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Convergence of modified Ishikawa's iteration process for asymptotically pseudocontractive mappings

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Abstract. We establish a common fixed point theorem for a finite family of continuous uniformly L-Lipschitzian asymptotically pseudocontractive mappings to prove the strong convergence of modified Ishikawa's method provided that the interior of the set of common fixed points is nonempty, wherein the compactness assumption is not imposed either on the mappings or on the space. Moreover, the computation of closed convex set for each iteration is not required. The results obtained in this paper are improvements over many results that have been proved for this class of nonlinear mappings.

1. Introduction

Let C be a nonempty subset of a real Hilbert space H. A mapping $T : C \to H$ is called

1. nonexpansive, if
$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C,$$
2. asymptotically nonexpansive [5], if for each $n \in \mathbb{N}$, there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that
$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall\, x, y \in C,$$
3. uniformly L-Lipschitzian, if for each $n \in \mathbb{N}$, there exists a positive constant L such that
$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall\, x, y \in C,$$
4. pseudocontractive, if
$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall\, x, y \in C,$$

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5. asymptotically pseudocontractive, if for each $n \in \mathbb{N}$, there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, x - y\rangle \leq k_n \|x - y\|^2, \quad \forall\, x, y \in C.$$

Note that the above inequality can be equivalently written as

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|(I - T^n)x - (I - T^n)y\|^2, \quad \forall\, x, y \in C. \tag{1}$$

Every nonexpansive mapping is uniformly L-Lipschitzian with $L = 1$ and hence every asymptotically nonexpansive mapping with the sequence $\{k_n\}$ is also uniformly L-Lipschitzian with $L = \sup_{n\in\mathbb{N}} k_n$.

Remark 1.1. *1. If T is an asymptotically nonexpansive mapping, then for all $x, y \in C$, we have*

$$\langle T^n x - T^n y, x - y\rangle \leq \|T^n x - T^n y\|.\|x - y\| \leq k_n\|x - y\|^2, \ \ n \geq 1,$$

so that every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

2. Rhoades [13] showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

3. The asymptotically pseudocontractive mappings were introduced by Schu [14].

In recent years, Mann and Ishikawa iterative schemes [6, 8] have been studied extensively by many authors. Let H be a real Hilbert space and C be a nonempty subset of H. Let $T : C \to C$ be a mapping.

(a) The Mann iteration process is defined by the sequence $\{x_n\}_{n\geq 1}$,

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}_{n\geq 1}$ is a sequence in $[0, 1]$.

(b) The sequence $\{x_n\}_{n\geq 1}$ defined by

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ are sequences in $[0, 1]$, is known as the Ishikawa iteration process.

Recently, some authors considered the so called modified Mann iteration, respectively modified Ishikawa iteration, by replacing the operator T by its n-th iterate T^n, that is, the modified Ishikawa iteration is defined by the sequence $\{x_n\}_{n\geq 1}$,

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \ n \geq 1. \end{cases} \tag{2}$$

For $\beta_n = 0$, the modified Ishikawa iteration reduces to the modified Mann iteration.

In [14], Schu proved the following result:

Theorem 1.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H, $T : C \to C$ a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1,\infty)$; $q_n = 2k_n - 1$, $\forall n \in \mathbb{N}$; $\sum(q_n^2 - 1) < \infty$; $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$; $\epsilon < \alpha_n < \beta_n \leq b$, $\forall n \in \mathbb{N}, \epsilon > 0$ and $b \in (0, L^{-2}[\sqrt{1+L^2}-1])$; $x_1 \in C$, $\forall n \in \mathbb{N}$ and define*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \; n \geq 1. \tag{3}$$

Then $\{x_n\}$ converges to some fixed point of T.

