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New Weyl-type Theorems - I

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Abstract. In this paper we introduce the new spectral properties (W_E) and (UW_{E_a}). An operator T satisfies property (W_E) (resp. (UW_{E_a}), if its spectrum (resp. its approximate spectrum) is the disjoint union of its Weyl spectrum and its isolated eigenvalues(resp. of its upper semi-Weyl spectrum and its isolated eigenvalues in its approximate spectrum). The main purpose of the paper is to study relationship between the properties (W_E), (UW_{E_a}) and other Weyl-type theorems.

1. Introduction

Let X be a Banach space, and let $L(X)$ be the Banach algebra of all bounded linear operators acting on X. For $T \in L(X)$, we will denote by $N(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $R(T)$ the range of T, by $\beta(T)$ its defect and by T^* the adjoint of T. We will denote also by $\sigma(T)$ the spectrum of T and by $\sigma_a(T)$ the approximate point spectrum of T. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. If $T \in L(X)$ is either upper or lower semi Fredholm, then T is called a semi-Fredholm operator, and the index of T is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. An operator $T \in L(X)$ is called a Weyl operator if it is a Fredholm operator of index zero. The Weyl spectrum $\sigma_W(T)$ of T is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I$ is not a Weyl operator$\}$.

For a bounded linear operator T and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$, viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. a lower) semi-B-Fredholm operator. A semi-B- Fredholm operator T is an upper or a lower semi-B-Fredholm operator, and in this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [11]. Moreover if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator, see [2]. An operator $T \in L(X)$ is said to be a B-Weyl operator [4], if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I$ is not a B-Weyl operator$\}$.

The ascent $a(T)$ of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$, and the descent $\delta(T)$ of T, is defined by $\delta(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, with $\inf \emptyset = \infty$.

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According to [13], a complex number λ is a pole of the resolvent of T if and only if $0 < \max(a(T-\lambda I), \delta(T-\lambda I)) < \infty$. Moreover, if this is true, then $a(T-\lambda I) = \delta(T-\lambda I)$. An operator T is called Drazin invertible if 0 is a pole of T. The Drazin spectrum $\sigma_D(T)$ of T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Drazin invertible$\}$.

Define also the set $LD(X)$ by $LD(X) = \{T \in L(X) : a(T) < \infty$ and $R(T^{a(T)+1})$ is closed$\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\}$. Following [10], an operator $T \in L(X)$ is said to be left Drazin invertible if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$.

Let $SF_+(X)$ be the class of all upper semi-Fredholm operators and $SF_+^-(X) = \{T \in SF_+(X) : \mathrm{ind}(T) \leq 0\}$. The upper semi-Weyl spectrum $\sigma_{SF_+^-}(T)$ of T is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(X)\}$. Similarly is defined the upper semi-B-Weyl spectrum $\sigma_{SBF_+^-}(T)$ of T.

An operator $T \in L(X)$ is called upper semi-Browder if it is upper semi-Fredholm operator of finite ascent, and is called Browder if it is a Fredholm of finite ascent and descent. The upper semi-Browder spectrum $\sigma_{ub}(T)$ of T is defined by $\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not upper semi-Browder$\}$, and the Browder spectrum $\sigma_b(T)$ of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Browder$\}$.

An operator $T \in L(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \longrightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has this property at every $\lambda \in \mathbb{C}$. (See [15] for more details about this concept).

Below, we recall the list of all symbols and notations we will use:

