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Tensor product of n-isometries III

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Abstract. A Banach space operator $T \in B(\mathcal{X})$ is $A(m,p)$-isometric for some $A \in B(\mathcal{X})$, integer $m \geq 1$ and $p \in (0,\infty)$ if $\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}||AT^ix||^p = 0$ for all $x \in \mathcal{X}$. If $S \in B(\mathcal{X})$ is $A_1(m,p)$-isometric and $T \in B(\mathcal{Y})$ is $A_2(n,p)$-isometric for some $A_1 \in B(\mathcal{X})$, $A_2 \in B(\mathcal{Y})$ and integer $n \geq 1$, then the tensor product $S \otimes T$ is $(A_1 \otimes A_2)(m+n-1,p)$-isometric. An (m,p)-isometric operator is isometric iff it is normaloid; if $\mathcal{X} = \mathcal{H}$ is a Hilbert space and $A \in B(\mathcal{H})$ is bounded below, then an $A(m,2)$-isometric operator $T \in B(\mathcal{H})$ is A-isometric (i.e., $||ATx|| = ||Ax||$ for all $x \in \mathcal{H}$) iff it is similar to an isometry. It is proved that if $S \in B(\mathcal{X})$ is $A(m,p)$-isometric for some left invertible operator $A \in B(\mathcal{X})$, then S can not be supercyclic.

1. Introduction

An operator $T \in B(\mathcal{H})$, the algebra of operators (equivalently, bounded linear transformations) on a complex infinite dimensional Hilbert space $\mathcal{H}$ into itself, is an m-isometry for some integer $m \geq 1$ if

$$\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}T^{*i}T^i = 0 \Longleftrightarrow \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}||T^ix||^2 = 0$$

for all $x \in \mathcal{H}$ [2, 8]. Generalising [5–7, 13], a Banach space operator $T \in B(\mathcal{X})$ is an (m,p)-isometry, for some integer $m \geq 1$ and $p \in (0,\infty)$, if

$$\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}||T^ix||^p = 0$$

for all $x \in \mathcal{X}$. Properties of (m,p)-isometric operators have been studied by a number of authors in the recent past, see references (especially, [6, 13]). Thus (m,p)-isometric operators are bounded below with the approximate point spectrum contained in the unit circle $\partial\mathbf{D}$ (in the complex plane). Let $T \in B(\mathcal{X})$ be an (m,p)-isometry. If we define $\beta_k^{(p)}(T,x) : \mathcal{X} \longrightarrow \mathbb{R}$ by

$$\beta_k^{(p)}(T,x) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}||T^ix||^p \tag{1}$$

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for all $x \in \mathcal{X}$ and let $n^{(k)}$ denote the (descending Pochhammer) symbol $n^{(k)} = 1$ if $n = 0$, $n^{(k)} = 0$ if $n > 0$ and $k > n$ and $n^{(k)} = \binom{n}{k}k!$ if $n > 0$ and $k \leq n$, then [6, Proposition 2.1]

$$\|T^n x\|^p = \sum_{k=0}^{m-1} n^{(k)} \beta_k^{(p)}(T, x) \tag{2}$$

for all integers $n \geq 0$ and $x \in \mathcal{X}$. In particular,

$$\beta_{m-1}^{(p)}(T, x) = \frac{\|T^n x\|^p}{\binom{n}{m-1}(m-1)!} \geq 0$$

with equality if and only if T is $(m-1, p)$-isometric.

Given Banach space operators $S \in B(\mathcal{X})$ and $T \in B(\mathcal{Y})$, let $S \otimes T \in B(\mathcal{X} \otimes \mathcal{Y})$ denote the tensor product of S and T. If $\mathcal{X} = \mathcal{Y} = \mathcal{H}$, S is an m-isometry and T is an n-isometry, then $S \otimes T \in B(\mathcal{H} \otimes \mathcal{H})$ is an $(m+n-1)$-isometry [10, 11]. The (combinatorial) argument of the proof of [10, Theorem 2.10] is independent of the hypothesis that S and T are Hilbert space operators or that the index $p = 2$: Keeping in mind the fact that $\|(S \otimes T)^i(x \otimes y)\| = \|S^i x\|\|T^i y\|$, the argument indeed proves that "if $S \in B(\mathcal{X})$ is (m, p)-isometric and $T \in B(\mathcal{Y})$ is (n, p)-isometric, then $S \otimes T$ is $(m+n-1, p)$-isometric". (A proof of this follows also from [7, Theorem 3.3].)

