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Strong convergence of composite implicit iteration process for asymptotically quasi-nonexpansive mappings

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Abstract. In this paper, we give a necessary and sufficient condition for strong convergence of general composite implicit iteration process to a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings in the framework of Banach spaces. Also we establish some strong convergence theorems by using *condition* $(\overline{C})$ and semi-compactness in the framework of uniformly convex Banach spaces.

1. Introduction

Let C be a nonempty subset of a real Banach space E. Let $T\colon C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a mapping $T\colon C \to C$ is said to be:

(1) asymptotically nonexpansive if there exists a sequence $\{h_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\left\|T^n x - T^n y\right\| \quad \leq \quad h_n \left\|x - y\right\| \tag{1.1}$$

for all $x, y \in C$ and $n \geq 1$.

(2) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{h_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\left\|T^n x - p\right\| \quad \leq \quad h_n \left\|x - p\right\| \tag{1.2}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

The class of asymptotically nonexpansive mappings which is an important generalization of that of nonexpansive mappings was introduced by Goebel and Kirk [4] who proved that every asymptotically

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nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. In 1973, Petryshyn and Williamson [11] gave necessary and sufficient conditions for Mann iterative sequence [9] to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [3] extended the results of Petryshyn and Williamson [11] and gave necessary and sufficient conditions for Ishikawa [5] iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

Liu [8] extended the results of [3, 11] and gave necessary and sufficient conditions for Ishikawa iterative sequence with errors to converge to fixed point of asymptotically quasi-nonexpansive mappings.

In 2001, Xu and Ori [20] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H. Let C be a nonempty subset of H. Let $T_1, T_2, \dots, T_N$ be self-mappings of C and suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \dots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0, 1)$, $x_0 \in C$:

$$\begin{aligned}
x_1 &= t_1 x_0 + (1 - t_1) T_1 x_1, \\
x_2 &= t_2 x_1 + (1 - t_2) T_2 x_2, \\
&\vdots \\
x_N &= t_N x_{N-1} + (1 - t_N) T_N x_N, \\
x_{N+1} &= t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1}, \\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1 \tag{1.3}$$

where $T_k = T_{k \bmod N}$. (Here the mod N function takes values in $\{1, 2, \dots, N\}$). And they proved the weak convergence of the process (1.3).

The aim of this paper is to propose modified general composite implicit iteration process and to prove necessary and sufficient conditions for strong convergence of the iteration process to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings.

2. Preliminaries

Let C be a closed convex subset of a real Banach space E. Let $T\colon C \to C$ be a mapping.

Definition 2.1. *The modified Mann iteration scheme $\{x_n\}$ is defined by*

$$x_1 \in C,$$

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \quad n \geq 1 \tag{2.1}$$

where $\{\alpha_n\}$ is a suitable sequence in $[0, 1]$.

Definition 2.2. *The modified Ishikawa iteration scheme $\{x_n\}$ is defined by*

$$\begin{aligned}
y_n &= (1 - \beta_n) x_n + \beta_n T^n x_n, \quad n \geq 1 \\
x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T^n y_n, \quad n \geq 1
\end{aligned} \tag{2.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are some suitable sequences in $[0, 1]$.

Definition 2.3. *Let $\{T_1, T_2, \ldots, T_N\}$ be a family of asymptotically quasi-nonexpansive self mappings of C into itself. Let $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ is the set of all fixed points of T_i for each $i \in \{1, 2, \ldots, N\}$. Let $\{\alpha_n\}$ a real sequence in $(0, 1)$ and $x_0 \in C$, then the iterative sequence $\{x_n\}$ defined by*

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1, \\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1}, \\
&\vdots
\end{aligned} \tag{2.3}$$

is called the modified implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$.