$E(T)$: eigenvalues of T that are isolated in the spectrum $\sigma(T)$ of T,
$E^0(T)$: eigenvalues of T of finite multiplicity that are isolated in the spectrum $\sigma(T)$ of T,
$E_a(T)$: eigenvalues of T that are isolated in the approximate point spectrum $\sigma_a(T)$ of T,
$E_a^0(T)$: eigenvalues of T of finite multiplicity that are isolated in the spectrum $\sigma_a(T)$ of T,
$\Pi(T)$: poles of T,
$\Pi^0(T)$: poles of T of finite rank,
$\Pi_a(T)$: left poles of T,
$\Pi_a^0(T)$: left poles of T of finite rank,
$\sigma_b(T)$: Browder spectrum of T,
$\sigma_D(T)$: Drazin spectrum of T,
$\sigma_{ub}(T)$: upper semi-Browder spectrum of T,
$\sigma_{BW}(T)$: B-Weyl spectrum of T,
$\sigma_W(T)$: Weyl spectrum of T,
$\sigma_{SF_+^-}(T)$: upper semi-Weyl spectrum of T,
$\sigma_{SBF_+^-}(T)$: upper semi-B-Weyl spectrum of T,
$\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$,
$\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$,
$\Delta(T) = \sigma(T) \setminus \sigma_W(T)$,
$\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$.
$\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$: Browder's theorem holds for T,
$\sigma(T) \setminus \sigma_W(T) = E^0(T)$: Weyl's theorem holds for T,
$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$: generalized Browder's theorem holds for T,
$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$: generalized Weyl's theorem holds for T,
$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$: a-Browder's theorem holds for T,
$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$: a-Weyl's theorem holds for T,
$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$: generalized a-Browder's theorem holds for T,
$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$: generalized a-Weyl's theorem holds for T.

The paper is organized as follows. In the second section, we introduce the property (W_E). Then we prove that $T \in L(X)$ satisfies property(W_E) if and only if T satisfies generalized Weyl's theorem and $\sigma_{BW}(T) = \sigma_W(T)$. We also give conditions for the equivalence of property (WE) and the property (Bw) introduced in [14]. In the case of isoloid operators, we study the preservation of property (W_E) under functional calculus (resp.

under finite rank commuting perturbations).
In the third section, and in a similar way to the second, we introduce and study the property (UW_{E_a}). We prove that if T satisfies property (UW_{E_a}), then T satisfies generalized a-Weyl's theorem and satisfies also property (W_E). Moreover, we prove under the hypothesis $\sigma_{SF_+^-}(T) = \sigma_W(T)$, that T satisfies property (UW_{E_a}) if and only if T satisfies property (W_E) and $E(T) = E_a(T)$. Preservation of the property (UW_{E_a}) under functional calculus or finite rank commuting perturbation is also considered.
Our motivation in studying such properties is the analysis of the structure of the spectrum of a bounded linear operator acting on a Banach space X. An operator satisfying a Weyl-type property has a well-given partition of its spectrum as disjoint union of two of its distinguished parts. The original idea leading to a partition of the spectrum goes back to the famous paper by H. Weyl [17]. More recently, several authors had worked in this direction, see for example [1] and [16].
Hereafter, the symbol $\bigsqcup$ stands for disjoint union, while $iso(A), acc(A)$ means respectively isolated points and accumulation points of a given subset A of $\mathbb{C}$.

This paper will be followed by a second one, in which we will consider a "Browder-type" version of the results obtained.

2. Property (W_E)

Definition 2.1. *A Bounded linear operator $T \in L(X)$ is said to satisfy property (W_E), if its spectrum is the disjoint union of its Weyl spectrum and its isolated eigenvalues, that is $\sigma(T) = \sigma_W(T) \bigsqcup E(T)$.*

Example 2.2. *Recall that the Volterra operator V on $L^2([0,1])$ is defined by $V(f)(x) = \int_0^x f(t)dt$, for $f \in L^2([0,1])$. It is well known that $\sigma(V) = \{0\}$, $\sigma_W(V) = \{0\}$, $E(V) = \emptyset$. Hence property (W_E) is satisfied by V.*

Theorem 2.3. *Let $T \in L(X)$. Then T satisfies property (W_E) if and only if T satisfies generalized Weyl's theorem and $\sigma_{BW}(T) = \sigma_W(T)$.*

Proof. Suppose that T satisfies property (W_E), then $\sigma(T) = \sigma_W(T) \bigsqcup E(T)$. Thus $\lambda \in E(T) \Longleftrightarrow \lambda \in iso\sigma(T) \cap \sigma_W(T)^C \Longleftrightarrow \lambda \in \Pi^0(T)$, where $\sigma_W(T)^C$ is the complement of the Weyl spectrum of T. This implies $\Pi(T) = \Pi^0(T) = E^0(T) = E(T)$, and T satisfies Weyl's theorem. As $E(T) = \Pi(T)$, from [6, Theorem 2.9], it follows that T satisfies generalized Weyl's theorem. We also have $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T) = \sigma(T) \setminus E(T) = \sigma_W(T)$.

