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Error Estimates for the Eigenvalue Problem for Compact Operators

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Abstract. The paper deals with approximations of the eigenvalues of compact operators in a Hilbert space by the eigenvalues of finite matrices. Namely, let $(a_{jk})_{j,k=1}^{\infty}$ be the matrix representation of a compact A in an orthonormal basis, and $A_n = (a_{jk})_{j,k=1}^{n}$. A priori estimates are established for the quantity $\sup_{\mu\in\sigma(A)} \min_{\lambda\in\sigma(A_n)} |\lambda - \mu|$, where $\sigma(A)$ is the spectrum of A.

1. Introduction and statement of the main result

The literature devoted to approximations of the eigenvalues of various concrete operators is very rich, cf. the interesting papers [1–3, 6, 8] and references given therein. Besides, in many cases the error estimates are suggested. At the same time, to the best of our knowledge, such estimates for approximations of the eigenvalues of compact operators in a Hilbert space by the eigenvalues of finite matrices were not investigated in the available literature.

Let H be a separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and the unit operator I; A is a compact operator acting in H. In the sequel $(a_{jk})_{j,k=1}^{\infty}$ is the matrix representation of A in an orthonormal basis $\{e_k\}_{k=1}^{\infty}$, and $A_n = (a_{jk})_{j,k=1}^{n}$ is the $n \times n$-matrix; $\lambda_k(A)$ are the eigenvalues of A taken with their multiplicities. In this paper we establish a priori error estimates for the approximation of the eigenvalues of A by the eigenvalues of A_n.

Below $\sigma(A)$ denotes the spectrum of A, $R_\lambda(A)= (A-I\lambda)^{-1}$ $(\lambda \notin \sigma(A))$ is the resolvent; $\rho(A,\lambda) = \inf_{s\in\sigma(A)} |\lambda - s|$ the distance between $\lambda \in \mathbb{C}$ and $\sigma(A)$. For an integer $p \geq 2$, let SN_p be the Schatten - von Neumann ideal of compact operators A in H with the finite norm $N_p(A) = [Trace\ (AA^*)^{p/2}]^{1/p}$, where A^* is the adjoint of A, cf. [5].

Furthermore, put

$$g(A_n) = [N_2^2(A_n) - \sum_{k=1}^{n} |\lambda_k(A_n)|^2\ ]^{1/2},$$

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The following relations are checked in [4, Section 2.1].

$$g^2(A_n) \le N_2^2(A_n) - |Trace\ A_n^2| \text{ and } g^2(A_n) \le \frac{N_2^2(A_n - A_n^*)}{2} = 2N_2^2(A_{nI}),$$

where $A_{nI} = (A_n - A_n^*)/2i$. If A_n is a normal matrix: $A_n A_n^* = A_n^* A_n$, then $g(A_n) = 0$. Since A is compact we have

$$q_n := \|A_n - A\| \to 0.$$

Denote by $r(q_n)$ the unique positive root of the algebraic equation

$$z^n = q_n \sum_{j=0}^{n-1} \frac{z^{n-j-1} g^j(A_n)}{\sqrt{j!}}. \tag{1.1}$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *For any $\mu \in \sigma(A)$ and a natural n, either there is an eigenvalue $\lambda(A_n)$ of the $n \times n$-matrix A_n satisfying $|\mu - \lambda(A_n)| \le r(q_n)$, or $|\mu - a_{jj}| \le r(q_n)$ for some $j > n$. If in addition, $A \in SN_2$, then $r(q_n) \to 0$ as $n \to \infty$.*

This theorem is proved in the next section. In Section 3, we suggest another error estimate, which tends to zero, provided $A \in SN_p, p \ge 2$. That estimate is rougher than $r(q_n)$.

Put

$$P_n(x) = \sum_{j=0}^{n-1} \frac{x^{j+1} g^j(A_n)}{\sqrt{j!}} \quad (x \ge 0).$$

Thanks to [4, Lemma 1.6.1] we have $r(q_n) \le \zeta(q_n)$, where

$$\zeta(q_n) = \begin{cases} \sqrt[n]{q_n P_n(1)} & \text{if } q_n P_n(1) \le 1, \\ q_n P_n(1) & \text{if } q_n P_n(1) \ge 1 \end{cases}.$$

Thus in Theorem 1.1 one can replace $r(q_n)$ by $\zeta(q_n)$.