Since each $n \geq 1$ can be written as $n = (k-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Hence the above iteration process can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} x_n, \quad \forall n \geq 1. \tag{2.4}$$

Now, we define a new modified general composite implicit iteration process as follows:

Definition 2.4. *Let $\{T_1, T_2, \ldots, T_N\}$ be a family of asymptotically quasi-nonexpansive self mappings of C into itself where C is a closed, convex subset of a real Banach space E with $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $x_0 \in C$ be a given point, then the iterative sequence $\{x_n\}$ defined by*

$$\begin{aligned}
x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} y_n, \quad \forall n \geq 1 \\
y_n &= a_n x_n + b_n x_{n-1} + c_n T_{i(n)}^{k(n)} x_n + d_n T_{i(n)}^{k(n)} x_{n-1}, \quad \forall n \geq 1
\end{aligned} \tag{2.5}$$

where $\{\alpha_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \in [0, 1]$, $a_n + b_n + c_n + d_n = 1$ and $n = (k-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$.

Remark 2.5. *By proper selection of the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ it can be seen that the modified Mann iteration scheme, modified Ishikawa iteration scheme, and modified implicit iteration process can easily be obtained from (2.5).*

In the sequel we need the following lemmas to prove our main results.

Lemma 2.6. *(see [18]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \qquad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. *(Schu [15]) Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$, for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main Results

Theorem 3.1. *Let E be a real uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences $\{h_n^{(i)}\}$, $1 \le i \le N$ satisfying $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, where $h_n = \max\{h_n^{(1)}, h_n^{(2)}, \dots, h_n^{(N)}\}$. Let $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2.5), where $\{\alpha_n\}$ be a sequence of real numbers in $(\rho, 1-\rho)$ for some $\rho \in (0,1)$. Then,*

(a) $\lim_{n\to\infty} \left\|x_n - p\right\|$ *exists for all* $p \in \mathcal{F}$.

(b) $\lim_{n\to\infty} d(x_n, \mathcal{F})$ *exists, where* $d(x_n, \mathcal{F}) = \inf_{p\in\mathcal{F}} \left\|x_n - p\right\|$.

(c) $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$, *for all* $1 \le l \le N$.

Proof. Let $p \in \mathcal{F}$. Using (2.5), we have

$$\begin{aligned}
\left\|y_n - p\right\| &= \left\|a_n x_n + b_n x_{n-1} + c_n T_{i(n)}^{k(n)} x_n + d_n T_{i(n)}^{k(n)} x_{n-1} - p\right\| \\
&\le a_n \left\|x_n - p\right\| + b_n \left\|x_{n-1} - p\right\| + c_n \left\|T_{i(n)}^{k(n)} x_n - p\right\| \\
&\quad + d_n \left\|T_{i(n)}^{k(n)} x_{n-1} - p\right\| \\
&\le (a_n + c_n h_{k(n)}) \left\|x_n - p\right\| + (b_n + d_n h_{k(n)}) \left\|x_{n-1} - p\right\|. \qquad (3.1)
\end{aligned}$$

Using (2.5) and (3.1), we have

$$\begin{aligned}
\left\|x_n - p\right\| &= \left\|\alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} y_n - p\right\| \\
&\le \alpha_n \left\|x_{n-1} - p\right\| + (1-\alpha_n) \left\|T_{i(n)}^{k(n)} y_n - p\right\| \\
&\le \alpha_n \left\|x_{n-1} - p\right\| + (1-\alpha_n) h_{k(n)} \left\|y_n - p\right\| \\
&\le \alpha_n \left\|x_{n-1} - p\right\| + (1-\alpha_n) h_{k(n)} \Big[(a_n + c_n h_{k(n)}) \\
&\quad \times \left\|x_n - p\right\| + (b_n + d_n h_{k(n)}) \left\|x_{n-1} - p\right\| \Big] \\
&= \Big[\alpha_n + (1-\alpha_n) h_{k(n)} (b_n + d_n h_{k(n)})\Big] \left\|x_{n-1} - p\right\| \\
&\quad + (1-\alpha_n) h_{k(n)} (a_n + c_n h_{k(n)}) \left\|x_n - p\right\| \\
&\le \Big[\alpha_n + (1-\alpha_n)(1+\theta_n)(b_n + d_n(1+\theta_n))\Big] \left\|x_{n-1} - p\right\| \\
&\quad + (1-\alpha_n)(1+\theta_n)(a_n + c_n(1+\theta_n)) \left\|x_n - p\right\| \qquad (3.2)
\end{aligned}$$

