} \neq A^{[\dagger]} \circ A$ and so A is not J-EP.*

**Example 3.4.** *Let $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It can be easily seen that $A^{[\dagger]} = (1/4)A$, which proves that A is J-EP. However A is not EP.*

The following are consequences of the above definition.

**Theorem 3.5.** *If A is J-EP, then $A^{[\#]}$ exists. If $A^{[\dagger]} = A^{[\#]}$, then A is J-EP and vice versa.*

The following example shows that $A^{[\#]}$ may exist without $A$ being $J$-EP.

**Example 3.6.** *By taking $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we see that $A^{[\#]}$ exists, although A is not J-EP.*

We now have the following theorem.

**Theorem 3.7.** *Let $A, B$ be square matrices. We then have the following :*
*(a) If $AJ = JA$, then $A$ is EP if and only if $A$ is $J$-EP.*
*(b) If $AJ = JA$, then $AB$ is EP if and only if $A \circ B$ is $J$-EP.*
*(c) $A$ is $J$-EP if and only if $Ra(A^{[2]}) = Ra(A^{[*]})$.*

*Proof.* (a) : By definition, we note that $A \circ A^{[\dagger]} = AJJA^\dagger J = AA^\dagger J$. On the other hand, $A^{[\dagger]} \circ A = JA^\dagger JJA = JA^\dagger A = A^\dagger JA = A^\dagger AJ = AA^\dagger J$, as $AJ = JA$ and that $A$ is EP. Thus $A$ is $J$-EP.

Conversely, assume that $A$ is $J$-EP. We shall prove that $R(A) = R(A^*)$. Let $y \in R(A)$. Then, $y = AA^\dagger y = A \circ A^{[\dagger]} \circ y = A^{[\dagger]} \circ A \circ y$. Note that $Ra(A^{[\dagger]} \circ A) = Ra(I \circ A^\dagger A)$. Therefore, $y = I \circ A^\dagger A \circ x = JA^\dagger AJx = A^\dagger Ax \in R(A^*)$. Thus, $R(A) \subseteq R(A^*)$. On the other hand, if $y \in R(A^*)$, then $y = A^\dagger Ay = I \circ A^\dagger A \circ x$ (as $AJ = JA$). This means $y \in Ra(I \circ A^\dagger A) = Ra(A^{[\dagger]} \circ A)$ and so, $y = A^{[\dagger]} \circ A \circ u = A \circ A^{[\dagger]} \circ u = AA^\dagger u \in R(A)$. Thus, $R(A^*) \subseteq R(A)$. This completes the proof.

(b) : Suppose $AJ = JA$. We then have $(A \circ B)^{[\dagger]} = J(AB)^\dagger$. Therefore, $(A \circ B)$ is $J$-EP if and only if $J(AB)(AB)^\dagger = J(AB)^\dagger(AB)$ if and only if $AB$ is EP.

(c) : As pointed out earlier, $A$ is $J$-EP if and only if $AJ$ is EP. Therefore, $A$ is $J$-EP if and only if $R((AJ)^2) = R((AJ)^*) = R(JA^*)$. However, $Ra(A^{[2]}) = R((AJ)^2)$ and $Ra(A^{[*]}) = R((JA^*))$. The conclusion now follows. An alternate proof can be given as follows :

An alternate proof :

Suppose $A$ is $J$-EP. Then, $A^{[\#]}$ exists and so $Ra(A) = Ra(A^{[2]})$. Moreover, $Ra(A) = Ra(A^{[*]})$ and so one way is proved. For the converse, suppose $Ra(A^{[*]}) = Ra(A^{[2]})$. It is clear that $Ra(A^{[2]}) \subseteq Ra(A)$. Let $y = A \circ x \in Ra(A)$. From the lemma on linear equations (Lemma 2.2, [10]), we see that $x = A^{[\dagger]} \circ y + z, z \in Nu(A)$; that is, $x - z = A^{[\dagger]} \circ y \in Ra(A^{[\dagger]}) = Ra(A^{[*]}) = Ra(A^{[2]})$ and so, $x - z = A^{[2]} \circ u$. Consequently, $y = A \circ x = A^{[3]} \circ u \in Ra(A^{[2]})$. Therefore, $Ra(A) = Ra(A^{[2]}) = Ra(A^{[*]})$. Therefore, $R((AJ)(AJ)^\dagger) = R((AJ)^\dagger(AJ))$, which is the same as saying that $AJ$ is EP. $\square$

The following examples (Examples 3.8, 3.9 and 3.10) show that the commutativity assumption in the above theorem cannot be dispensed with.