2. Proof of Theorem 1.1

First let us prove that $r(q_n) \to 0$ as $n \to \infty$, provided $A \in SN_2$. To this end rewrite (1.1) as

$$1 = q_n \sum_{j=0}^{n-1} \frac{g^j(A_n)}{z^{j+1}\sqrt{j!}}.$$

Clearly, $g(A_n) \le N_2(A_n) \le N_2(A)$ and thus

$$1 \le q_n \sum_{j=0}^{\infty} \frac{N_2^j(A)}{r^{j+1}(q_n)\sqrt{j!}}.$$

Since $q_n \to 0$, hence $r(q_n) \to 0$.

Furthermore, we will consider A as a perturbation of the operator $C_n = S_n + A_n$ where $S_n = diag(a_{jj})_{k=n+1}^{\infty}$. Put $Q_n = \sum_{k=1}^{n} (., e_k) e_k$. Then $A_n = Q_n A Q_n$ and $S_n = (I - Q_n) S_n = S_n (I - Q_n)$. Clearly, $S_n A_n = A_n S_n = 0$ and consequently

$$\sigma(C_n) = \sigma(A_n) \cup \{a_{jj}\}_{k=n+1}^{\infty}. \tag{2.1}$$

Thus

$$\|R_\lambda(C_n)\| = \max\{\|Q_n R_\lambda(A_n)\|, \|(I - Q_n) R_\lambda(S_n)\|\}. \tag{2.2}$$

Thanks to Corollary 2.1.2 from [4],

$$\|Q_n R_\lambda(A_n)\| \le \sum_{k=0}^{n-1} \frac{g^k(A_n)}{\sqrt{k!}\rho^{k+1}(A_n,\lambda)} \quad (\lambda \notin \sigma(A_n)). \tag{2.3}$$

Rewrite (2.3) as

$$\|Q_n R_\lambda(A_n)\| \le P_n(1/\rho(A_n,\lambda)). \tag{2.4}$$

Now (2.4) and (2.2) imply the inequality

$$\|R_\lambda(C_n)\| \le \max\{P_n(1/\rho(A_n,\lambda)), 1/\rho(S_n,\lambda)\}. \tag{2.5}$$

But due to (2.1), $\rho(A_n,\lambda) \ge \rho(C_n,\lambda)$ and $\rho(S_n,\lambda) \ge \rho(C_n,\lambda)$. In addition, $P_n(x) \ge x$ for $x \ge 0$. Thus

$$\|R_\lambda(C_n)\| \le P_n(1/\rho(C_n,\lambda)). \tag{2.6}$$

Furthermore, for two bounded operators A and $\tilde{A}$, the spectral variation $sv_A(\tilde{A})$ of $\tilde{A}$ with respect to A is defined by

$$sv_A(\tilde{A}) := \sup_{\mu\in\sigma(\tilde{A})} \inf_{\lambda\in\sigma(A)} |\lambda - \mu|.$$

Assume that

$$\|R_\lambda(A)\| \le \phi(1/\rho(A,\lambda)) \quad (\lambda \notin \sigma(A)),$$

where $\phi(x)$ is a monotonically increasing non-negative continuous function of a non-negative variable x, such that $\phi(0) = 0$ and $\phi(\infty) = \infty$. Then due to Lemma 8.4.2 [4], the inequality $sv_A(\tilde{A}) \le z(\phi, q)$ is true, where $z(\phi, q)$ is the a unique positive root of the equation

$$1 = q\phi(1/z). \tag{2.7}$$

Now (2.6) implies $sv_{C_n}(A) \le r_n(q_n)$. According to (2.1) this proves the theorem. □

3. The case $A \in SN_{2p}$, $p > 1$

Assume that

$$A \in SN_{2p}, p = 2, 3, \ldots \tag{3.1}$$

Let

$$n = jp \text{ with integers } p \ge 1, j > 1. \tag{3.2}$$

Denote by $\hat{r}_p(q_n)$ the unique positive root of the algebraic equation

$$z^n = q_n \sum_{m=0}^{p-1} \sum_{k=0}^{j} z^{n-pk-m-1} \frac{N_{2p}^{kp+m}(2A_n)}{\sqrt{k!}}. \tag{3.3}$$

Theorem 3.1. *Let condition (3.1) hold. Then for any $\mu \in \sigma(A)$ and a natural $n = pj$, either there is an eigenvalue $\lambda(A_n)$ of the $n\times n$-matrix A_n satisfying $|\mu - \lambda(A_n)| \le \hat{r}_p(q_n)$, or $|\mu - a_{jj}| \le \hat{r}_p(q_n)$ for some $j > n$. Moreover, $\hat{r}_p(q_n) \to 0$ as $n \to \infty$.*

To prove this theorem we need the following result.