We consider here $A(m, p)$-isometries, where, for an operator $A \in B(\mathcal{X})$, $T \in B(\mathcal{X})$ is $A(m, p)$-isometric if

$$\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \|AT^i x\|^p = 0$$

for all $x \in \mathcal{X}$. Evidently, an $I(m, p)$-isometry is an (m, p)-isometry; if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then

$$\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \|AT^i x\|^p = 0 \Longleftrightarrow \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \||A|T^i x\|^p = 0$$

for all $x \in \mathcal{H}$, and an $A(m, 2)$-isometry is an $|A|^2 - m$-isometry of [4]. Let $[S, T] = ST - TS$ denote the commutant of $S, T \in B(\mathcal{X})$. We prove in the following that if S is an $A_1(m, p)$-isometry and T is an $A_2(n, p)$-isometry for some $A_1, A_2 \in B(\mathcal{X})$, and $[S, T] = [A_1, A_2] = [A_1, T] = [A_2, S] = 0$, then ST is an $A_1A_2(m+n-1, p)$-isometry. If, instead, S is an $A(m, p)$-isometry and T is an $A(n, p)$-isometry for some $A \in B(\mathcal{X})$, then $[S, T] = 0$ implies ST is an $A(m+n-1, p)$-isometry. Translated to $A - m$-isometries, this generalises [4, Theorem 3.1], and translated to tensor products and left-right multiplication operators we have the following: (i) If $S \in B(\mathcal{X})$ is an $A_1(m, p)$-isometry and $T \in B(\mathcal{Y})$ is an $A_2(n, p)$-isometry for some $A_1 \in B(\mathcal{X})$ and $A_2 \in B(\mathcal{Y})$, then $S \otimes T \in B(\mathcal{X} \otimes \mathcal{Y})$ is an $(A_1 \otimes A_2)(m+n-1, p)$-isometry; (ii) if $S \in B(\mathcal{X})$ is an $A_1(m, p)$-isometry and $T^* \in B(\mathcal{Y}^*)$ is an $A_2^*(n, p)$-isometry for some $A_1 \in B(\mathcal{X})$ and $A_2^* \in B(\mathcal{Y}^*)$, then the left-right multiplication operator $L_S R_T \in B(B(\mathcal{Y}, \mathcal{X}))$, $L_S R_T(Z) = SZT$, is a $(L_{A_1} R_{A_2})(m+n-1, p)$-isometry.

An important property of (m, p)-isometric operators, which follows from (1) and (2), is that to each $x \in \mathcal{X}$ there corresponds a positive integer n_0 such that $\|T^n x\|^p \leq \|T^{n+1} x\|^p$ all $n \geq n_0$. Hence, in general, the spectral radius $r(T) = 1$ of an (m, p)-isometry T is less than its norm $\|T\|$. We prove that an (m, p)-isometry T is normaloid, i.e. $r(T) = \|T\|$, if and only if it is isometric. This generalises some results from [9]. For Hilbert space operators $A, T \in B(\mathcal{H})$ such that A is bounded below and T is $A(m, 2)$-isometric, it is seen that T is A-isometric if and only if it is similar to an isometry. Finally, we prove that if an operator $S \in B(\mathcal{X})$ is $A(m, p)$-isometric for some left invertible operator $A \in B(\mathcal{X})$, then S can not be supercyclic.

2. Results

Unless otherwise stated, we assume in the following that A, S are operators in $B(\mathcal{X})$ such that S is an $A(m, p)$-isometry. The following facts about $A(m, p)$-isometries are easily derived (argue as in [6, 13].

(a). Defining $\beta_k^{(p)}(A,S,x)$ by replacing $\|T^ix\|^p$ by $\|AS^ix\|^p$ in the right-hand-side of (1), we have from (2) that

$$\|AS^nx\|^p = \sum_{k=0}^{m-1} n^{(k)}\beta_k^{(p)}(A,S,x) \tag{3}$$

for all $n \geq 0$ and $x \in \mathcal{X}$. Furthermore,

$$\beta_{m-1}^{(p)}(A,S,x) = \frac{\|AS^nx\|^p}{\binom{n}{m-1}(m-1)!} \geq 0 \tag{4}$$

with equality if and only if S is $A(m-1,p)$-isometric.