In [9], Ofoedu proved the following result:

Theorem 1.3. *Let C be a nonempty closed convex subset of a real Banach space E and $T : C \to C$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ and $p \in F(T) = \{x \in C : Tx = x\}$. Let $\{\alpha_n\} \subset [0,1]$ be such that $\sum_{n\geq 1} \alpha_n = \infty$, $\sum_{n\geq 1} \alpha_n^2 < \infty$, and $\sum_{n\geq 1} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively defined by* (3).

Suppose there exists a strictly increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$\langle T^n x - p, j(x-p)\rangle \leq k_n\|x-p\|^2 - \psi(\|x-p\|), \; \forall\, x \in C,$$

$j(x-p) \in J(x-p)$, where $J : E \to 2^{E^*}$ be the normalized duality mapping. Then $\{x_n\}$ converges strongly to $p \in F(T)$.

Many authors (e.g. [2, 4]) have studied the two-mapping case of iterative schemes for different kinds of mappings.

In [2], Chang et al. proved the following result:

Theorem 1.4. *Let C be a nonempty closed convex subset of a real Banach space E and $T_i : C \to C$, $i = 1,2$ be two uniformly L_i-Lipschitzian asymptotically pseudocontractive mappings with sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ and $F(T_1) \cap F(T_2) \neq \emptyset$, where $F(T_i)$ is the set of fixed points of T_i in C and let p be a point in $F(T_1) \cap F(T_2)$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ be two sequences such that $\sum_{n\geq 1} \alpha_n = \infty$, $\sum_{n\geq 1} \alpha_n^2 < \infty$, and $\sum_{n\geq 1} \alpha_n(k_n-1) < \infty$. For arbitrary $x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_n \\ \;\; y_n = (1-\beta_n)x_n + \beta_n T_2^n x_n, \; n \geq 1. \end{cases}$$

Suppose there exists a strictly increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$\langle T_i^n x - p, j(x-p)\rangle \leq k_n\|x-p\|^2 - \psi(\|x-p\|), \; \forall\, x \in C, \; i = 1,2.$$

Then $\{x_n\}$ converges strongly to $p \in F(T_1) \cap F(T_2)$.

Further Rafiq [11] introduced and analyzed a class of multistep iterative schemes for families of asymptotically pseudocontractive mappings T_i, $i = 1,2,...,N$ having bounded ranges and showed the strong convergence of the sequence to the common fixed point of T_i.

In a finite dimensional Hilbert space, the Mann and Ishikawa iteration have only weak convergence, in general. Chidume and Mutanguda [3], gave an example on which Mann iterative sequence failed to converge to a fixed point of a Lipschitz pseudocontractive mappings. Zhou [18], Yao et al. [16] and Tang et al. [15] proved the results for hybrid Ishikawa algorithm, hybrid Mann algorithm and for another algorithm respectively for Lipschitz pseudocontractive mapping to obtain the strong convergence. But it is worth mentioning that these schemes were not that easy to compute. The iterations $\{x_n\}$ in the above papers were generated by projection of initial point x_0 on to the intersection of closed convex sets C_n and Q_n for each $n \geq 1$, which is not easy to compute.

This brings to the question that whether it is possible to obtain the strong convergence of Ishikawa's scheme (not hybrid) to a fixed point of Lipschitz pseudocontractive mappings.

In 2011 Zegeye et al. [17] proved a result which justifies and answers to the above question. The result is as follows:

Theorem 1.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to C$, $i = 1, 2, ..., N$, be a finite family of Lipschitz pseudocontractive mappings with Lipschitz constants L_i for $i = 1, 2, ..., N$, respectively. Assume that the interior of $F = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T_n x_n; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n \end{cases}$$

where $T_n =: T_{n(mod\ N)}$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying the following conditions:

(i) $\alpha_n \leq \beta_n$, $\forall\, n \geq 0$;

(ii) $\liminf_{n\to\infty} \alpha_n = \alpha > 0$ *and*

(iii) $\sup_{n\geq 1} \beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$, *for* $L =: \max\{L_i : i = 1, 2, ..., N\}$.

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.