where $h_{k(n)} - 1 = \theta_n, \quad k(n) \ge 1$.
Now, we have

$$\begin{aligned}
\alpha_n + (1-\alpha_n)(1+\theta_n)(b_n + d_n(1+\theta_n)) &\le \alpha_n + (1 - \alpha_n + \theta_n) \\
&\quad \times (b_n + d_n + d_n\theta_n) \\
&\le \alpha_n + (1 - \alpha_n + \theta_n)(1+\theta_n) \\
&\le \alpha_n + 1 - \alpha_n + 2\theta_n - \alpha_n\theta_n + \theta_n^2 \\
&\le 1 + t_n \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}(1-\alpha_n)(1+\theta_n)(a_n+c_n(1+\theta_n)) &\leq (1-\alpha_n+\theta_n) \\ &\quad \times(a_n+c_n+c-n\theta_n) \\ &\leq (1-\alpha_n+\theta_n)(1+\theta_n) \\ &\leq 1-\alpha_n+2\theta_n-\alpha_n\theta_n+\theta_n^2 \\ &\leq 1-\alpha_n+t_n, \end{aligned} \tag{3.4}$$

where $t_n = 2\theta_n - \alpha_n\theta_n + \theta_n^2$.

Substituting (3.3) and (3.4) in (3.2), we have

$$\begin{aligned}\|x_n-p\| &\leq (1+t_n)\|x_{n-1}-p\| + (1-\alpha_n+t_n)\|x_n-p\| \\ &\leq \frac{1+t_n}{\alpha_n}\|x_{n-1}-p\| + \frac{t_n}{\alpha_n}\|x_n-p\| \\ &\leq \frac{1+t_n}{1-\rho}\|x_{n-1}-p\| + \frac{t_n}{1-\rho}\|x_n-p\| \\ &\leq \frac{1+t_n}{1-\rho-t_n}\|x_{n-1}-p\| \\ &\leq \Big(1+\frac{2t_n+\rho}{1-\rho-t_n}\Big)\|x_{n-1}-p\|. \end{aligned} \tag{3.5}$$

Since, by assumption,

$$\sum_{n=1}^{\infty}\theta_n = \sum_{n=1}^{\infty}(h_{k(n)}-1) < \infty,$$

we have

$$\sum_{n=1}^{\infty} t_n < \infty.$$

Thus $t_n \to 0$ as $n \to \infty$. Therefore there exists a positive integer n_0 such that $t_n \leq (1-\rho)/2$ for all $n \geq n_0$.

It follows from (3.5) that

$$\begin{aligned}\|x_n-p\| &\leq \Big[1+\frac{2(2t_n+\rho)}{1-\rho}\Big]\|x_{n-1}-p\| \\ &= (1+\delta_n)\|x_{n-1}-p\|, \end{aligned} \tag{3.6}$$

where $\delta_n = 2(2t_n+\rho)/(1-\rho)$.

Taking infimum over all $p \in \mathcal{F}$, we have

$$d(x_n,\mathcal{F}) \leq (1+\delta_n)d(x_{n-1},\mathcal{F}). \tag{3.7}$$

It follows from Lemma 2.6 that

$$\lim_{n\to\infty}\|x_n-p\| \quad \text{and} \quad \lim_{n\to\infty} d(x_n,\mathcal{F}) \quad \text{exist.}$$

Without loss of generality, we can assume that

$$\lim_{n\to\infty} \left\|x_n - p\right\| = r, \tag{3.8}$$

where $r \geq 0$ is some number. Since $\left\{ \left\|x_n - p\right\| \right\}$ is a convergent sequence and so $\{x_n\}$ is a bounded sequence in C.