(b). $A(m,p)$-isometries are generally not bounded below: consider operators $A, T \in B(\mathcal{X})$ such that T is not bounded below and $AT = A$, when it is seen that T is $A(m,p)$-isometric for all $m \geq 1$. If, however, $0 \notin \sigma_a(A)$, the approximate point spectrum of A, then $\sigma_a(S)$ is contained in the boundary $\partial\mathbf{D}$ of the unit disc $\mathbf{D}$. In particular, if $0 \notin \sigma_a(A)$, then S is bounded below, $\sigma(S) = \overline{\mathbf{D}}$ if S is not invertible and $\sigma(S) \subseteq \partial\mathbf{D}$ if S is invertible.

(c). S is $A(n,p)$-isometric for all integers $n \geq m$. In particular, if we let $t \in \{0, 1, \cdots, n-1\}$ with $n > 1$, then

$$\sum_{j=0}^{m+n-1} (-1)^{m+n-1-j}\binom{m+n-1}{j} \prod_{i=0}^{t}(j-i)\|AS^jx\|^p = 0 \tag{5}$$

for all $x \in \mathcal{X}$ (*cf.* [7, Lemma 3.2]). Clearly, if S is invertible, then S^{-1} is $A(n,p)$-isometric for all integers $n \geq m$.

(d). Let k be a non-negative integer. If we let $\widehat{\prod_{j=0}^{m-1}}(k-j) = k(k-1)\cdots(k-j+1)(k-j-1)\cdots(k-m+1)$ ($= \prod_{j=0}^{m-1}(k-j)$ but with the factor $(k-j)$ missing from the product), then a straightforward verification for values $0 \leq k \leq m-1$ coupled with [13, Proposition 3.7(i)] shows that

$$\|AS^kx\|^p = \sum_{j=0}^{m-1} \frac{(-1)^{m-1-j}}{j!(m-1-j)!} \widehat{\prod_{j=0}^{m-1}}(k-j)\|AS^jx\|^p$$

for all integers $k \geq 0$ and $x \in \mathcal{X}$. Evidently, given integers $k \geq 0$ and $m > 1$, there exists a finite sequence $\{a_{ji}\}_{i=0}^{m-1}$ such that

$$\widehat{\prod_{j=0}^{m-1}}(k-j) = a_{j0} + \sum_{i=1}^{m-1} a_{ji} \prod_{t=0}^{i-1}(k-t).$$

Hence, for all k and $x \in \mathcal{X}$,

$$\|AS^kx\|^p = \sum_{j=0}^{m-1} \frac{(-1)^{m-1-j}}{j!(m-1-j)!}\{a_{j0} + \sum_{i=1}^{m-1} a_{ji} \prod_{t=0}^{i-1}(k-t)\}\|AS^jx\|^p. \tag{6}$$

We prove next our main result. Our proof of the theorem borrows from the argument of the proof of [7, Theorem 3.3].

Theorem 2.1. *If $S \in B(\mathcal{X})$ is $A_1(m,p)$-isometric and $T \in B(\mathcal{X})$ is $A_2(n,p)$-isometric for some operators $A_1, A_2 \in B(\mathcal{X})$ such that $[S,T] = [A_1,A_2] = [A_1,T] = [A_2,S] = 0$, then S_1S_2 is $A_1A_2(m+n-1,p)$-isometric. If, instead, S is $A(m,p)$-isometric and T is $A(n,p)$-isometric for some operator $A \in B(\mathcal{X})$, then $[S,T] = 0$ implies ST is $A(m+n-1,p)$-isometric.*