Conversely, if T satisfies generalized Weyl's theorem, that is $\sigma(T) = \sigma_{BW}(T) \bigsqcup E(T)$, and $\sigma_{BW}(T) = \sigma_W(T)$, then $\sigma(T) = \sigma_W(T) \bigsqcup E(T)$, and T satisfies property (W_E). □

Remark 2.4. *From Theorem 2.3, if $T \in L(X)$ satisfies property (W_E) then it satisfies generalized Weyl's theorem. However, the converse is not true in general as seen by the following example:*
Let $X = \ell^2(\mathbb{N})$, let $B = \{e_i \mid e_i = (\delta_i^j)_{j\in\mathbb{N}}, i \in \mathbb{N}\}$ be the canonical basis of $\ell^2(\mathbb{N})$. Let E be the subspace of $\ell^2(\mathbb{N})$ generated by the set $\{e_i \mid 1 \leq i \leq n\}$. Let P be the orthogonal projection on E. Then $\sigma(P) = \{0,1\}, \sigma_W(P) = \{0\}$, $\sigma_{BW}(P) = \emptyset$ and $E(P) = \{0,1\}$. So $\Delta^g(P) = E(P)$, i.e. P satisfies generalized Weyl's theorem. But P does not satisfy property (W_E), since $\sigma(P) \setminus \sigma_W(P) \neq E(P)$.

Remark 2.5. *It follows from the proof of Theorem 2.3 that if $T \in L(X)$ satisfies property (W_E), then $\Pi^0(T) = E^0(T) = E(T) = \Pi(T)$.*

The equality of the Weyl spectrum and the B-Weyl spectrum establish a link between generalized Weyl's theorem and property (W_E). In the following lemma, we give a sufficient condition for such equality, which in turn implies equivalence of property (W_E) and generalized Weyl's theorem.

Proposition 2.6. *Let $T \in L(X)$. If iso $\sigma_W(T) = \emptyset$, then $\sigma_W(T) = \sigma_{BW}(T)$. In this case, T satisfies property (W_E) if and only if T satisfies generalized Weyl's theorem.*

Proof. From Theorem 2.3, it's enough to show that if $iso\,\sigma_W(T) = \emptyset$, then $\sigma_W(T) = \sigma_{BW}(T)$. So if $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_{BW}(T)$, then $T - \lambda I$ is a B-Weyl operator. From [5, Remark A, iii)] if η is small enough and $|\eta| > 0$, then $T - \lambda I - \eta I$ is a Weyl operator. As $iso\,\sigma_W(T) = \emptyset$, then $\lambda \notin \sigma_W(T)$. Therefore $\sigma_W(T) \subset \sigma_{BW}(T)$. As we have always $\sigma_{BW}(T) \subset \sigma_W(T)$, then $\sigma_{BW}(T) = \sigma_W(T)$.
Then if $\sigma_W(T) = \sigma_{BW}(T)$, it is clear that T satisfies property (W_E) if and only if T satisfies generalized Weyl's theorem. □

In [14], Gupta and Kashyap introduced a new variant of generalized Weyl's theorem called the property(Bw). An operator $T \in L(X)$ satisfies property (Bw) if $\Delta^g(T) = E^0(T)$ or equivalently $\sigma(T) = \sigma_{BW}(T) \bigsqcup E^0(T)$. In the following theorem we establish a relationship between property(W_E) and property(Bw).

Theorem 2.7. *Let $T \in L(X)$. Then T satisfies property (W_E) if and only if T satisfies property(Bw) and $E(T) = E^0(T)$.*

Proof. Suppose that T satisfies property(W_E). Then from Theorem 2.3, $\sigma_{BW}(T) = \sigma_W(T)$, and from Remark 2.5, $E(T) = E^0(T)$. Hence $\Delta^g(T) = E^0(T)$ and so T satisfies also property (Bw).