It is our purpose in this paper to prove strong convergence of modified Ishikawa's scheme (not hybrid) to a common fixed point of a finite family of continuous uniformly L-Lipschitzian asymptotically pseudocontractive mappings provided the interior of the set of common fixed points is nonempty. No compactness assumption is imposed either on one of the mappings or on C. Moreover, the computation of closed and convex set C_n for each $n \geq 1$ is not required. The results obtained in this paper improve on and extend the results of Zegeye et al. [17].

2. Preliminaries

In the sequel we shall need the following definitions and lemmas:

Let H be a real Hilbert space. A function $\phi : H \times H \to \mathbb{R}$ defined by

$$\phi(x, y) := \|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \text{ for } x, y \in H \tag{4}$$

is studied by Alber [1], Kamimura and Takahashi [7] and Reich [12].

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \text{ for } x, y \in H.$$

The function ϕ also has the following property:

$$\phi(y, x) = \phi(z, x) + \phi(y, z) + 2\langle z - y, x - z\rangle \text{ for all } x, y, z \in H. \tag{5}$$

In what follows, we shall make use of the following:

Lemma 2.1. *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in [0, 1]$, the following inequality holds:*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2.$$

Lemma 2.2. *[14] Let H be a Hilbert space and C be a nonempty convex subset of H. Let $L > 0$ and $T : C \to C$ be a uniformly L-Lipschitzian map. Let $\{x_n\}_{n\geq 1}$ be the sequence defined by (2). If $c_n = \|T^n x_n - x_n\|$, $\forall\, n \in \mathbb{N}$, then*

$$\|x_n - Tx_n\| \leq c_n + c_{n-1}[L(1 + 3L + 2L^2)], \ \ \forall\, n \in \mathbb{N}.$$

Lemma 2.3. *[10] Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum\limits_{n=1}^{\infty} \delta_n < \infty$ and $\sum\limits_{n=1}^{\infty} b_n < \infty$, then $\lim\limits_{n\to\infty} a_n = 0$.

3. Main Results

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to C$, $i = 1, 2, ..., N$, be a finite family of continuous uniformly L-Lipschitzian asymptotically pseudocontractive mappings with sequence $\{k_n^{(i)}\} \subseteq [1, \infty)$ satisfying $\lim\limits_{n\to\infty} k_n^{(i)} = 1$ and Lipschitz constants L_i for $i = 1, 2, ..., N$, respectively. Assume that the interior of $F = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n T_i^k x_n; \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_i^k y_n \end{aligned} \tag{6}$$

where $n = (k-1)N + i$, for each $n \geq 1$, $i = 1, 2, \ldots, N$, $k \geq 1$ is a positive integer with $k \to \infty$ as $n \to \infty$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, where $q_n = 2k_n - 1$ for each $n \geq 1$, and $k_n = \max\{k_n^{(i)} : i = 1, 2, \ldots, N\}$;

(ii) $a \leq \alpha_n \leq \beta_n \leq b$ for some $a > 0$ and some $b \in \left(0, \frac{\sqrt{1+L^2}-1}{L^2}\right)$ for $L =: \max\{L_i : i = 1, 2, ..., N\}$.

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.

Proof. Suppose that $p \in F(T)$. Then from equations (1), (6) and by Lemma 2.1, we have that

$$\begin{aligned} \|y_n - p\|^2 &= (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|T_i^k x_n - p\|^2 - \beta_n(1 - \beta_n)\|T_i^k x_n - x_n\|^2 \\ &\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n \left\{q_n\|x_n - p\|^2 + \|x_n - T_i^k x_n\|^2\right\} \\ &\quad - \beta_n(1 - \beta_n)\|T_i^k x_n - x_n\|^2 \\ &\leq q_n\|x_n - p\|^2 + \beta_n^2\|T_i^k x_n - x_n\|^2, \end{aligned} \tag{7}$$