It follows from (3.1) and (3.8) that

$$\lim_{n\to\infty} \left\|y_n - p\right\| \leq r,$$

which further gives that

$$\begin{aligned} \limsup_{n\to\infty} \left\|T_{i(n)}^{k(n)} y_n - p\right\| &\leq \limsup_{n\to\infty} h_{k(n)} \left\|y_n - p\right\| \\ &\leq r. \end{aligned} \tag{3.9}$$

Also from (2.5), we have

$$\lim_{n\to\infty} \left\|\alpha_n(x_{n-1} - p) + (1-\alpha_n)(T_{i(n)}^{k(n)} y_n - p)\right\| = \lim_{n\to\infty} \left\|x_n - p\right\| = r. \tag{3.10}$$

Lemma 2.7 and (3.8) - (3.10) imply that

$$\lim_{n\to\infty} \left\|T_{i(n)}^{k(n)} y_n - x_{n-1}\right\| = 0. \tag{3.11}$$

Again from (2.5) and (3.11), we have

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| \leq \lim_{n\to\infty} (1-\alpha_n) \left\|T_{i(n)}^{k(n)} y_n - x_{n-1}\right\| = 0. \tag{3.12}$$

and thus

$$\lim_{n\to\infty} \left\|x_n - x_{n+j}\right\| = 0, \quad \forall j = 1, 2, \ldots, N. \tag{3.13}$$

On the other hand, from (3.11) and (3.12), we have

$$\begin{aligned} \lim_{n\to\infty} \left\|x_n - T_{i(n)}^{k(n)} y_n\right\| &\leq \lim_{n\to\infty} \Big[\|x_n - x_{n-1}\| \\ &\quad + \left\|x_{n-1} - T_{i(n)}^{k(n)} y_n\right\| \Big] \\ &= 0. \end{aligned} \tag{3.14}$$

Now,

$$\begin{aligned} \left\|T_{i(n)}^{k(n)} x_n - x_n\right\| &\leq \|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)} y_n - x_{n-1}\right\| \\ &\quad + \left\|T_{i(n)}^{k(n)} y_n - T_{i(n)}^{k(n)} x_n\right\| \\ &\leq \|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)} y_n - x_{n-1}\right\| \\ &\quad + h_{k(n)} \left\|y_n - x_n\right\| \\ &\leq \|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)} y_n - x_{n-1}\right\| \\ &\quad + h_{k(n)} [\left\|y_n - x_{n-1}\right\| + \|x_n - x_{n-1}\|] \\ &= (2+\theta_n) \|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)} y_n - x_{n-1}\right\| \\ &\quad + (1+\theta_n) \left\|y_n - x_{n-1}\right\|. \end{aligned} \tag{3.15}$$