Proof. If we define $f(x) = f(A_1, A_2, S, T, x)$ by

$$\begin{aligned} f(x) &= \sum_{k=0}^{m+n-1} (-1)^{m+n-1-k} \binom{m+n-1}{k} \|A_1 A_2 (ST)^k x\|^p \\ &= \sum_{k=0}^{m+n-1} (-1)^{m+n-1-k} \binom{m+n-1}{k} \|A_1 S^k (A_2 T^k x)\|^p, \end{aligned}$$

then using equality (6), the fact that T is an $A_2(m+n-1,p)$-isometry and identity (5) we have:

$$\begin{aligned} f(x) &= \sum_{k=0}^{m+n-1} (-1)^{m+n-1-k} \binom{m+n-1}{k} [\sum_{j=0}^{m-1} \frac{(-1)^{m-1-j}}{j!(m-1-j)!} \{a_{j0} + \sum_{i=1}^{m-1} a_{ji} \prod_{t=0}^{i-1} (k-t) \|A_1 S^j (A_2 T^k x)\|^p] \\ &= \sum_{j=0}^{m-1} \frac{(-1)^{m-1-j}}{j!(m-1-j)!} [\sum_{k=0}^{m+n-1} (-1)^{m+n-1-k} \binom{m+n-1}{k} \{a_{j0} + \sum_{i=1}^{m-1} a_{ji} \prod_{t=0}^{i-1} (k-t)\} \|A_2 T^k (A_1 S^j x)\|^p] \\ &= \sum_{j=0}^{m-1} \frac{(-1)^{m-1-j}}{j!(m-1-j)!} [\sum_{i=1}^{m-1} a_{ji} \{ \sum_{k=0}^{m+n-1} (-1)^{m+n-1-k} \binom{m+n-1}{k} \prod_{t=0}^{i-1} (k-t) \|A_2 T^k (A_1 S^j x)\|^p \}] \\ &= 0. \end{aligned}$$

Thus ST is $A_1 A_2(m+n-1,p)$-isometric.

If $A_1 = A_2 = A$, then the argument above goes through with $\|A_1 A_2 (ST)^k\|^p$ replaced by $\|A(ST)^k x\|^p$. This completes the proof. □

Theorem 2.1 extends [4, Theorem 3.1] and [7, Theorem 3.3]. We consider next applications of Theorem 2.1 to tensor products and the elementary operator of left-right multiplication, starting with tensor products.

Tensor products. Given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X} \overline{\otimes} \mathcal{Y}$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$; let, for $S \in B(\mathcal{X})$ and $T \in B(\mathcal{Y})$, $S \otimes T \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$ denote the tensor product operator defined by S and T. It is easily seen that $S \in B(\mathcal{X})$ is $A(m,p)$-isometric for some $A \in B(\mathcal{X})$ (resp., $T \in B(\mathcal{Y})$ is $A(m,p)$-isometric for some $A \in B(\mathcal{Y})$) if and only if $S \otimes I \in B(\mathcal{X} \otimes \mathcal{Y})$ is $(A \otimes I)(m,p)$-isometric (resp., $(I \otimes T) \in B(\mathcal{X} \otimes \mathcal{Y})$ is $(I \otimes A)(m,p)$-isometric). Trivially, the operators $C \otimes I$ and $I \otimes D$ commute for all $C \in B(\mathcal{X})$ and $D \in B(\mathcal{Y})$.

Corollary 2.2. *If $S \in B(\mathcal{X})$ is $A_1(m,p)$-isometric and $T \in B(\mathcal{Y})$ is $A_2(n,p)$-isometric for some operators $A_1 \in B(\mathcal{X})$ and $A_2 \in B(\mathcal{Y})$, then $S \otimes T$ is $(A_1 \otimes A_2)(m+n-1,p)$-isometric.*

Proof. Apply Theorem 2.1 to the $(A_1 \otimes I)(m,p)$-isometric operator $S \otimes I \in B(X \otimes Y)$ and $(I \otimes A_2)(n,p)$-isometric operator $I \otimes T \in B(X \otimes Y)$. □

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Left-right multiplication operator $\triangle_{ST}$.