and,

$$\begin{aligned} \|y_n - T_i^k y_n\|^2 &= (1 - \beta_n)\|x_n - T_i^k y_n\|^2 + \beta_n\|T_i^k x_n - T_i^k y_n\|^2 - \beta_n(1 - \beta_n)\|T_i^k x_n - x_n\|^2 \\ &\leq (1 - \beta_n)\|x_n - T_i^k y_n\|^2 + \beta_n L^2\|x_n - y_n\|^2 - \beta_n(1 - \beta_n)\|T_i^k x_n - x_n\|^2 \\ &= (1 - \beta_n)\|x_n - T_i^k y_n\|^2 + \beta_n^3 L^2\|x_n - T_i^k x_n\|^2 - \beta_n(1 - \beta_n)\|x_n - T_i^k x_n\|^2. \end{aligned} \tag{8}$$

Using (7) and (8), we have

$$\begin{aligned}\|T_i^k y_n - p\|^2 &\le q_n\|y_n - p\|^2 + \|y_n - T_i^k y_n\|^2\\ &\le q_n^2\|x_n - p\|^2 + q_n\beta_n^2\|T_i^k x_n - x_n\|^2 + \beta_n^3 L^2\|x_n - T_i^k x_n\|^2\\ &\quad + (1-\beta_n)\|x_n - T_i^k y_n\|^2 - \beta_n(1-\beta_n)\|x_n - T_i^k x_n\|^2\\ &= q_n^2\|x_n - p\|^2 - \beta_n\left(1 - \beta_n - q_n\beta_n - \beta_n^2 L^2\right)\|x_n - T_i^k x_n\|^2\\ &\quad + (1-\beta_n)\|x_n - T_i^k y_n\|^2. \end{aligned} \tag{9}$$

Using (6) and (9), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T_i^k y_n - p\|^2 - \alpha_n(1-\alpha_n)\|T_i^k y_n - x_n\|^2\\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n q_n^2\|x_n - p\|^2 + \beta_n^2\|x_n - T_i^k x_n\|^2\\ &\quad - \alpha_n\beta_n\left(1 - \beta_n - q_n\beta_n - \beta_n^2 L^2\right)\|x_n - T_i^k x_n\|^2\\ &\quad + \alpha_n(1-\beta_n)\|x_n - T_i^k y_n\|^2 - \alpha_n(1-\alpha_n)\|T_i^k y_n - x_n\|^2\\ &\le q_n^2\|x_n - p\|^2 - \alpha_n\beta_n\left(1 - \beta_n - q_n\beta_n - \beta_n^2 L^2\right)\|x_n - T_i^k x_n\|^2. \end{aligned} \tag{10}$$

By assumption, we see that there exists n_0 such that

$$1 - \beta_n - q_n\beta_n - \beta_n^2 L^2 \ge \frac{1 - 2b - L^2 b^2}{2} > 0\,,\ \forall n \ge n_0\,.$$

Hence from (10), we have

$$\|x_{n+1} - p\|^2 \le \left(1 + (q_n^2 - 1)\right)\|x_n - p\|^2\,.$$

Therefore by Lemma 2.3, $\lim_{n\to\infty}\|x_n - p\|$ exists and in particular, $\{\|x_n - p\|\}$ is bounded. This implies that $\{x_n\}$, $\{T_i^k x_n\}$ and hence, $\{y_n\}$ are bounded. Furthermore, from (5) we also have that

$$\phi(p, x_n) = \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - p, x_n - x_{n+1}\rangle.$$

This implies that

$$\langle x_{n+1} - p, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}(\phi(p, x_n) - \phi(p, x_{n+1})). \tag{11}$$

From (4) we have $\phi(x, y) = \|x - y\|^2$. Therefore, we have

$$\phi(p, x_n) - \phi(p, x_{n+1}) \ge 0.$$

Hence it follows from (11) that

$$\langle x_{n+1} - p, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) \ge 0.$$