Again, by using (2.5), we have

$$\begin{aligned}
\|y_n - x_{n-1}\| =& \|a_n x_n + b_n x_{n-1} + c_n T_{i(n)}^{k(n)} x_n \\
& + d_n T_{i(n)}^{k(n)} x_{n-1} - x_{n-1}\| \\
=& \|a_n x_n + c_n T_{i(n)}^{k(n)} x_n + d_n T_{i(n)}^{k(n)} x_{n-1} \\
& - (1 - b_n) x_{n-1}\| \\
=& \|a_n x_n + c_n T_{i(n)}^{k(n)} x_n + d_n T_{i(n)}^{k(n)} x_{n-1} \\
& - (a_n + c_n + d_n) x_{n-1}\| \\
=& \|c_n\Big(T_{i(n)}^{k(n)} x_n - x_n\Big) + d_n\Big(T_{i(n)}^{k(n)} x_{n-1} - x_n\Big) \\
& + (a_n + c_n + d_n)(x_n - x_{n-1})\| \\
\leq& c_n \|T_{i(n)}^{k(n)} x_n - x_n\| + d_n \|T_{i(n)}^{k(n)} x_{n-1} - x_n\| \\
& + (a_n + c_n + d_n)\|x_n - x_{n-1}\| \\
\leq& c_n \|T_{i(n)}^{k(n)} x_n - x_n\| + d_n \Big[\|T_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} x_n\| \\
& + \|T_{i(n)}^{k(n)} x_n - x_n\|\Big] + (a_n + c_n + d_n)\|x_n - x_{n-1}\| \\
\leq& (c_n + d_n)\|T_{i(n)}^{k(n)} x_n - x_n\| + d_n(1 + \theta_n)\,\|x_{n-1} - x_n\| \\
& + (a_n + c_n + d_n)\|x_n - x_{n-1}\| \\
\leq& (c_n + d_n)\|T_{i(n)}^{k(n)} x_n - x_n\| + (a_n + c_n + d_n(2 + \theta_n))\|x_n - x_{n-1}\| \\
\leq& (c_n + d_n)\|T_{i(n)}^{k(n)} x_n - x_n\| + (1 - b_n + d_n + d_n\theta_n)\|x_n - x_{n-1}\|. \qquad (3.16)
\end{aligned}$$

Substituting (3.16) into (3.15), we get

$$\begin{aligned}
\left\|T_{i(n)}^{k(n)} x_n - x_n\right\| \quad \leq \quad & (2 + \theta_n)\,\|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)} y_n - x_{n-1}\right\| \\
& + (1 + \theta_n)\Big[(c_n + d_n)\left\|T_{i(n)}^{k(n)} x_n - x_n\right\| \\
& + (1 - b_n + d_n + d_n\theta_n)\,\|x_n - x_{n-1}\|\,\Big] \\
\leq \quad & (3 + d_n + d_n t_n + 2\theta_n)\,\|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)} y_n - x_{n-1}\right\| \\
& + (1 + \theta_n)(c_n + d_n)\left\|T_{i(n)}^{k(n)} x_n - x_n\right\|.
\end{aligned} \qquad (3.17)$$

Since $c_n + d_n \leq b < 1$, the above inequality becomes

$$\begin{aligned}
[1 - (1 + \theta_n)b]\left\|T_{i(n)}^{k(n)} x_n - x_n\right\| \quad \leq \quad & (3 + d_n + d_n t_n + 2\theta_n)\,\|x_n - x_{n-1}\| \\
& + \left\|T_{i(n)}^{k(n)} y_n - x_{n-1}\right\|.
\end{aligned} \qquad (3.18)$$

From (3.11), (3.13) and (3.18), we have

$$\lim_{n\to\infty} \left\|T_{i(n)}^{k(n)} x_n - x_n\right\| = 0. \qquad (3.19)$$

Since $\{T_l : 1 \leq l \leq N\}$ is uniformly L-Lipschitzian. Also since any positive integer $n > N$ can be written as

$n = (k(n) - 1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$, we have

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq \left\|x_{n-1} - T_{i(n)}^{k(n)} y_n\right\| + \left\|T_{i(n)}^{k(n)} y_n - T_n x_n\right\| \\ &\leq D_n + L\left\|T_{i(n)}^{k(n)-1} y_n - x_n\right\| \\ &\leq D_n + L\Big[\left\|T_{i(n)}^{k(n)-1} y_n - x_{n-1}\right\| + \|x_{n-1} - x_n\|\Big] \end{aligned} \tag{3.20}$$

where $D_n = \left\|x_{n-1} - T_{i(n)}^{k(n)} y_n\right\|$.