Recall from [12, Page 50] that a pair $\langle \mathcal{X}, \tilde{\mathcal{X}} \rangle$ of Banach spaces is a *dual pairing* if either $\tilde{\mathcal{X}} = \mathcal{X}^*$ or $\mathcal{X} = \tilde{\mathcal{X}}^*$. Let $x \otimes y'$, $x \in \mathcal{X}$ and $y' \in \mathcal{Y}^*$, denote the rank one operator $\mathcal{Y} \to \mathcal{X}$, $y \to \langle y, y' \rangle x$. An *operator ideal* J between Banach spaces $\mathcal{Y}$ and $\mathcal{X}$ is a linear subspace of $B(\mathcal{Y}, \mathcal{X})$ equipped with a Banach norm α such that
(i) $x \otimes y' \in J$ and $\alpha(x \otimes y') = \|x\|\|y\|$;
(ii) $\triangle_{ST}(A) = L_S R_T(A) = SAT$ and $\alpha(SAT) \leq \|S\|\alpha(A)\|T\|$
for all $x \in \mathcal{X}$, $y' \in \mathcal{Y}^*$, $A \in J$, $S \in B(\mathcal{X})$ and $T \in B(\mathcal{Y})$ [12, Page 51]. Thus defined, each J is a tensor product relative to the dual pairings $\langle \mathcal{X}, \mathcal{X}^* \rangle$ and $\langle \mathcal{Y}^*, \mathcal{Y} \rangle$ and the bilinear mappings

$$\mathcal{X} \times \mathcal{Y}^* \to J, \quad (x, y') \to x \otimes y',$$
$$B(\mathcal{X}) \times B(\mathcal{Y}^*) \to B(J), \quad (S, T^*) \to S \otimes T^*,$$

where $S \otimes T^*(A) = SAT$ $(= L_S R_T(A) = \triangle_{S,T}(A))$. The following corollary is now evident from Theorem 2.1 and Corollary 2.2.

Corollary 2.3. *Let $S, A_1 \in B(\mathcal{X})$ and $T^*, A_2^* \in B(\mathcal{Y}^*)$. If S is $A_1(m,p)$-isometric and T^* is $A_2^*(n,p)$-isometric, then $\triangle_{S,T}$ is $\triangle_{A_1,A_2}(m+n-1,p)$-isometric.*

Isometric (m,p)-isometries. If $S \in B(\mathcal{X})$ is an invertible $(2,p)$-isometry, then $\|Sx\| \geq \|x\|$ and (S^{-1} is $(2,p)$-isometric, and hence $\beta_1^{(p)}(S^{-1},x) \geq 0$ implies) $\|S^{-1}y\| \geq \|y\|$ for all $x, y \in \mathcal{X}$. Letting $y = Sx$, this implies $\|Sx\| = \|x\|$ for all $x \in \mathcal{X}$. Hence "an invertible operator $S \in B(\mathcal{X})$ is $(2,p)$-isometric if and only if it is isometric". This does not extend to non–invertible $(2,p)$-isometries [13, Example 1.2] and to invertible (m,p)-isometries for $m \geq 3$ [13, Example 4.10(i)]. Recall, [6, Proposition 3.1] (also see [1]), that if $S \in B(\mathcal{X})$ is an (m,p)-isometry and $x \in \mathcal{X}$, then there is a natural number $n_0 = n_0(x)$ such that $\|S^n x\| \leq \|S^{n+1}x\|$ for all $n \geq n_0$. This implies that an (m,p)-isometry S is either power bounded or $\|S^n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Power bounded (m,p)-isometries are normaloid.

Theorem 2.4. *If $S \in B(\mathcal{X})$ is (m,p)-isometric, then the following statements are mutually equivalent:*
(i) S is normaloid (i.e., $r(S) = \|S\|$).
(ii) S is power bounded (i.e., $\sup_n \|S^n\| < \infty$).
(iii) S is a contraction (i.e., $\|S\| \leq 1$).
(iv) S^k is a contraction for some natural number k.
(v) S is isometric.

Proof. The hypothesis S is (m,p)-isometric implies $\|S^n x\|^p = \sum_{k=0}^{m-1} n^{(k)} \beta_k^{(p)}(S,x)$ for all $x \in \mathcal{X}$, and so

$$0 \leq \beta_{m-1}^{(p)}(S,x) = \lim_{n\to\infty} \frac{\|S^n x\|^p}{n^{m-1}} \leq \lim_{n\to\infty} \frac{\|S\|^{np}\|x\|^p}{n^{m-1}}. \tag{7}$$