**Example 3.8.** *Let $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then, $A$ is EP, $A^\dagger = (1/4)A$. However, $A$ is not $J$-EP as $A \circ A^{[\dagger]} = (1/2)\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, while $A^{[\dagger]} \circ A = (1/2)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. Note that $AJ \neq JA$.*

**Example 3.9.** *Let $J$ be as in Example 3.8 and $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. Then, $A = A^{[*]}$ and so $A^{[\#]}$ exists. Moreover, $A^{[\dagger]} = A^{[\#]} = (1/4)A$. Therefore, $A$ is $J$-EP. But $A$ is not EP. Note again that $AJ \neq JA$.*

**Example 3.10.** *Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $AB = 0$, which is EP. However, $A \circ B = -2A$, which is not $J$-EP. Note that $AJ \neq JA$. On the other hand, let $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $A \circ B = A$, which is $J$-EP. However, $AB = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$, which is not EP. In this case also, $AJ \neq JA$.*

The following result was proved by Meenakshi in relation to sums of EP matrices being EP.

**Theorem 3.11.** *(Theorem 1, [8]) Let $A_i$ $(i = 1 \ldots m)$ be EP. Then, $A := A_1 + \ldots + A_m$ is EP if any one of the following equivalent conditions hold :*

*(i)* $N(A) \subseteq N(A_i)$ *for each i.*

*(ii)* $rank\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} = rank(A).$

We then have the following theorem.

**Theorem 3.12.** *Let* $A_1, \dots, A_m$ *be J-EP and let* $A := A_1 + \dots + A_m$*. Suppose* $Nu(A) \subseteq Nu(A_i)$ *for each i and that* $A_i \circ A_j = 0$ *for* $i \neq j$*. Then A is J-EP.*

*Proof.* Assume that $Nu(A) \subseteq Nu(A_i)$ for each $i$. Since $A_iJ$ is EP for all $i$, this means that $A_1J + \dots + A_mJ$ is EP, by Theorem 3.11. Therefore, from Remarks 3.2, we infer that $A_1 + \dots + \dots A_m$ is $J$-EP. Alternately, from [7], we see that $R(J(A_1 + \dots + A_m)^*) = R(((A_1 + \dots + A_m)J)^2)$. The left hand side of this last equation is nothing but $Ra((A_1 + \dots + A_m)^{[*]})$, whereas the right hand side equals $Ra(A_1^{[2]} + \dots + A_m^{[2]})$ (after simplification using the fact that $A_i \circ A_j = 0$ for $i \neq j$). Note that $Ra(A_1^{[2]} + \dots + A_m^{[2]}) = Ra(A^{[2]})$, as $A_i \circ A_j = 0$ for $i \neq j$. Thus, $A = A_1 + \dots + A_m$ is $J$-EP. □