From (3.11) we have $D_n \to 0$ as $n \to \infty$. Again

$$\begin{aligned} \left\|T_{i(n)}^{k(n)-1} y_n - x_{n-1}\right\| &\leq \left\|T_{i(n)}^{k(n)-1} y_n - T_{i(n-N)}^{k(n)-1} x_{n-N}\right\| \\ &\quad + \left\|T_{i(n-N)}^{k(n)-1} x_{n-N} - T_{i(n-N)}^{k(n)-1} y_{n-N}\right\| \\ &\quad + \left\|T_{i(n-N)}^{k(n)-1} y_{n-N} - x_{(n-N)-1}\right\| \\ &\quad + \left\|x_{(n-N)-1} - x_n\right\|. \end{aligned} \tag{3.21}$$

Since for each $n > N$, $n = (n - N)(mod\ N)$, and $n = (k(n) - 1) + i(n)$, we have $n - N = (k(n-1) - 1)N + i(n) = (k(n-N) - 1)N + i(n-N)$, that is, $k(n-N) = k(n) - 1$ and $i(n-N) = i(n)$. Therefore from (3.21), we have

$$\begin{aligned} \left\|T_{i(n)}^{k(n)-1} y_n - x_{n-1}\right\| &\leq \left\|T_{i(n)}^{k(n)-1} y_n - T_{i(n)}^{k(n)-1} x_{n-N}\right\| \\ &\quad + \left\|T_{i(n-N)}^{k(n)-1} x_{n-N} - T_{i(n-N)}^{k(n)-1} y_{n-N}\right\| \\ &\quad + \left\|T_{i(n-N)}^{k(n)-1} y_{n-N} - x_{(n-N)-1}\right\| \\ &\quad + \left\|x_{(n-N)-1} - x_n\right\| \\ &\leq L\left\|y_n - x_{n-N}\right\| + L\left\|x_{n-N} - y_{n-N}\right\| \\ &\quad + D_{n-N} + \left\|x_{(n-N)-1} - x_n\right\|. \end{aligned} \tag{3.22}$$

By (3.20), (3.21) and (3.22), we have

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq D_n + L^2(\left\|y_n - x_{n-N}\right\| + \left\|x_{n-N} - y_{n-N}\right\|) \\ &\quad + L(D_{n-N} + \left\|x_{(n-N)-1} - x_n\right\| + \|x_n - x_{n-1}\|) \\ &\leq D_n + L^2(\left\|y_n - x_n\right\| + \|x_n - x_{n-N}\| + \left\|x_{n-N} - y_{n-N}\right\|) \\ &\quad + L(D_{n-N} + \left\|x_{(n-N)-1} - x_n\right\| + \|x_n - x_{n-1}\|) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.23}$$

It follows from (3.13) and (3.23) that

$$\begin{aligned} \lim_{n\to\infty} \|x_n - T_n x_n\| &\leq \lim_{n\to\infty} [\|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|] \\ &= 0. \end{aligned} \tag{3.24}$$

Consequently, for any $l = 1, 2, \ldots, N$, from (3.13) and (3.23), we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - T_{n+l} x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| \\ &\quad + \|T_{n+l} x_{n+l} - T_{n+l} x_n\| \\ &\leq (1 + L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| \\ &\to 0 \quad \text{as} \quad n \to \infty, \end{aligned} \tag{3.25}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T_{n+l}x_n\| = 0 \text{ for all } l = 1, 2, \ldots, N.$$

Since for each $l = 1, 2, \ldots, N$, $\left\{ \|x_n - T_{n+l}x_n\| \right\}$ is a subset of $\bigcup_{l=1}^{N} \left\{ \|x_n - T_{n+l}x_n\| \right\}$, we have

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0,$$

for all $l = 1, 2, \ldots, N$. This completes the proof. □

Next, we prove necessary and sufficient conditions for the strong convergence of the iteration process (2.5) to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings.