Lemma 3.2. *Let A_n be a linear operator acting in a Euclidean space $\mathbb{C}^n$ with $n = jp$ and integers $p \geq 1, j > 1$. Then*

$$\|R_\lambda(A_n)\| \leq \sum_{m=0}^{p-1}\sum_{k=0}^{j} \frac{N_{2p}^{kp+m}(2A_n)}{\rho^{pk+m+1}(A_n,\lambda)\sqrt{k!}} \quad (\lambda \notin \sigma(A_n)). \tag{3.4}$$

Proof. Due to the Schur theorem, cf. [7],

$$A_n = D + V \quad (\sigma(A_n) = \sigma(D))$$

where D is a normal matrix and V is a nilpotent matrix. Besides, D and V have the same invariant subspaces, and V is called the nilpotent part of A_n. Thanks to [4, Lemma 6.8.3],

$$\|R_\lambda(A_n)\| \leq \sum_{m=0}^{p-1}\sum_{k=0}^{j} \frac{N_{2p}^{kp+m}(V)}{\rho^{pk+m+1}(A_n,\lambda)\sqrt{k!}} \quad (\lambda \notin \sigma(A_n))$$

where V is the nilpotent part of A_n. But it is not hard to check that $N_{2p}(V) \leq N_{2p}(2A_n)$, cf [4, page 90, formula (8.8)]. This proves the lemma. □

Proof of Theorem 3.1: First let us prove that $\hat{r}_p(q_n) \to 0$ as $n \to \infty$, provided (3.1) holds. To this end rewrite (3.3) as

$$1 = q_n \sum_{m=0}^{p-1}\sum_{k=0}^{j} z^{-pk-m-1}\frac{N_{2p}^{kp+m}(2A_n)}{\sqrt{k!}}.$$

Hence

$$1 \leq q_n \sum_{m=0}^{p-1}\sum_{k=0}^{\infty} \hat{r}_p^{-pk-m-1}(q_n)\frac{N_{2p}^{kp+m}(2A)}{\sqrt{k!}}.$$

Since $q_n \to 0$, we have $\hat{r}_p(q_n) \to 0$.

Furthermore, rewrite (3.4) as

$$\|R_\lambda(A_n)\| \leq \hat{P}_{n,p}(1/\rho(A_n,\lambda)), \tag{3.5}$$

where

$$\hat{P}_{n,p}(x) = \sum_{m=0}^{p-1}\sum_{k=0}^{\infty} x^{pk+m+1}\frac{N_{2p}^{kp+m}(2A_n)}{\sqrt{k!}} \quad (x \geq 0).$$

Now (3.5) and (2.2) imply the inequality

$$\|R_\lambda(C_n)\| \leq \max\{\hat{P}_{n,p}(1/\rho(A_n,\lambda)), 1/\rho(S_n,\lambda)\}.$$

But due to (2.1) , we have $\rho(A_n,\lambda) \geq \rho(C_n,\lambda)$ and $\rho(S_n,\lambda) \geq \rho(C_n,\lambda)$. In addition, $\hat{P}_{n,p}(x) \geq x$ for $x \geq 0$. Thus,

$$\|R_\lambda(C_n)\| \leq \hat{P}_{n,p}(1/\rho(C_n,\lambda)). \tag{3.6}$$

Due to the above mentioned Lemma 8.4.2 [4], the inequality

$$sv_{C_n}(A) \leq \hat{r}_p(q_n).$$

holds. According to (2.1) this proves the theorem. □

Furthermore, again use [4, Lemma 1.6.1]; we obtain $\hat{r}_p(q_n) \leq \hat{\zeta}_p(q_n)$, where

$$\hat{\zeta}_p(q_n) = \begin{cases} \sqrt[n]{q_n\hat{P}_{n,p}(1)} & \text{if } q_n\hat{P}_{n,p}(1) \leq 1, \\ q_n\hat{P}_{n,p}(1) & \text{if } q_n\hat{P}_{n,p}(1) \geq 1 \end{cases}.$$

Thus in Theorem 3.1 one can replace $\hat{r}_p(q_n)$ by $\hat{\zeta}_p(q_n)$.

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