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Spectral properties of m-isometric operators

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Abstract. We study spectral properties of an m-isometric operator and show that an m-isometric operator has the single valued extension property. Also we show if an m-isometric operator T is invertible and paranormal, then T is a unitary operator. Next we give a new proof of $T^*T - I \geq 0$ if T is a 2-isometry.

1. Introduction

J. Agler and M. Stankus introduced an m-isometry [1], [2] and [3]. Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be a set of all bounded linear operators on $\mathcal{H}$. Let ${}_mC_k$ be the binomial coefficient. An operator $T \in B(\mathcal{H})$ is said to be an m-isometry if $\displaystyle\sum_{k=0}^{m}(-1)^k \; {}_mC_k \, T^{*m-k}T^{m-k} = 0$. It is known that an isometry ($m = 1$) admits Wold decomposition and has many interesting properties. m-Isometries are not only a natural extension of an isometry, but they are also important in the study of Dirichlet operators and some other classes of operators. For an operator T, we denote

$$B_m(T) = T^{*m}T^m - \; {}_mC_1 \cdot T^{*m-1}T^{m-1} + \cdots + (-1)^{m-1} \cdot I.$$

Hence T is an m-isometry if and only if $B_m(T) = 0$.

Agler and Stankus [1] proved the following interesting result. If T is an m-isometry, then $B_{m-1}(T) \geq 0$ ([1], Proposition 1.5). We think their proof of $B_{m-1}(T) \geq 0$ is fantastic. First we extend some results obtained by S.M. Patel [6] to m-isometric operators. Next we give a new proof of $B_1(T) \geq 0$ for a 2-isometric operator T. This is a known result, however we think our proof would be interesting.

Let $\sigma(T)$ and $r(T)$ be the spectrum and the spectral radius of T, respectively. Also let $\mathbf{T} = \{z : |z| = 1\}$. For an m-isometry T, the following theorem holds.

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Theorem A [1]. *Let T be an m-isometry. Then the following assertions hold.*
(1) *If z is an approximate eigenvalue of T, then $z \in \mathbf{T}$.*
(2) *If T is invertible, then $\sigma(T) \subset \mathbf{T}$.*
(3) *If T is not invertible, then $\sigma(T) = \{z : |z| \leq 1\}$.*

2. m-isometry

Theorem 1. *For an m-isometry T the following statements hold.*
(a) *If a is an eigenvalue of T, then $\overline{a}$ is an eigenvalue of T^*.*
(b) *Eigenvectors of T corresponding to distinct eigenvalues are orthogonal.*
(c) *If a is an approximate eigenvalue of T, then $\overline{a}$ is an approximate eigenvalue of T^*.*
(d) *If a, b are distinct approximate eigenvalues of T, and $\{x_n\}, \{y_n\}$ sequences of unit vectors such that $(T-a)x_n \longrightarrow 0$ and $(T-b)y_n \longrightarrow 0$, then $\langle x_n, y_n\rangle \longrightarrow 0$.*

Proof.
(a) Let $Tx = ax$ $(x \neq 0)$. Then it holds

$$0 = \left(\sum_{j=0}^{m} (-1)^j {}_mC_j T^{*(m-j)} T^{m-j}\right) x = (aT^* - 1)^m x.$$

Since $|a| = 1$, we have $(T^* - \overline{a})^m x = 0$ and hence $\overline{a}$ is an eigenvalue of T^*.
(b) Let a, b be distinct eigenvalues of T and x, y be corresponding eigenvectors. It holds

$$0 = \left\langle \left(\sum_{j=0}^{m} (-1)^j {}_mC_j T^{*(m-j)} T^{m-j}\right) x, y \right\rangle = (a\overline{b} - 1)^m \langle x, y\rangle.$$