If S is normaloid, then (the fact that $\sigma(S) \subseteq \overline{\mathbf{D}}$ implies) $r(S) = \|S\| = \|S^n\| = 1$ (for all natural numbers n), and hence

$$\beta_{m-1}^{(p)}(S,x) = 0,$$

i.e., S is $(m-1,p)$-isometric. Repeating the argument we eventually have that S is $(1,p)$-isometric (equivalently, isometric). Since an isometry is necessarily normaloid, we have proved $(i) \Longleftrightarrow (v)$. Evidently, (v) implies (ii), (iii) and (iv). If $\sup_n \|S^n\| \leq c < \infty$, then (by (7) above) $\beta_{m-1}^{(p)}(S,x) = 0$. Repeating the argument (as above) it follows that T is isometric, and hence (ii) implies (v). That (iii) and (iv) imply (v) being evident, the proof is complete. □

Theorem 2.4 partially extends to $A(m,p)$-isometric S: If S is $A(m,p)$ isometric and either of the conditions (i) to (iv) of Theorem 2.4 is satisfied, then S is A-isometric. An A-isometric operator S need not be isometric (indeed, even be bounded below or have spectrum in the closed unit disc): Consider $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, when it is seen that $AS = A$ (implies S is A-isometric) and S is not even a contraction. As stated above, a condition on A ensuring S is a bounded below $A(m,p)$-isometry with $\sigma(S) \subseteq \overline{\mathbf{D}}$ is that $0 \notin \sigma_a(A)$. Now choose $\mathcal{X} = \mathcal{H}$ to be a Hilbert space, and let $S \in B(\mathcal{H})$ be an $A(m,2)$-isometry for some $A \in B(\mathcal{H})$. Then, for all $x \in \mathcal{H}$,

$$\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \|S^i x\|^2 = 0 \Longleftrightarrow \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} S^{*i}|A|^2 S^i = 0$$

$$\Longleftrightarrow \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \||A|S^i x\|^2 = 0.$$

Suppose additionally that A is injective. Then, [3, Page 1464], there exists an operator $S^\sharp \in B(\mathcal{H})$ such that $|A|^2 S^\sharp = S^*|A|^2$. Thus, if S is $A(m,2)$-isometric, then

$$\sum_{i=0}^{m} (-1)^{m-i}\binom{m}{i} S^{*i}|A|^2 S^i = 0 \Longleftrightarrow \sum_{i=0}^{m} (-1)^{m-i}\binom{m}{i} S^{\sharp^i} S^i = 0$$

(which implies that S is left invertible). The operator $S^\sharp$ is (the so called) $|A|^2$-adjoint of S [3] and is given by $S^\sharp = |A|^{2^\dagger} S^*|A|^2$, where $C^\dagger$ represents the Moore–Penrose inverse of the operator C. It is clear from the foregoing that a sufficient condition for S to be A-isometric is that S is normaloid.

Let us assume now that $0 \notin \sigma_a(A)$, $A \in B(\mathcal{H})$, and $S \in B(\mathcal{H})$ is $A(m,2)$-isometric. Then $|A|^2$ is invertible, $\sigma(S) = \overline{\mathbf{D}}$ if S is not invertible, $\sigma(S) \subseteq \partial\mathbf{D}$ if S is invertible, and the norms $\|x\|^2 = (x,x)$ and $\|x\|_A^2 = (Ax, Ax)$, $x \in \mathcal{H}$, are equivalent. If S is A-isometric, then there exists an isometry V such that $|A|^{-1}V|A| = S$ (i.e., S is similar to an isometry) [3, Corollary 3.4]. Thus, if S is A-isometric, then S is power bounded (indeed, similar to an isometry). Since S is power bounded implies S is A-isometric, we have:

Corollary 2.5. *Let $A, S \in B(\mathcal{H})$. If $0 \notin \sigma_a(A)$ and S is $A(m,2)$-isometric, then S is A-isometric if and only if it is similar to an isometry.*