**Remark 3.13.** *If A is J-EP and* $A_i \circ A_j = 0$ *for* $i \neq j$ *then,* $R((AJ)^*) = R((A_1J)^2 + \dots + (A_mJ)^2)$*, by Remark 3.2. The right hand side of the above is contained in* $R((A_1J)^2) + \dots + R((A_mJ)^2) = \sum Ra(A_i^{[2]})$*. The left hand side is* $Ra(A^{[*]}) = Ra(A^{[2]})$*. An interesting problem in linear algebra is to study rank additivity results. For some recent results, refer [13] and the references cited therein. On the other hand, range additivity is less studied. Results connecting these two themes was studied by Baksalary, Semrl and Styan [1]. In particular, they proved the following : If* $A_1, A_2$ *and* $A_3$ *are matrices and* $A = A_1 + A_2 + A_3$ *and if every {1}- inverse of A is also a {1}-inverse of each of the* $A_i$*'s, then* $R(A) = \bigoplus R(A_i)$ *(Refer Theorems 4 and 5, [1]). Thus, in the setting of Theorem 3.12, if* $m = 3$ *and if every {1}-inverse of* $(A_1J)^2 + (A_2J)^2 + (A_3J)^2$ *is a {1}-inverse of* $(A_iJ)^2$ *for each* $i = 1, 2, 3$*, then* $Ra(A^{[2]}) = \sum Ra(A_i^{[2]})$*.*

We now investigate the reverse order law with respect to the Moore-Penrose inverse in the indefinite setting. It is known that in the Euclidean setting, if $A$ and $B$ are EP and if $R(A) = R(B)$, then $(AB)^\dagger = B^\dagger A^\dagger$ (See Theorem 7.2.4, Chapter 7, [5]). The following theorem is an analogue of the above.

**Theorem 3.14.** *If A and B are J-EP with* $Ra(A) = Ra(B)$*, then* $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$*.*

*Proof.* Since $A$ and $B$ are $J$-EP, $AJ$ and $BJ$ are EP. Also, $Ra(A) = Ra(B)$ implies that $R(AJ) = R(BJ)$. Therefore, by Theorem 7.2.4, Chapter 7, [5], we see that $(AJBJ)^\dagger = (BJ)^\dagger (AJ)^\dagger$. On simplifying, we get $(AJB)^\dagger = B^\dagger JA^\dagger$. The conclusion follows by noting that the reverse order law $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$ holds if and only if $(AJB)^\dagger = B^\dagger JA^\dagger$. □

The example below shows that the assumption $Ra(A) = Ra(B)$ in the above theorem cannot be dropped.

**Example 3.15.** *Let* $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$*. In this case, A and B are J-EP.* $Ra(A)$ *and* $Ra(B)$ *are spanned by the vectors* $(1, 1)^t$ *and* $(1, 0)^t$*, respectively. Also,* $A \circ B = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$ *and* $(A \circ B)^{[\dagger]} = (1/2)\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$*. On the other hand,* $B^{[\dagger]} \circ A^{[\dagger]} = (1/4)\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$*. Thus,* $(A \circ B)^{[\dagger]} \neq B^{[\dagger]} \circ A^{[\dagger]}$*.*

**Remark 3.16.** *It follows from Theorem 3.7 (b) that if A and B are such that* $JA^*A = A^*AJ$*, then* $A^*ABB^*$ *is EP if and only if* $A^*A \circ BB^*$ *is J-EP. However,* $A^*ABB^*$ *is EP if and only if* $(AB)^\dagger = B^\dagger A^\dagger$*. We thus have the following :*

**Theorem 3.17.** *Let A be such that* $AJ = JA$*. Then,* $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$ *if and only if* $A^*A \circ BB^*$ *is J-EP.*

*Proof.* $J$-EP-ness of $A^*A \circ BB^*$ is equivalent to $A^*ABB^*$ is EP, as $JA^*A = A^*AJ$ (from the above remark). However, from the above remark, $A^*ABB^*$ is EP if and only if $(AB)^\dagger = B^\dagger A^\dagger$. Since $AJ = JA$, from this we get the reverse order law $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$. □

The following example illustrates that even if $AJ = JA$, neither of the conclusions in Theorem 3.17 need hold.

**Example 3.18.** *Let* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ *and let* $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Then,* $AJ = JA, A^*A \circ BB^* = 2\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$, *is not J-EP. Since* $AJ = JA$, *we see that* $B^{[\dagger]} \circ A^{[\dagger]} = JB^\dagger A^\dagger = (1/4)\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. *On the other hand, one can see that* $(A \circ B)^{[\dagger]} = (1/10)\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

It is interesting to note, however, that $A \circ B$ can be $J$-EP without $A^*A \circ BB^*$ being $J$-EP, even if $AJ = JA$, as the following example shows.