Since $a \neq b$ and $|b| = 1$, $a - b = (a\overline{b} - 1)b$ and $a\overline{b} - 1 \neq 0$. Hence, $\langle x, y\rangle = 0$.
(c) Let $\{x_n\}$ be a sequence of unit vectors such that $\lim(T - a)x_n = 0$. Then since

$$\left(\sum_{j=0}^{m} (-1)^j {}_mC_j T^{*(m-j)} T^{m-j}\right) x_n = 0,$$

$$0 = \lim_{n\to\infty} \left(\sum_{j=0}^{m} (-1)^j {}_mC_j T^{*(m-j)} T^{m-j}\right) x_n = \lim_{n\to\infty} (aT^* - 1)^m x_n.$$

Therefore, since $|a| = 1$,

$$\lim_{n\to\infty} a^m (T^* - \overline{a})^m x_n = 0$$

and $\overline{a}$ is an approximate eigenvalue of T^*.
(d) Since

$$0 = \lim_{n\to\infty} \left\langle \left(\sum_{j=0}^{m} (-1)^j {}_mC_j T^{*(m-j)} T^{m-j}\right) x_n, y_n \right\rangle = (a\overline{b} - 1)^m \lim_{n\to\infty} \langle x_n, y_n\rangle,$$

we have $\langle x_n, y_n\rangle \longrightarrow 0$. □

An operator T is said to have the single valued extension property if, for every open subset $\mathcal{U}$ of $\mathbb{C}$, an analytic function $f : \mathcal{U} \longrightarrow \mathcal{H}$ satisfies $(T - \lambda)f(\lambda) = 0 \quad \forall\lambda \in \mathcal{U}$, then $f(\lambda) = 0 \quad \forall\lambda \in \mathcal{U}$. Uchiyama and Tanahashi [7] studied the spectral condition (d) in Theorem 1 and proved that if T has the spectral condition (d), then T has the single valued extension property. Hence we have the following result.

Theorem 2. *An m-isometric operator T has the single valued extension property.*

An operator T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|$ for all unit vector $x \in \mathcal{H}$. It is well known that if T is paranormal, then $r(T) = \|T\|$. Moreover, if there exists T^{-1}, then T^{-1} is also paranormal. The following lemma can be easily obtained. For the completeness, we give a proof.

Lemma 1. *Let T be invertible and paranormal. If $\sigma(T) \subset \mathbf{T}$, then T is a unitary operator.*

Proof. Since $\|T\| = r(T) = 1$, $T^*T \leq I$. Therefore, $T^{*-1}T^*TT^{-1} \leq T^{*-1}T^{-1}$. Hence, $I \leq T^{*-1}T^{-1}$. Since T^{-1} is paranormal and $\sigma(T^{-1}) \subset \mathbf{T}$, $\|T^{-1}\| = r(T^{-1}) = 1$, that is, $T^{*-1}T^{-1} \leq I$. Therefore, $T^{*-1}T^{-1} = I$ and $T^*T = I$. Since T is invertible, T is a unitary operator. □

If T is an invertible m-isometry, then $\sigma(T) \subset \mathbf{T}$. Hence, we have the following result.

Theorem 3. *Let T be an m-isometry. If T is invertible and paranormal, then T is a unitary operator.*

Theorem 3 holds for a $*$-paranormal operator. An operator $T \in B(\mathcal{H})$ is said to be $*$-paranormal if

$$\|T^*x\|^2 \leq \|T^2x\|\|x\| \text{ for all } x \in \mathcal{H}.$$

Theorem 4. *Let T be an m-isometry. If T is invertible and $*$-paranormal, then T is a unitary operator.*

For the proof of Theorem 4, we prepare the following.

Theorem B [8]. *Let T be $*$-paranormal. Then $\|T\| = r(T)$.*
Moreover, if T is invertible, then $\|T^{-1}\| \leq r(T^{-1})^3 \cdot r(T)^2$.

First we give other result.

Lemma 2. *Let T be invertible and satisfy*

$$\|T^2x\|^3 \leq \|T^3x\|^2 \cdot \|x\| \text{ for every } x \in \mathcal{H}.$$

Then $\|T\|^2 \leq \|T^{-1}\| \cdot r(T)^3$.