Conversely assume that T satisfies property (Bw) and $E(T) = E^0(T)$. As we have $\Delta(T) \subseteq \Delta^g(T)$, then $\Delta(T) \subseteq E(T)$. Now if $\lambda \in E(T)$, as $E(T) = E^0(T)$, then $\lambda \in \Delta^g(T)$. Hence $T - \lambda I$ is a B-Weyl operator and $\alpha(T - \lambda I) < \infty$. So by [8, Lemma 2.4], $T - \lambda I$ is a Weyl operator and $\lambda \in \Delta(T)$. Consequently $\Delta(T) = E(T)$ and T satisfies property (W_E). □

In general, we cannot expect that property (W_E) holds for an operator satisfying property(Bw) and generalized Browder's theorem, as shown by the following example.

Example 2.8. *Let $Q \in L(\ell^2(\mathbb{N}))$ defined by: $Q(x_0, x_1, ...) = (\frac{1}{2}x_1, \frac{1}{3}x_2, ...)$, for $(x_n)_n \in \ell^2(\mathbb{N})$ and let $N \in L(\ell^2(\mathbb{N}))$ be a nilpotent operator. Let $T = Q \oplus N$, then T is quasi-nilpotent operator but not a nilpotent one. Thus $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T) = \{0\}$, $E(T) = \{0\}$ and $E^0(T) = \Pi(T) = \Pi^0(T) = \emptyset$. So T satisfies generalized Browder theorem and property(Bw), but does not satisfy property (W_E).*

Definition 2.9. *An operator $T \in L(X)$ is said to be isoloid if $iso\,\sigma(T) = E(T)$. T is of stable sign index if for all λ, $\mu \in \mathbb{C}$ such that $T - \lambda I$ and $T - \mu I$ are B-Fredholm operators, then index$(T - \lambda I)$ and index$(T - \mu I)$ have the same sign.*

Theorem 2.10. *Let $T \in L(X)$ be an isoloid operator of stable sign index. If T satisfies property (W_E) and if f is an anlaytic function in a neighborhood of the spectrum $\sigma(T)$ of T, which is not constant on any connected component of $\sigma(T)$, then $f(T)$ satisfies property (W_E).*

Proof. Since T is isoloid, then from [7, Lemma 2.9], we have $\sigma(f(T)) \setminus E(f(T)) = f(\sigma(T) \setminus E(T))$. As T satisfies property (W_E), then $f(\sigma(T) \setminus E(T)) = f(\sigma_W(T))$. As T is of stable sign index, using a similar proof as in [7, Theorem 2.4], we have $f(\sigma_W(T)) = \sigma_W(f(T))$. Hence $\sigma(f(T)) \setminus E(f(T)) = \sigma_W(f(T))$, and $f(T)$ satisfies property (W_E). □

Theorem 2.11. *Let $T \in L(X)$ be an isoloid operator and let F be a finite rank operator commuting with T. If T satisfies property (W_E), then $T + F$ satisfies property (W_E).*

Proof. Since T satisfies property (W_E), it satisfies generalized Weyl's theorem. Since T is also isoloid, from [7, Theorem 3.4], $T + F$ satisfies generalized Weyl's theorem. Moreover as F is of finite rank, we have $\sigma_W(T) = \sigma_W(T + F)$ and from [5, Theorem 4.3] we have $\sigma_{BW}(T) = \sigma_{BW}(T + F)$. As T satisfies property (W_E), from Theorem 2.3, we have $\sigma_W(T) = \sigma_{BW}(T)$. Hence $\sigma_W(T + F) = \sigma_{BW}(T + F)$. As we know already that $T + F$ satisfies generalized Weyl's theorem, then $T + F$ satisfies property (W_E). □

3. Property (UW_{E_a})

Definition 3.1. *A Bounded linear operator $T \in L(X)$ is said to satisfy property (UW_{E_a}) if $\Delta_a(T) = E_a(T)$, or in other words if its approximate spectrum is the disjoint union of its upper semi-Weyl spectrum and its isolated eigenvalues in its approximate spectrum, that is $\sigma_a(T) = \sigma_{SF_+^-}(T) \bigsqcup E_a(T)$.*