**Example 3.19.** *Let* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Then* $C := A^*A \circ BB^*$ *is given by the matrix* $C = 4\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ *and one can easily check that C is not J-EP. Now,* $A \circ B = AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, *which is J-EP. Note that* $AJ = JA$.

We now pass on to the reverse order law for triple products. We first prove the following theorem, which is a simple application of Theorem 3.17.

**Theorem 3.20.** *Let* $A, B$ *and* $C$ *be square matrices and* $J$ *be a weight such that* $AJ = JA, (A \circ B)J = J(A \circ B)$. *Further, assume that* $A^*ABB^*$ *and* $(AB)^*(AB)CC^*$ *are EP. Then,* $(A \circ B \circ C)^{[\dagger]} = C^{[\dagger]} \circ B^{[\dagger]} \circ A^{[\dagger]}$.

*Proof.* Let us observe first that the conditions $(A \circ B)J = J(A \circ B)$ and EP-ness of $(AB)^*(AB)CC^*$ together imply that $(A \circ B \circ C)^{[\dagger]} = C^{[\dagger]} \circ (A \circ B)^{[\dagger]}$ (see Remarks 3.16 and Theorem 3.17). On the other hand, from the conditions $AJ = JA$ and the EP-ness of $A^*ABB^*$ we see that $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$. Combining these two, we get the desired result. □

A well known result of Hartwig [6] says that for matrices $A, B, C$ for which the product $ABC$ is defined, the reverse order law $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ holds if and only if $Q = P^\dagger$ and that $A^*APQ$ and $QPCC^*$ are EP, where $P = A^\dagger ABCC^\dagger$ and $Q = CC^\dagger B^\dagger A^\dagger A$ (see (ii) of Theorem 1, [6]). We prove below that for square matrices $A, B, C$ and a weight $J$, if $BJ = JB$, then the reverse order law for the indefinite product $A \circ B \circ C$ holds if and only if the reverse order law for the triple product $ABC$ holds; that is,

**Theorem 3.21.** $(A \circ B \circ C)^{[\dagger]} = C^{[\dagger]} \circ B^{[\dagger]} \circ A^{[\dagger]}$ *if and only if* $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$, *assuming* $BJ = JB$.

*Proof.* Assume that $BJ = JB$ and that $(A \circ B \circ C)^{[\dagger]} = C^{[\dagger]} \circ B^{[\dagger]} \circ A^{[\dagger]}$. Then, we have $J(AJBJC)^\dagger J = JC^\dagger JB^\dagger JA^\dagger J$, which in turn implies that $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$. The converse can be established in a similar manner. □

As the following example shows, the commutativity assumption $BJ = JB$ cannot be dispensed with in the above theorem.

**Example 3.22.** *Let* $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Let* $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. *It can be easily seen that* $(A \circ B \circ C)^{[\dagger]} = (-1/4)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, *whereas* $C^{[\dagger]} \circ B^{[\dagger]} \circ A^{[\dagger]} = (-1/16)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. *Note that* $BJ \neq JB$.

Baksalary, Hauke and Johnson considered polynomials in two variables, evaluated at $A$ and $\tilde{A}$ with $\tilde{A} \in \{A^=, A^\dagger, A^*\}$, in which $A^=$ denotes a reflexive generalized inverse of $A$ and obtained characterizations of EP matrices (Refer Theorems 1 and 2, [2]). It is natural to investigate an analogue of the same in indefinite inner product spaces with respect to the indefinite matrix product. Let us recall an interesting result. A problem posed in issue 34 of IMAGE - The Bulletin of the International Linear Algebra Society [14], was to prove the equivalence of the following two statements for a square matrix $A$ : (1) $A + A^\dagger = 2AA^\dagger$ (2) $A + A^\dagger = AA^\dagger + A^\dagger A$. In issue 35 of IMAGE, in addition to the equivalence of (1) and (2) above, it was shown that each of them implies EP-ness of $A$. Another related problem is to characterize all square matrices such that $A + A^\dagger = 2AA^\dagger$. The solution to this problem is given in the following.