Proof. By the definition of $*$-paranormality, we have

$$\|T^2x\|^3 \leq \|T^3x\|^2 \cdot \|x\|,$$

$$\|T^3x\|^3 \leq \|T^4x\|^2 \cdot \|Tx\|,$$

$$\|T^4x\|^3 \leq \|T^5x\|^2 \cdot \|T^2x\|,$$

$$\|T^5x\|^3 \leq \|T^6x\|^2 \cdot \|T^3x\|,$$

$$\|T^6x\|^3 \leq \|T^7x\|^2 \cdot \|T^4x\|,$$

$$\vdots$$

$$\|T^nx\|^3 \leq \|T^{n+1}x\|^2 \cdot \|T^{n-2}x\|,$$

$$\|T^{n+1}x\|^3 \leq \|T^{n+2}x\|^2 \cdot \|T^{n-1}x\|,$$

$$\|T^{n+2}x\|^3 \leq \|T^{n+3}x\|^2 \cdot \|T^nx\|.$$

Therefore, for $n \geq 2$,

$$\|T^2x\|^2 \cdot \|T^{n+1}x\| \cdot \|T^{n+2}x\| \leq \|x\| \cdot \|Tx\| \cdot \|T^{n+3}x\|^2.$$

Next it holds

$$\|T^2x\|^2 \cdot \|T^3x\| \cdot \|T^4x\| \leq \|x\| \cdot \|Tx\| \cdot \|T^5x\|^2,$$
$$\|T^2x\|^2 \cdot \|T^4x\| \cdot \|T^5x\| \leq \|x\| \cdot \|Tx\| \cdot \|T^6x\|^2,$$
$$\|T^2x\|^2 \cdot \|T^5x\| \cdot \|T^6x\| \leq \|x\| \cdot \|Tx\| \cdot \|T^7x\|^2,$$
$$\vdots$$
$$\|T^2x\|^2 \cdot \|T^{n+1}x\| \cdot \|T^{n+2}x\| \leq \|x\| \cdot \|Tx\| \cdot \|T^{n+3}x\|^2,$$
$$\|T^2x\|^2 \cdot \|T^{n+2}x\| \cdot \|T^{n+3}x\| \leq \|x\| \cdot \|Tx\| \cdot \|T^{n+4}x\|^2,$$
$$\|T^2x\|^2 \cdot \|T^{n+3}x\| \cdot \|T^{n+4}x\| \leq \|x\| \cdot \|Tx\| \cdot \|T^{n+5}x\|^2.$$

Therefore, for $n \geq 2$,

$$\|T^2x\|^{2(n+1)} \cdot \|T^3x\| \cdot \|T^4x\|^2 \leq \|x\|^{n+1} \cdot \|Tx\|^{n+1} \cdot \|T^{n+4}x\| \cdot \|T^{n+5}x\|^2$$
$$\leq \|x\|^{n+1} \cdot \|Tx\|^{n+1} \cdot \|T^{n+1}x\| \cdot \|T^{n+1}x\|^2 \cdot \|T\|^{11}.$$

Hence,

$$\|T^2x\|^2 \cdot \|T^3x\|^{\frac{1}{n+1}} \cdot \|T^4x\|^{\frac{2}{n+1}} \leq \|x\| \cdot \|Tx\| \cdot \|T^{n+1}x\|^{\frac{3}{n+1}} \cdot \|T\|^{\frac{11}{n+1}}$$
$$\leq \|x\| \cdot \|Tx\| \cdot \|T^{n+1}\|^{\frac{3}{n+1}} \cdot \|x\|^{\frac{3}{n+1}} \cdot \|T\|^{\frac{11}{n+1}}.$$