Theorem 3.2. *Let $T \in L(X)$. Then T satisfies property (UW_{E_a}) if and only if T satisfies generalized a-Weyl's theorem and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.*

Proof. Suppose that T satisfies property (UW_{E_a}), then $\sigma_a(T) = \sigma_{SF_+^-}(T) \bigsqcup E_a(T)$. Thus $\lambda \in E_a(T) \iff \lambda \in iso\sigma_a(T) \cap \sigma_{SF_+^-}(T)^C \iff \lambda \in \Pi_a^0(T)$, where $\sigma_{SF_+^-}(T)^C$ is the complement of the upper semi- Weyl spectrum. This implies $\Pi_a(T) = \Pi_a^0(T) = E_a^0(T) = E_a(T)$, and T satisfies a-Weyl's theorem. Moreover as $\Pi_a(T) = E_a(T)$, then from [6, Theorem 2.10], T satisfies generalized a-Weyl's theorem. We also have $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi_a(T) = \sigma_a(T) \setminus E_a(T) = \sigma_{SF_+^-}(T)$.

Conversely, if T satisfies generalized a-Weyl's theorem, that is $\sigma_a(T) = \sigma_{SBF_+^-}(T) \bigsqcup E_a(T)$, and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}$, then $\sigma_a(T) = \sigma_{SF_+^-} \bigsqcup E_a(T)$, and so T satisfies property (UW_{E_a}). $\square$

The following example shows that there exists operators satisfying generalized a-Weyl's theorem but not property (UW_{E_a}).

Example 3.3. *Let T be defined on $\ell^2(\mathbb{N})$ by*

$$T(x_1, x_2, x_3, ...) = (0, \frac{1}{2}x_1, 0, 0, ...).$$

Then $\sigma_a(T) = \sigma_{SF_+^-}(T) = \{0\}$ and $E_a(T) = \{0\}$. As T is nilpotent, then $\sigma_{SBF_+^-}(T) = \emptyset$. So $\Delta_a^g(T) = E_a(T)$, and T satisfies generalized a-Weyl's theorem, but T does not satisfy property (UW_{E_a}).

Remark 3.4. *From Theorem 3.2, if $T \in L(X)$ satisfies property (UW_{E_a}), then $\Pi_a^0(T) = E_a^0(T) = \Pi_a(T) = E_a(T)$.*

Theorem 3.5. *Suppose that $T \in L(X)$. If T satisfies property (UW_{E_a}), then T satisfies property (W_E).*

Proof. Assume that T satisfies property (UW_{E_a}). Then from Theorem 3.2, T satisfies generalized a-Weyl's theorem and $\Pi_a^0(T) = \Pi_a(T)$. Hence from [10, Theorem 3.7], T satisfies generalized Weyl's theorem. Let us show that $\Pi^0(T) = \Pi(T)$. Indeed, If $\lambda \in \Pi(T)$, as $\Pi(T) \subseteq \Pi_a(T)$ and since $\Pi_a^0(T) = \Pi_a(T)$, then $\lambda \in \Pi_a^0(T)$. This implies that $\alpha(T - \lambda I) < +\infty$. Therefore $\lambda \in \Pi^0(T)$. As we know that $\Pi^0(T) \subseteq \Pi(T)$, then $\Pi^0(T) = \Pi(T)$. Consequently T satisfies generalized Weyl's theorem and $\Pi^0(T) = \Pi(T)$. Hence $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T) = \sigma(T) \setminus \Pi^0(T) = \sigma_W(T)$. From Theorem 2.3, T satisfies property (W_E). $\square$

The converse of Theorem 3.5 does not hold in general as shown by the following example.