**Lemma 3.23.** *(Lemma on page 2336, [2]) Let $A \in M_n(\mathbb{C})$. Then $A + A^\dagger = 2AA^\dagger$ if and only if $A$ is an EP matrix and $A^3 - 2A^2 + A = 0$.*

Let us point out that if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $A$ is invertible and hence trivially $J$-EP. In this case, $A^{[-1]} = A$ and hence, the equation $A + A^{[\dagger]} = A \circ A^{[\dagger]} + A^{[\dagger]} \circ A$ is not satisfied. Note that the other equation $A^{[3]} - 2A^{[2]} + A = 0$ is not satisfied, too. However, a generalization of the above Lemma does hold in indefinite inner product spaces, as we prove below.

**Lemma 3.24.** *Let $A$ be a square matrix such that $A + A^{[\dagger]} = 2A \circ A^{[\dagger]}$. Then, $A$ is $J$-EP and satisfies the equation $A^{[3]} - 2A^{[2]} + A = 0$. The converse is also true.*

*Proof.* The equation $A + A^{[\dagger]} = 2A \circ A^{[\dagger]}$ can be written as $A^{[\dagger]} = A \circ (2A^{[\dagger]} - J)$. From this it follows that $Ra(A^{[\dagger]}) \subseteq Ra(A)$ and hence $R((AJ)^\dagger) = R(JA^\dagger) \subseteq R(AJ)$, proving that $AJ$ is EP. Thus, $A$ is $J$-EP. Now, premultiplying and postmultiplying the equation $A + A^{[\dagger]} = 2A \circ A^{[\dagger]}$ by $A$, we get $A^{[3]} - 2A^{[2]} + A = 0$.

Conversely, if $A$ is $J$-EP, then by premultiplying and postmultiplying the equation $A^{[3]} - 2A^{[2]} + A = 0$ by $A^{[\dagger]}$, we get the result. □

Let $P(A, A^{[\dagger]}) = A \circ Q_1(A, A^{[\dagger]}) + A^{[\dagger]} \circ Q_2(A, A^{[\dagger]})$, where $Q_i(A, A^{[\dagger]})$ are polynomials in two variables evaluated at $A$ and $A^{[\dagger]}$, with multiplication with respect to the indefinite matrix product. We then have the following analogue of Corollary in page 2338 of [2].

**Theorem 3.25.** *Let $P(A, A^{[\dagger]}) = 0$, where $P(A, A^{[\dagger]})$ is as above with the condition that $Q_i(A, A^{[\dagger]})$ is non-singular for at least one i. Then, $A$ is $J$-EP. Consequently, $P(A, A^{[\dagger]})$ can be expressed as an annihilating polynomial of $A$.*

*Proof.* Suppose $P(A, A^{[\dagger]}) = 0$ and that $Q_2(A, A^{[\dagger]})$ is non-singular. Then $A \circ Q_1(A, A^{[\dagger]}) \circ Q_2(A, A^{[\dagger]})^{[-1]} = -A^{[\dagger]}$. This means, $Ra(A^{[\dagger]}) \subseteq Ra(A)$ and so $AJ$ is EP. Therefore, $A$ is $J$-EP. The second statement follows from Corollary 3 in Chapter 4 in [3]. □

Nonsingularity of the $Q_i$ for at least one $i$ is crucial in the above proof. Also, it may be the case that $A$ is $J$-EP with neither of the $Q_i$ being non-singular. The following example illustrates this.

