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On the ℓ^1-direct sums of normed algebras with pointwise multiplication

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Abstract. Let Γ be a set and $\mathfrak{A}_\gamma$ be a unital normed algebra for each $\gamma \in \Gamma$. Here, we first show that any left multiplier on $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ is described by a function in $\ell^\infty-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$. We then characterize compact and weakly compact left multipliers on $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ in terms of functions in $c_0-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$.

1. Introduction

Let Γ be a set and $\mathfrak{A}_\gamma$ be a unital normed algebra for each $\gamma \in \Gamma$. As usual, let $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ be the space of all functions $f : \Gamma \longrightarrow \bigcup_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ with

$$\|f\|_1 = \sum_{\gamma\in\Gamma}\|f(\gamma)\| < \infty.$$

Let also $\ell^\infty-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ denote the space of all functions $f : \Gamma \longrightarrow \bigcup_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ with

$$\|f\|_\infty := \sup_{\Gamma\in\Gamma}\|f(\gamma)\| < \infty.$$

For each $\varphi \in \ell^\infty-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$, the linear operator $\Lambda_\varphi : \ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma \longrightarrow \ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ defined by $\Lambda_\varphi(f) = \varphi f$ for all $f \in \ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ is a left multiplier; that is, $\Lambda_\varphi(fg) = \Lambda_\varphi(f)\, g$ for all $f, g \in \ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$; see [2] for details.

The remarkable fact on left multipliers on $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ is that there are no other examples. In fact, if u_γ is the unit of $\mathfrak{A}_\gamma$ with $\|u_\gamma\| = 1$ and $\delta_\gamma^{u_\gamma} \in \ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ is the function defined by $\delta_\gamma^{u_\gamma}(\gamma) = u_\gamma$ and $\delta_\gamma^{u_\gamma} = 0$ otherwise, then any left multiplier $\Lambda : \ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma \longrightarrow \ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ is of the form Λ_φ for the function $\varphi \in \ell^\infty-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ defined by $\varphi(\gamma) = \Lambda(\delta_\gamma^{u_\gamma})(\gamma)$ for all $\gamma \in \Gamma$. Indeed, for each $\gamma \in \Gamma$,

$$\begin{aligned}\|\varphi(\gamma)\| &= \left\|\Lambda(\delta_\gamma^{u_\gamma})(\gamma)\right\| \le \sum_{\alpha\in\Gamma}\left\|\Lambda(\delta_\gamma^{u_\gamma})(\alpha)\right\| \\ &= \|\Lambda(\delta_\gamma^{u_\gamma})\|_1 \le \|\Lambda\|.\end{aligned}$$

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and so $\|\varphi\|_\infty \leq \|\Lambda\|$. Also, for each function with finite support $f \in \ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$, we have $f = \sum_{n=1}^m \delta_{\gamma_n}^{f(\gamma_n)}$ for some $\gamma_1, \cdots, \gamma_m \in \Gamma$ and thus

$$\Lambda(f) = \sum_{n=1}^{m} \Lambda(\delta_{\gamma_n}^{u_{\gamma_n}}\ \delta_{\gamma_n}^{f(\gamma_n)}) = \sum_{n=1}^{m} \Lambda(\delta_{\gamma_n}^{u_{\gamma_n}})\ \delta_{\gamma_n}^{f(\gamma_n)},$$

whence $\Lambda(f) = \varphi f$. This together with that Λ and Λ_φ are bounded on $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ and that functions with finite support are norm dense in $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ imply that $\Lambda = \Lambda_\varphi$; see [6] for more details.

Recently, we have characterized compact multipliers on $L^1(\Omega, \mathfrak{A})$, the space of all Bochner integrable functions f from a locally compact Hausdorff space Ω to a nontrivial normed algebra $\mathfrak{A}$; see [1]. There also has been considerable progress in the study of analysis on $L^1(\Omega, \mathfrak{A})$, and several authors have studied various aspects of the subject; see for example [3], [4], [5], [7], [8] [10] and [11]. Our purpose in this work is to present the following characterization of compact and weakly compact left multipliers on $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$.

Theorem 1.1. *Let Γ be a set, $\mathfrak{A}_\gamma$ be a unital normed algebra for each $\gamma \in \Gamma$ and $\varphi \in \ell^\infty-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$. Then the following assertions are equivalent.*

(a) $\varphi \in c_0-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$.
(b) Λ_φ *is compact on* $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$.
(c) Λ_φ *is weakly compact on* $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$.

Here $c_0-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ denotes the closed subalgebra of $\ell^\infty-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ consisting of all functions vanishing at infinity.

2. The proof of Theorem 1.1

We first show that (a) implies (b). For this end, suppose that $\varphi \in c_0-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$. Then $\mathrm{coz}(\varphi) := \{\gamma \in \Gamma : \varphi(\gamma) \neq 0\}$ is a $\sigma-$finite subset of Γ; indeed, if we set $F_n = \{\gamma \in \Gamma : \|\varphi(\gamma)\| \geq 1/n\}$ for all $n \geq 1$, then (F_n) is an increasing sequence of finite subsets in Γ and $\mathrm{coz}(\varphi) = \bigcup_{n=1}^\infty F_n$. Moreover,

$$\|\Lambda_{\chi_{F_n}^\varphi} - \Lambda_\varphi\| = \|\Lambda_{\chi_{\Gamma\setminus F_n}^\varphi}\| = \|\chi_{\Gamma\setminus F_n}^\varphi\|_\infty < \frac{1}{n},$$

where for each $E \subseteq \Gamma$, $\chi_E^\varphi : \Gamma \longrightarrow \bigcup_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ is defined by $\chi_E^\varphi(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$ and $\chi_E^\varphi(\gamma) = 0$ for all $\gamma \in \Gamma \setminus E$. Since $\Lambda_{\chi_{F_n}^\varphi}$ is a compact operator on $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ for all $n \geq 1$, it follows that Λ_φ is also compact.