Let $n \to \infty$, we have

$$\|T^2x\|^2 \leq \|x\| \cdot \|Tx\| \cdot r(T)^3.$$

Let x be $T^{-1}y$. Therefore, we have

$$\|Ty\|^2 \leq \|T^{-1}y\| \cdot \|y\| \cdot r(T)^3$$

and $\|T\|^2 \leq \|T^{-1}\| \cdot r(T)^3$. $\square$

If T is $*$-paranormal, then

$$\|Tx\|^4 = (T^*Tx, x)^2 \leq \|T^*Tx\|^2 \cdot \|x\|^2 \leq \|T^3x\| \cdot \|Tx\| \cdot \|x\|^2.$$

Therefore, it holds

$$(*) \qquad \|Tx\|^3 \leq \|T^3x\| \cdot \|x\|^2 \quad \text{for every } x \in \mathcal{H}.$$

If T is invertible and $*$-paranormal, by $(*)$ then we have

$$\|T^{-2}x\|^3 \leq \|x\| \cdot \|T^{-3}x\|^2 \quad \text{for every } x \in \mathcal{H}.$$

Let $S = T^{-1}$. Then the operator S satisfies

$$\|S^2x\|^3 \leq \|S^3x\|^2 \cdot \|x\| \quad \text{for every } \ x \in \mathcal{H}.$$

Therefore, it holds $\|S\|^2 \leq \|S^{-1}\| \cdot r(S)^3$.

Hence by Lemma 2, we have

Lemma 3. *Let T be invertible and $*$-paranormal. Then*

$$\|T^{-1}\|^2 \leq \|T\| \cdot r(T^{-1})^3.$$

By Theorem B and Lemma 3, for an invertible $*$-paranormal operator T we have

$$\|T^{-1}\| \leq r(T^{-1})^3 \cdot r(T)^2 \; = \; r(T^{-1})^3 \cdot \|T\|^2.$$

Thus it holds

$$\|T^{-1}\|^3 \leq \|T\|^3 \cdot r(T^{-1})^6$$

and

$$\|T^{-1}\| \leq \|T\| \cdot r(T^{-1})^2.$$

Proof of Theorem 4. Since T is invertible and m-isometric, $\sigma(T) \subset \mathbf{T}$. Hence, by Theorem B, $\|T\| = 1$. Since T is invertible and $*$-paranormal, then $\|T^{-1}\| = 1$ by Lemma 3. So it holds $\|T\| = \|T^{-1}\| = 1$. Hence T is a unitary operator. □

Remark 1. There exists an invertible $*$-paranormal operator T such that $r(T^{-1}) < \|T^{-1}\|$ (cf. [8]).

3. 2-isometry

In this section, we give a new proof of $T^*T - I \geq 0$ if T is a 2-isometry. For this we need the following lemmas.

Lemma 4. *For a constant $a \in \mathbb{R}$ and $m \in \mathbb{N}$, it holds*

$$-\sum_{k=1}^{m} (-1)^k \, {}_mC_k \, (1 + (m-k)a) = 1 + ma.$$

Proof. We have

$$-\sum_{k=1}^{m} (-1)^k \, {}_mC_k = 1 - \Big(\sum_{k=0}^{m} (-1)^k \, {}_mC_k \Big) = 1 - (1-1)^m = 1$$

and

$$\begin{aligned} -\sum_{k=1}^{m} (-1)^k \, {}_mC_k \, (m-k) &= -m \sum_{k=1}^{m} (-1)^k \, {}_mC_k + \sum_{k=1}^{m} (-1)^k \, k \, {}_mC_k \\ &= m + \sum_{k=1}^{m} m \, (-1)^k \, {}_{m-1}C_{k-1} = m + m \sum_{j=0}^{m-1} (-1)^{j+1} \, {}_{m-1}C_j \\ &= m - m \sum_{j=0}^{m-1} (-1)^j \, {}_{m-1}C_j = m - m(1-1)^{m-1} = m. \; \square \end{aligned}$$