Supercyclicity. An operator $S \in B(\mathcal{X})$, $\mathcal{X}$ necessarily separable, is supercyclic with supercyclic vector $x \in \mathcal{X}$ if scalar multiples of the orbit $\mathrm{Orb(S,x)} = \{\mathrm{x, Sx, S^2x, S^3x}, \cdots\}$ are dense in $\mathcal{X}$. Let S be $A(m,p)$-isometric for some $A \in B(\mathcal{X})$ such that $0 \notin \sigma_a(A)$. Assume that S is supercyclic with a supercyclic vector x. Then S has a dense range, and hence, since S is already (see above) bounded below, S is invertible (with S^{-1} an $A(m,p)$-isometry). Let $y \in \mathcal{X}$. Then there is a subsequence $\{n_i\}$ of the sequence of natural numbers and a sequence $\{\lambda_{n_i}\}$ of scalars such that $\lim_{i\to\infty} \lambda_{n_i} S^{n_i} x = y$. Recall that there exists a natural number n_0 such that $\|AS^n y\| \le \|AS^{n+1}y\|$ for all $n \ge n_0$ and $y \in \mathcal{X}$. Hence

$$\|Ay\| = \|\lim_{n\to\infty} \lambda_{n_i} AS^{n_i}x\| \le \|AS(\lim_{i\to\infty} \lambda_{n_i} S^{n_i}x)\| = \|ASy\|$$

for all $y \in \mathcal{X}$. Since S^{-1} is also supercyclic and $A(m,p)$-isometric, $\|Az\| \le \|AS^{-1}z\|$ for all $z \in \mathcal{X}$. Thus

$$\|Ay\| = \|ASy\| = \|AS^{-1}y\|$$

for all $y \in \mathcal{X}$.

Corollary 2.6. *Let $A \in B(\mathcal{X})$ and let $S \in B(\mathcal{X})$ be $A(m,p)$-isometric. If A is left invertible, then S can not be supercylic.*

Proof. Suppose that S is supercyclic. Then, letting A^ℓ denote the left inverse of A, the argument above implies that $\|S^n\| \le \|A^\ell\|\|A\|$ for all integers n (including 0), i.e., S is doubly power bounded and hence a generalised scalar operator [14, Theorem 1.5.13]. Since generalised scalar operators (on a Banach space of dimension greater than 1) have a non-trivial closed invariant linear subspace [14, Proposition 1.5.11], we have a contradiction. □

References

[1] M. Faghih Ahmadi and K. Hedayatian, *Hypercyclicity and supercyclicity of m-isometric operators*, Rocky Mountain J. Math. **42**(2012), 15–23.

[2] J. Agler, M. Stankus, *n-isometric transformations of Hilbert space I*, Integral Equat. Oper. Th. **21**(4)(1995) 383–429.

[3] M. Laura Aries, Gustavo Corach and M. Celeste Gonzalez, *Partial isometries in semi–Hilbertian spaces*, Linear Alg. Appl. **428**(2008), 1460–1475.

[4] O. A. Mahmoud Sid Ahmed and A. Saadi, *A − m-isometric operators on semi–Hilbertian spaces*, Linear Alg. Appl. **436**(2012), 3930–3942.

[5] O. A. Mahmoud Sid Ahmed, *m-isometric operators on Banach spaces*, Asian-European J. Math. **3**(2010), 1–19.
[6] Frédéric Bayart, *m-isometries on Banach spaces*, Math. Nachr. **284**(2011), 2141–2147. (DOI 10.1002/mana.200910029)
[7] T. Bermudez, A. Martinón and J. A. Noda, *Products of m-isometrise*, Linear Alg. Appl. (2012) (DOI:org/10.1016/laa.2012.07.011)
[8] F. Botelho, J. Jamison, *Isometric properties of elementary operators*, Linear Alg. Appl. **432**(1)(2010) 357–365.
[9] M. Chō, S. Ôta, K. Tanahashi and A. Uchiyama, Spectral properties of m-isometric operators, Functional Analysis, Approximation and Computation **4**:2(2012), 33–39.
[10] B. P. Duggal, *Tensor product of n-isometries*, Linear Alg. Appl. **437**(2012), 307–318. (DOI:10.1016/j.laa.2012.02.017)
[11] B. P. Duggal, *Tensor product of n-isometries II*, Functional Analysis, Approximation and Computation **4:1**(2012), 27–32.
[12] J. Eschmeier, *Tensor products and elementary operators*, J. reine angew. Math. **390**(1988), 47–66.
[13] P. Hoffman, M. Mackey and M. Ó Searcóid, *On the second parameter of an* (m, p)*-isometry*, Integral Equat. Oper. Th. **71**(2011), 389–405.
[14] K.B. Laursen and M.M. Neumann, Introduction to local spectral theory, Clarendon Press, Oxford (2000).