**Example 3.26.** *Let $A$ and $J$ be as in Example 3.3. We know from Example 3.3 that $A$ is not $J$-EP. From the equation $A \circ A^{[\dagger]} \circ A = A$, we see that $A \circ (A^{[\dagger]} \circ A - J) = 0$. In this case, $Q_1(A, A^{[\dagger]}) = (1/2)\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ and $Q_2(A, A^{[\dagger]}) = 0$, thereby proving that neither of the $Q_i$ is non-singular. On the other hand, if $A$ and $J$ are as in Example 3.4, then $Q_1$ and $Q_2$ are given by the matrices $A^{[\dagger]}$ and $A$, respectively, which are clearly singular.*

## 4. Concluding Remarks

For matrices $A$ and $B$ of the same order, $A$ is said to be below $B$ in the star order, denoted by $A <^* B$, if $AA^* = BA^*$ and $A^*A = A^*B$. The star order, which is a partial order, was introduced by Drazin. The star order and other kinds of matrix partial orders were studied in connection with the reverse order law for generalized inverses. A good and an up to date reference on this topic is the monograph by S. K. Mitra, Bhimasankaram and Saroj Malik [12]. Among several interesting results concerning the star order, we state below two interesting results.

**Theorem 4.1.** *(Corollary 5.2.9, [12]) If A and B are matrices of the same order, then the following are eqiuvalent:* (1) $A <^* B$ (2) $A^\dagger <^* B^\dagger$ (3) $AA^\dagger B = A = BA^\dagger A = BA^\dagger B$ *and* (4) $A^\dagger AB = A^\dagger = B^\dagger AA^\dagger = B^\dagger AB^\dagger$.

**Theorem 4.2.** *(Theorem 5.4.3, [12]) Let A and B be mtrices of the same order such that $A <^* B$. Then, the following are equivalent:*
(1) *A is EP* (2) $A^\dagger B = BA^\dagger$ *and* (3) $AB^\dagger = B^\dagger A$.

One can now define the star order with respect to the indefinite matrix product. However, it turns out that the two notions coincide. This is because : we would want to define $A <^{[*]} B$ if $A \circ A^{[*]} = B \circ A^{[*]}$ and $A^{[*]} \circ A = A^{[*]} \circ B$. This, after simplification, yields $A <^* B$. One can now attempt to study reverse order laws and other partial orders with respect to the indefinite matrix product. An in-depth investigation of this is deferred for future study. Let us first present a generalization of Theorem 4.2 above.

**Theorem 4.3.** *Let A and B be square matrices of the same order such that $A <^* B$. Then, A is J-EP if and only if* $A^{[\dagger]} \circ B = B \circ A^{[\dagger]}$.

*Proof.* Suppose $A <^* B$ and that $A$ is $J$-EP. Since $A$ is $J$-EP, $AA^\dagger J = JA^\dagger A$. From $A <^* B$, we have that $AA^\dagger = BA^\dagger, A^\dagger A = A^\dagger B$. Therefore, $AA^\dagger J = BA^\dagger J$ and $JA^\dagger A = JA^\dagger B$. From this, we conclude that $BA^\dagger J = JA^\dagger B$, which is the same as saying $A^{[\dagger]} \circ B = B \circ A^{[\dagger]}$. For the converse, note that if $BA^\dagger J = JA^\dagger B$, then from $A <^* B$, we see that $AA^\dagger J = JA^\dagger A$. This implies that $A$ is $J$-EP. □

We end with the following Theorem.

**Theorem 4.4.** *Let A and B be square matrices of the same size. Then, we have the following: If* $A <^* B, AJ = JA, BJ = JB$ *and* $AB^* = B^*A$, *then* $A^{[\dagger]} \circ B = B \circ A^{[\dagger]} \Longrightarrow B^{[\dagger]} \circ A = A \circ B^{[\dagger]}$.