Lemma 5. *Let T be a 2-isometry. If $T^*Tx = (1+a)x$ for some constant $a \in \mathbb{R}$ and a non-zero vector $x \in \mathcal{H}$, then, for every m ($m \geq 2$),*

$$T^{*m}T^m x = (1 + ma)x.$$

Proof. Since T is a 2-isometry, it holds

$$T^{*2}T^2x = 2T^*Tx - x = (1+2a)x.$$

Hence, it holds for $m = 2$. We assume $T^{*k}T^kx = (1+ka)x$ for $k = 2, ..., m-1$. It is easy to see that T is an m-isometry for every $m \geq 2$. Since

$$T^{*m}T^m = -\sum_{k=1}^{m}(-1)^k\, {}_mC_k\, T^{*(m-k)}T^{m-k},$$

by Lemma 4 it holds

$$T^{*m}T^m x = \Big(-\sum_{k=1}^{m}(-1)^k\, {}_mC_k\, T^{*(m-k)}T^{m-k}\Big)x$$

$$= \Big(-\sum_{k=1}^{m}(-1)^k\, {}_mC_k\,(1+(m-k)a)\Big)x = (1+ma)x.$$

By induction, the proof is complete. □

Theorem 5. *Let T be a 2-isometry. Then $T^*T - I \geq 0$.*

Proof. We assume $T^*T - I \ngeq 0$. Since the operator $T^*T - I$ is hermitian, there exist $a < 0$ and a sequence $\{x_n\}$ of unit vectors such that $(T^*T - I - a)x_n \longrightarrow 0$ $(n \longrightarrow \infty)$. By Berberian's method [4], we can assume that there exists a non-zero vector x such that $(T^*T - I - a)x = 0$. Hence we have $T^*Tx = (1+a)x$. Since T is a 2-isometry, by Lemma 5 it holds

$$T^{*m}T^m x = (1+ma)x \ \text{ for every } m \geq 2.$$

Since $a < 0$ and $\|T^m x\|^2 = (1+ma)\|x\|^2$, it's a contradiction for some large $m \in \mathbb{N}$. □

Then we have the following corollary.

Corollary 1 [6]. *Let T be a 2-isometry. If T is power bounded, then T is an isometry.*

Proof. By the above theorem, $T^*T - I \geq 0$. Suppose T is power bounded. If T is not an isometry, there is $a > 0$ such that $a \in \sigma(T^*T - I)$. Therefore, similarly as in the proof of Theorem 5, we have $\|T^m\| \geq \sqrt{1+ma}$ for every $m \in \mathbb{N}$. This is a contradiction to our assumption. □

Remark 2. An operator T is called concave if $T^{*2}T^2 - 2T^*T + I \geq 0$. Then by [5], it holds $T^{*m}T^m \geq mT^*T - (m-1)I$ for every $m \geq 2$. Hence, for a concave operator T if $T^*Tx = (1+a)x$ for some constant $a \in \mathbb{R}$ and a non-zero vector $x \in \mathcal{H}$, then $\|T^m x\|^2 \geq (1+ma)\|x\|^2$ $(m \geq 2)$. Hence, a power bounded concave operator T is a contraction, i.e., $T^*T \leq I$.

Patel proved that a 2-isometry similar to a spectraloid operator (the spectral radius coincides with the numerical radius, i.e., $r(T) = w(T)$) is an isometry ([6], Corollary 2.6). We have the following result.

Theorem 6. *Let T be a 2-isometry. If the range of T is dense, then T is a unitary operator.*

Proof. Since $(T^*T - I)^2 \geq 0$ and $2T^*T - I = T^{*2}T^2$,

$$0 \leq (T^*T - I)^2 = (T^*T)^2 - 2T^*T + I = (T^*T)^2 - T^{*2}T^2 = T^*(TT^* - T^*T)T.$$

Since the range of T is dense, we have $TT^* - T^*T \geq 0$. Hence, T^* is hyponormal and $\|T\| = \|T^*\| = r(T^*) = r(T) = 1$. By Corollary 2.6 of [6], T is an isometry. Since T is an isometry, the range of T is closed and hence T is a unitary operator. □

The following result is a direct consequence of the above theorem.

Corollary 2 [1, Proposition 1.23]. *If T is an invertible 2-isometry, then T is a unitary operator.*

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