Theorem 3.2. *Let E be a real uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N} : C \to C$ be N asymptotically quasi-nonexpansive mappings with sequences $\{h_n^{(i)}\}$, $1 \le i \le N$ satisfying $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, where $h_n = \max\{h_n^{(1)}, h_n^{(2)}, \ldots, h_n^{(N)}\}$. Let $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2.5), where $\{\alpha_n\}$ be a sequence of real numbers in $(\rho, 1-\rho)$ for some $\rho \in (0,1)$. Then the sequence $\{x_n\}$ converges to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$ if and only if* $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Proof. If for some $p \in \mathcal{F}$, $\lim_{n\to\infty} \left\| x_n - p \right\| = 0$, then obviously $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Conversely, suppose $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, then we have $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Thus for any given $\varepsilon > 0$ there exists a positive integer N_1 such that for $n \ge N_1$, $d(x_n, \mathcal{F}) < \varepsilon/4$.

Since $\lim_{n\to\infty} \left\| x_n - p \right\|$ exists for all $p \in \mathcal{F}$, we have $\left\| x_n - p \right\| < K$, for all $n \ge 1$ and some positive number K.

Further $\sum_{n=1}^{\infty} \delta_n < \infty$ implies that there exists a positive integer N_2 such that $\sum_{j=n}^{\infty} \delta_j < \varepsilon/(4K)$ for all $n \ge N_2$. Let $N = \max\{N_1, N_2\}$. It follows from (3.6), that

$$\left\| x_n - p \right\| \le \left\| x_{n-1} - p \right\| + K\delta_n.$$

Now, for each $m, n \ge N$, we have

$$\begin{aligned}
\|x_n - x_m\| &\le \left\| x_n - p \right\| + \left\| x_m - p \right\| \\
&\le \left\| x_N - p \right\| + K \sum_{j=N+1}^{n} \delta_j + \left\| x_N - p \right\| \\
&\quad + K \sum_{j=N+1}^{n} \delta_j \\
&= 2 \left\| x_N - p \right\| + 2K \sum_{j=N+1}^{n} \delta_j \\
&< 2.\frac{\varepsilon}{4} + 2K.\frac{\varepsilon}{4K} = \varepsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. Thus, the completeness of E implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = p^*$. Now, we have to show that p^* is a common fixed point of the mappings $\{T_i : i = 1, 2, \ldots, N\}$. Indeed, we know that the set $\mathcal{F} = \cap_{i=1}^{N} F(T_i)$ is closed. From the continuity of $d(x, \mathcal{F}) = 0$ with $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ and $\lim_{n\to\infty} x_n = p^*$, we get $d(p^*, \mathcal{F}) = 0$, and so $p^* \in \mathcal{F}$, that is, p^* is a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$. This completes the proof. □

Remark 3.3. *Theorem 3.2 extends and improves the corresponding result of [3, 7, 8, 11, 17] and many others to the case of more general class of nonexpansive, quasi-nonexpansive and asymptotically nonexpansive mapping and general composite implicit iteration process for a finite family of mappings considered in this paper and no boundedness condition imposed on C.*

Recall that the following:

A mapping $T\colon C \to C$ where C is a subset of a Banach space E with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) [16] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in C$, $\|x - Tx\| \geq f(d(x, F(T)))$, where $d(x, F(T)) = \inf\left\{ \left\|x - p\right\| : p \in F(T)\right\}$.

A family $\{T_i\}_{i=1}^N$ of N self-mappings of C with $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy

(1) *condition* (B) on C ([2]) if there is a nondecreasing function $f\colon [0,1] \to [0,1]$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ and all $x \in C$ such that $\max_{1\leq l\leq N}\left\{ \|x - T_l x\| \right\} \geq f(d(x,\mathcal{F}))$;

(2) *condition* $(\overline{C})$ on C ([1]) if there is a nondecreasing function $f\colon [0,1] \to [0,1]$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ and all $x \in C$ such that $\|x - T_l x\| \geq f(d(x,\mathcal{F}))$, for at least one T_l, $l = 1, 2, \ldots, N$; or in other words at least one of the $T_l's$ satisfies *condition*(A).

Condition (B) reduces to condition (A) when all but one of the $T_i's$ are identities. Also conditions (B) and $(\overline{C})$ are equivalent (see [1]).