*Proof.* From Theorem 4.3, it follows that $A$ is $J$-EP, and hence EP (as $AJ = JA$). This, in turn, is equivalent to $A^\dagger B = BA^\dagger$ (Refer Theorem 4.2). As remarked earlier (Refer Remarks 3.2), $A$ is EP if and only if $A^\dagger$ is a polynomial in $A$. Combining this with the assumption $AB^* = B^*A$, we see that $A^\dagger B^* = B^*A^\dagger$. We then have $(A^{[\dagger]})^* A^{[\dagger]} BB^* = (A^\dagger)^* A^\dagger BB^* = (A^\dagger)^* BA^\dagger B^* = (B^*A^\dagger)^*(A^\dagger B^*) = (A^\dagger B^*)^*(A^\dagger B^*)$, which is Hermitian. From $AJ = JA$, we then have from Theorem 3.17 that $(A^{[\dagger]} \circ B)^{[\dagger]} = B^{[\dagger]} \circ A$. On the other hand, $B^*BA^\dagger(A^\dagger)^* = B^*A^\dagger B(A^\dagger)^* = (A^\dagger B^*)B(A^\dagger)^* = (B(A^\dagger)^*)^*(B(A^\dagger)^*)$, which is again Hermitian. Now, since $BJ = JB$, it follows from Theorem 3.17 that $B^{[\dagger]} \circ A = (A^{[\dagger]} \circ B)^{[\dagger]} = (B \circ A^{[\dagger]})^{[\dagger]} = A \circ B^{[\dagger]}$. □

**Acknowledgement :** I thank the anonymous referee for critical comments that has improved the manuscript to a great extent. Most of this work was done when the author was with the Mathematics Department at IISER - Kolkata. The author thanks IISER - Kolkata for support and facilities. I am also thankful to Professor Dragan S. Djordjević and Ms. Ivana Radojević, University of Nis, Serbia, for helpful comments on a preliminary version of this manuscript.

## References

[1] J. K. Baksalary, P. Semrl and G. P. H. Styan, A note on rank additivity and range additivity, Linear Algebra and its Applications, 237/238 (1996), 489–498.
[2] O. M. Baksalary, J. Hauke and C. R. Johnson, Polynomials satisfied by two linked matrices, Linear Algebra and its Applications, 429 (2008), 2335–2339.
[3] A. Ben-Israel and T. N. E. Greville, Generalized Inverses: Theory and Applications, ($2^{nd}$ Edition), Canadian Mathematical Society Books in Mathematics, Vol. 15, Springer Verlag, New York, 2003.
[4] J. Bognar, Indefinite inner product spaces, Springer Verlag, 1974.
[5] D. S. Djordjevic and V. Rakocevic, Lectures on generalized inverses, University of Nis, Faculty of Sciences and Mathematics, Nis, 2008.
[6] R. E. Hartwig, The reverse order law revisited, Linear Algebra and its Applications, 76 (1986), 241–246.
[7] S. Jayaraman, Characterization of EP matrices, Solution to Problem 43.1, IMAGE (Bulletin of the International Linear Algebra Society), Spring 2010 Issue.
[8] Ar. Meenakshi, On sums of EP matrices, Houston Journal of Mathematics, 9(1) (1983), 63–69.
[9] K. Ramanathan and K. C. Sivakumar, Theorems of the alternative over indefinite inner product spaces, Journal of Optimization Theory and Appllications, 137(1) (2008), 99–104.

[10] K. Ramanathan and K. C. Sivakumar, Nonnegative Moore-Penrose inverse of Gram matrices in an indefinite inner product space, Journal of Optimization Theory and Applications, 140(1) (2009), 189–196.

[11] K. Ramanathan, K. Kamaraj and K. C. Sivakumar, Indefinite product of matrices and applications to indefinite inner product spaces, Journal of Analysis, 12 (2004), 135–142.

[12] S. K. Mitra, P. Bhimasankaram and Saroj B. Malik, Matrix Partial Orders, Shorted Operators and Applications, Series in Algebra, Vol. 10, World Scientific, Singapore, 2010.

[13] Y. Tian, The Moore-Penrose inverse for sums of matrices under rank additivity conditions, Linear and Multilinear Algebra, 53(1) (2005), 45–65.

[14] G. Trenkler, A property for the sum of a matrix $A$ and its Moore-Penrose inverse $A^{\dagger}$, Problem 34.8, IMAGE (Bulletin of the International Linear Algebra Society), 34 (2005), 39.