That (b) implies (c) is trivial. To complete the proof, suppose that Λ_φ is weakly compact on $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$. Let $\mathcal{F}$ be the family of all finite subsets in Γ directed under upward inclusion. Suppose on the contrary that $\varphi \notin c_0-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$. Then we may find $\varepsilon_0 > 0$ with $\|\chi_{\Gamma\setminus F}^\varphi\|_\infty \geq \varepsilon_0$ for all $F \in \mathcal{F}$; that is, there exists $\gamma_F \in \Gamma \setminus F$ such that $|\varphi(\gamma_F)| \geq \varepsilon_0$. Thus $(\delta_{\gamma_F}^{\varphi(\gamma_F)})_{F\in\mathcal{F}}$ is a net in $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ bounded by $\|\varphi\|_\infty$, and there is $f_0 \in \ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ and a subnet $(\gamma_{F(\beta)})_{\beta\in B}$ of $(\gamma_F)_{F\in\mathcal{F}}$ such that

$$\delta_{\gamma_{F(\beta)}}^{\varphi(\gamma_{F(\beta)})} \to f_0$$

in the weak topology of $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$. In particular, since $\|\varphi(\gamma_{F(\beta)})\| \neq 0$, it follows from the Hahn-Banach theorem that there exists $\Phi_\beta \in \mathfrak{A}_\gamma^*$ such that

$$|\langle\, \Phi_\beta, \varphi(\gamma_{F(\beta)})\, \rangle| = \|\varphi(\gamma_{F(\beta)})\| \quad\text{and}\quad \|\Phi_\beta\| = 1.$$

The dual space of $\ell^1-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$ is equal to $\ell^\infty-\oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma^*$ with the duality

$$\langle\, \Psi, f\, \rangle = \sum_{\gamma\in\Gamma} \langle\, \Psi(\gamma), f(\gamma)\, \rangle$$

for all $\Psi \in \ell^\infty - \oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma^*$ and $f \in \ell^1 - \oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma$; see for example [2]. Define $\Phi \in \ell^\infty - \oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma^*$ by $\Phi(\gamma_{F(\beta)}) = \Phi_\beta$ and $\Phi(\gamma) = 0$ otherwise. Then $\Phi \in \ell^\infty - \oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma^*$, and therefore

$$\begin{aligned}
\langle \Phi, f_0 \rangle &= \lim_\beta \langle \Phi, \delta_{\gamma_{F(\beta)}}^{\varphi(\gamma_{F(\beta)})} \rangle \\
&= \lim_\beta \sum_{\gamma\in\Gamma} \langle \Phi(\gamma), \delta_{\gamma_{F(\beta)}}^{\varphi(\gamma_{F(\beta)})}(\gamma) \rangle \\
&= \lim_\beta \langle \Phi(\gamma_{F(\beta)}), \varphi(\gamma_{F(\beta)}) \rangle \\
&= \lim_\beta \langle \Phi_\beta, \varphi(\gamma_{F(\beta)}) \rangle
\end{aligned}$$

and hence $|\langle \Phi, f_0 \rangle| = \lim_\beta \|\varphi(\gamma_{F(\beta)})\| \geq \varepsilon_0$. It follows that $f_0(\gamma_0) \neq 0$ for some $\gamma_0 \in \Gamma$ and thus there is $\beta_0 \in B$ such that $\gamma_0 \in F(\beta)$ for all $\beta \geq \beta_0$; in particular, $\gamma_0 \neq \gamma_{F(\beta)}$ for all $\beta \geq \beta_0$. Now, invoke the Hahn-Banach theorem to conclude that there exists $\Psi_{\gamma_0} \in \mathfrak{A}_{\gamma_0}^*$ such that $|\langle \Psi_{\gamma_0}, f_0(\gamma_0) \rangle| = \|f_0(\gamma_0)\|$ and $\|\Psi_{\gamma_0}\| = 1$; see [9]. Define $\Psi_0 \in \ell^\infty - \oplus_{\gamma\in\Gamma}\mathfrak{A}_\gamma^*$ by $\Psi_0(\gamma_0) = \Psi_{\gamma_0}$ and $\Psi_0(\gamma) = 0$ for all $\gamma \in \Gamma \setminus \{\gamma_0\}$. Therefore

$$\begin{aligned}
\|f_0(\gamma_0)\| &= |\langle \Psi_{\gamma_0}, f_0(\gamma_0) \rangle| = |\langle \Psi_0(\gamma_0), f_0(\gamma_0) \rangle| \\
&= |\sum_{\gamma\in\Gamma} \langle \Psi_0(\gamma), f_0(\gamma) \rangle| = |\langle \Psi_0, f_0 \rangle| \\
&= \lim_\beta |\langle \Psi_0, \delta_{\gamma_{F(\beta)}}^{\varphi(\gamma_{F(\beta)})} \rangle| = \lim_\beta |\sum_{\gamma\in\Gamma} \langle \Psi_0(\gamma), \delta_{\gamma_{F(\beta)}}^{\varphi(\gamma_{F(\beta)})}(\gamma) \rangle| \\
&= \lim_\beta |\langle \Psi_0(\gamma_0), \delta_{\gamma_{F(\beta)}}^{\varphi(\gamma_{F(\beta)})}(\gamma_0) \rangle| = 0;
\end{aligned}$$

that is, $f_0(\gamma_0) = 0$. This contradiction completes the proof. □

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