Senter and Dotson [16] established a relation between *condition* (A) and *demicompactness*. They actually showed that the *condition* (A) is weaker than demicompactness for a nonexpansive mapping.

Every compact operator is demicompact. Since every completely continuous mapping $T\colon C \to C$ is continuous and demicompact, so it satisfies *condition* (A).

Therefore to study strong convergence of $\{x_n\}$ defined by (2.5) we use *condition* $(\overline{C})$ instead of the complete continuity of the mappings $T_1, T_2, \ldots, T_N$.

Theorem 3.4. *Let E be a real uniformly convex Banach space, C be a nonempty closed convex subset of E. Let* $\{T_i\}_{i=1}^N\colon C \to C$ *be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings as in Theorem 3.1 and satisfying* condition $(\overline{C})$, $\{\alpha_n\}$ *be a sequence as in Theorem 3.1. Let* $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. *Then the sequence* $\{x_n\}$ *defined by (2.5), converges to a common fixed point of the mappings* $\{T_1, T_2, \ldots, T_N\}$.

Proof. By Theorem 3.1, we know that $\lim_{n\to\infty} \left\|x_n - p\right\|$ and $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exist. Let one of $T_i's$, say T_u, $u \in \{1, 2, \ldots, N\}$ satisfy condition (A), also by Theorem 3.1, we have $\lim_{n\to\infty} \|x_n - T_u x_n\| = 0$, so we have $\lim_{n\to\infty} f(d(x_n, \mathcal{F})) = 0$. By the property of f and the fact that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists, we have $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By Theorem 3.2, we obtain $\{x_n\}$ converges strongly to a common fixed point in $\mathcal{F}$. This completes the proof. □

Remark 3.5. *(1) Theorem 3.4 extends and improves the corresponding result of [6, 12–14, 19] and many others to the case of more general class of nonexpansive, quasi-nonexpansive and asymptotically nonexpansive mapping and general composite implicit iteration process for a finite family of mappings considered in this paper and no boundedness condition imposed on C.*

(2) Theorem 3.4 also extends and improves the corresponding result of [17] to the case of general composite implicit iteration process for a finite family of mappings considered in this paper and no boundedness condition imposed on C.

(3) Theorem 3.4 also extends and improves the corresponding result of [1] to the case of more general class of asymptotically nonexpansive mapping and general composite implicit iteration process for a finite family of mappings considered in this paper.

For our next result, we shall need the following definition:

Definition 3.6. *Let C be a nonempty closed subset of a Banach space E. A mapping $T\colon C \longrightarrow C$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in C such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = x \in C$.*

Theorem 3.7. *Let E be a real uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N\colon C \to C$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings as in Theorem 3.1 such that one of the mappings in $\{T_1, T_2, \ldots, T_N\}$ is semi-compact, $\{\alpha_n\}$ be a sequence as in Theorem 3.1. Let $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (2.5), converges to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$.*

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \ldots, N\}$. By Theorem 3.1, we have $\lim_{n\to\infty}\left\|x_n - T_{i_0}x_n\right\| = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{n_j\to\infty} x_{n_j} = p \in C$. Now again by Theorem 3.1 we have

$$\lim_{n_j\to\infty}\left\|x_{n_j} - T_l x_{n_j}\right\| = 0,$$

for all $l \in \{1, 2, \ldots, N\}$. So $\left\|p - T_l p\right\| = 0$ for all $l \in \{1, 2, \ldots, N\}$. This implies that $p \in \mathcal{F}$, also $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By Theorem 3.2, we obtain $\{x_n\}$ converges strongly to a common fixed point in $\mathcal{F}$. This completes the proof. □

Remark 3.8. *Theorem 3.7 extends and improves the corresponding result of [10, 15] to the case of more general class of asymptotically nonexpansive mapping and general composite implicit iteration process for a finite family of mappings considered in this paper and no boundedness condition imposed on C.*

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