Example 3.6. *Let T be the operator given by the direct sum of the unilateral right shift R on $\ell^2(\mathbb{N})$, and the quasinilpotent operator S defined on $\ell^2(\mathbb{N})$, by $S(x_1, x_2, x_3, ...) = (x_2/2, x_3/3, ...)$ for all $x = (x_1, x_2, x_3, ...) \in \ell^2(\mathbb{N})$. Then $\sigma(T) = D(0,1)$, where $D(0,1)$ is the closed unit disc in $\mathbb{C}$ and $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ the unit circle of $\mathbb{C}$. Furthermore $\sigma_W(T) = D(0,1), E_a(T) = \{0\}$ and $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T) = C(0,1) \cup \{0\}$, while $\Pi_a(T) = \emptyset$ since $a(T) = a(S) = \infty$. Hence, T does not satisfies property (UW_{E_a}). But T satisfies property (W_E) because $E(T) = \emptyset$.*

In the following theorem, we give a sufficient conditions under which the property (UW_{E_a}) and (W_E) are equivalent.

Theorem 3.7. *Let $T \in L(X)$ be such that $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Then the following statements are equivalent:*
(i) T satisfies property (UW_{E_a});
(ii) T satisfies property (W_E) and $E(T) = E_a(T)$.

Proof. (*i*) ⇒ (*ii*) Suppose that T satisfies property (UW_{E_a}). Then from Theorem 3.5, T satisfies property (W_E). So it is sufficient to prove that $E(T) = E_a(T)$. Let $\lambda \in E_a(T)$ be arbitrary. Since T satisfies property (UW_{E_a}), then $T - \lambda I$ is an upper semi-Fredholm with negative index. As $\sigma_{SF_+^-}(T) = \sigma_W(T)$, then $T - \lambda I$ is a Weyl operator. As T satisfies property (W_E), then $\lambda \in E(T)$. As we have always that $E(T) \subseteq E_a(T)$, then $E(T) = E_a(T)$.
(*ii*) ⇒ (*i*) Suppose that T satisfies property(W_E) and $E(T) = E_a(T)$. If $\lambda \in E_a(T)$, then $\lambda \in E(T)$. Since T satisfies property(W_E), then $\lambda \notin \sigma_W(T)$ and so $\lambda \notin \sigma_{SF_+^-}(T)$. We have also $\lambda \in \sigma_a(T)$. Indeed if $\lambda \notin \sigma_a(T)$, as $T - \lambda I$ is a Weyl operator, then $\alpha(T - \lambda I) = \beta(T - \lambda I) = 0$. Hence $\lambda \notin \sigma(T)$, which is a contradiction. Therefore $\lambda \in \Delta_a(T)$, and $E_a(T) \subseteq \Delta_a(T)$. Conversely if $\lambda \in \Delta_a(T)$, then $T - \lambda I$ is an upper semi-Fredholm operator such that $ind(T - \lambda I) \leq 0$. By our assumption $T - \lambda I$ is a Weyl operator. As T satisfies property (W_E), then $\lambda \in E(T)$. So $\lambda \in E_a(T)$. Finally $E_a(T) = \Delta_a(T)$ and T satisfies property (W_E). □

Remark 3.8. *The condition $\sigma_{SF_+^-}(T) = \sigma_W(T)$ is always satisfied if T^* has the SVEP. Of course in this case if $\lambda \notin \sigma_{SF_+^-}(T)$, then $T - \lambda I$ is a semi-Fredholm operator with negative index. As T^* has the SVEP, then from [9, Corollary 2.7] , the descent $\delta(T - \lambda I)$ is finite. Therefore $\lambda \notin \sigma_W(T)$. As we have always $\sigma_W(T) \subset \sigma_{SF_+^-}(T)$, then $\sigma_W(T) = \sigma_{SF_+^-}(T)$,*

Definition 3.9. *An operator $T \in L(X)$ is said to be finitely a-isoloid if $\text{iso}\, \sigma_a(T) = E_a^0(T)$, and is said to be finitely a-polaroid, if $\text{iso}\, \sigma_a(T) = \Pi_a^0(T)$.*

Theorem 3.10. *Let $T \in L(X)$ be a finitely a-isoloid operator. Then T satisfies property (UW_{E_a}) if and only if T satisfies a-Weyl's theorem.*

Proof. suppose that T satisfies property (UW_{E_a}), then from Theorem 3.2 and [10, Theorem 3.11], T satisfies a-Weyl's theorem. Conversely, if T satisfies a-Weyl's theorem, then $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$. Now let $\lambda \in E_a(T)$ be arbitrary given, then $\lambda \in \text{iso}\, \sigma_a(T)$. Since T is finitely a-isoloid, it implies that $\lambda \in E_a^0(T)$ and $E_a(T) \subseteq E_a^0(T)$. As we have always $E_a^0(T) \subseteq E_a(T)$, then $E_a(T) = E_a^0(T)$. Consequently $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a(T)$ and T possesses property (UW_{E_a}). □

Lemma 3.11. *Let $T \in L(X)$ be a finitely a-polaroid operator and f an anlaytic function in a neighborhood of the spectrum $\sigma(T)$ of T, which is not constant on any connected component of $\sigma(T)$, then $E_a(f(T)) = \Pi_a^0(f(T))$.*

Proof. Since T is a finitely a-polaroid, then T is a-polaroid and $E_a(T) = \Pi_a^0(T)$. From [6, Theorem 3.5], we have $\sigma_a(f(T)) \setminus E_a(f(T)) = f(\sigma_a(T) \setminus E_a(T)) = f(\sigma_a(T) \setminus \Pi_a^0(T)) = f(\sigma_{ub}(T))$. As $\sigma_{ub}(T)$ satisfies the spectral mapping theorem, see [3, Corollary 3.9], then $f(\sigma_{ub}(T)) = \sigma_{ub}(f(T))$. Hence $\sigma_a(f(T)) \setminus E_a(f(T)) = \sigma_{ub}(f(T)) = \sigma_a(f(T)) \setminus \Pi_a^0(f(T))$. Therefore $E_a(f(T)) = \Pi_a^0(f(T))$. □

Theorem 3.12. *Let $T \in L(X)$ be a finitely a-polaroid operator and f an anlaytic function in a neighborhood of the spectrum $\sigma(T)$ of T, which is not constant on any connected component of $\sigma(T)$. Then $f(T)$ satisfies property (UW_{E_a}) if and only if $f(T)$ satisfies a-Weyl's theorem.*

Proof. The direct sense is obvious. Now if $f(T)$ satisfies a-Weyl's theorem, then $f(T)$ satisfies a-Browder's theorem. Since T is finitely a-polaroid operator, then from Lemma 3.11 we have $E_a(f(T)) = \Pi_a^0(f(T))$. Since a finitely a-polaroid is finitely a-isoloid, then from Theorem 3.10, $f(T)$ satisfies property (UW_{E_a}). □

Theorem 3.13. *Let H be a Hilbert space and let $T \in L(H)$ be a finitely a-polaroid operator. If F is a finite rank operator commuting with T, then T satisfies property (UW_{E_a}) if and only if $T + F$ satisfies property (UW_{E_a}).*

Proof. Since T is finitely a-polaroid operator, then from [6, Lemma 3.9], $T + F$ is an a-polaroid operator. Assume that T satisfies property (UW_{E_a}). From Theorem 3.10, it follows that T satisfies a-Weyl's theorem. Now from [6, Corollary 3.10], $T + F$ satisfies a-Weyl's theorem. We know from [12, Theorem 3.2] that $acc\, \sigma_a(T) = acc\, \sigma_a(T + F)$. Let $\lambda \in E_a(T + F)$, then $\lambda \notin acc\, \sigma_a(T)$. Since T is finitely a-polaroid, it follows that $T - \lambda$ is invertible or $\lambda \in \Pi_a^0(T)$. In the two cases we have $\lambda \in \Pi_a^0(T + F)$. Thus $T + F$ is finitely a-polaroid. As $T + F$ satisfies a-Weyl's theorem, again from Theorem 3.10, $T + F$ satisfies property (UW_{E_a}). For the converse, observe that $T = T + F - F$. Moreover if $T + F$ satisfies property (UW_{E_a}), then $T + F$ is a finitely a-polaroid operator and F commutes withy $T + F$. □

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