Moreover, since the interior of F is nonempty, there exists $p^* \in F$ and $r > 0$ such that $p^* + rh \in F$ whenever $\|h\| \le 1$. Thus we get,

$$0 \le \langle x_{n+1} - (p^* + rh), x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n),$$

from which we obtain that

$$2r\langle h, x_n - x_{n+1}\rangle \le \phi(p^*, x_n) - \phi(p^*, x_{n+1}),$$

and hence

$$\langle h, x_n - x_{n+1}\rangle \leq \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})).$$

Since h with $\|h\| \leq 1$ is arbitrary, we have

$$\|x_n - x_{n+1}\| \leq \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})).$$

So, if $n > m$, then we have

$$\begin{aligned}\|x_m - x_n\| &= \|x_m - x_{m+1} + x_{m+1} - ... - x_{n-1} + x_{n-1} - x_n\| \\ &\leq \sum_{i=m}^{n-1} \|x_i - x_{i+1}\| \\ &\leq \frac{1}{2r}\sum_{i=m}^{n-1}(\phi(p^*, x_i) - \phi(p^*, x_{i+1})) \\ &= \frac{1}{2r}(\phi(p^*, x_m) - \phi(p^*, x_n)).\end{aligned}$$

But we know that $\{\phi(p^*, x_n)\}$ converges. Therefore, we obtain that $\{x_n\}$ is a Cauchy sequence. Since C is a closed subset of H, there exists $x^* \in C$, such that $x_n \to x^* \in C$.
Now, from (10) we have that

$$\|x_{n+1} - p\|^2 \leq q_n^2\|x_n - p\|^2 - \alpha_n\beta_n\left(1 - \beta_n - q_n\beta_n - \beta_n^2 L^2\right)\|x_n - T_i^k x_n\|^2.$$

This implies that

$$\frac{a^2(1 - 2b - L^2b^2)}{2}\|x_n - T_i^k x_n\|^2 \leq q_n^2\|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{12}$$

Using (12), we have

$$\begin{aligned}\frac{a^2(1 - 2b - L^2b^2)}{2}\sum\|x_n - T_i^k x_n\|^2 &\leq \sum(q_n^2\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ &= \sum(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) + \sum(q_n^2 - 1)\|x_n - p\|^2 < \infty\,,\end{aligned}$$

from which it follows that

$$\lim_{n\to\infty}\|x_n - T_i^k x_n\| = 0.$$

Since, for each $n \geq 1$, we have $n = (k-1)N + i$, thus

$$\lim_{n\to\infty}\|x_n - T_n^k x_n\| = 0.$$

Now

$$\|T_n^k x_n - x_{n-1}\| \leq \|T_n^k x_n - x_n\| + \|x_n - x_{n-1}\|\,.$$

So, we have

$$\lim_{n\to\infty}\|T_n^k x_n - x_{n-1}\| = 0\,. \tag{13}$$

Let $\sigma_n = \|T_n^k x_n - x_{n-1}\|$. Then from (13), we have $\sigma_n \to 0$. Since for each $n > N$, we have $T_n = T_{n-N}$ and

$$\begin{aligned}\|x_{n-1} - T_n x_n\| &\leq \|x_{n-1} - T_n^k x_n\| + \|T_n^k x_n - T_n x_n\| \\ &\leq \sigma_n + L\|T_n^{k-1} x_n - x_n\| \\ &\leq \sigma_n + L\{\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\| \\ &\quad + \|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_n\|\} \\ &= \sigma_n + L\{\|T_{n-N}^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\| \\ &\quad + \|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_n\|\} \\ &= \sigma_n + L^2\|x_n - x_{n-N}\| + L\sigma_{n-N} + L\|x_{(n-N)-1} - x_n\|\,.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0\,. \tag{14}$$

It follows from (14), that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| \leq \lim_{n\to\infty}\{\|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|\} = 0\,.$$

Consequently, for any $i = 1, 2, \ldots N$, we have

$$\begin{aligned}\|x_n - T_{n+i} x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| + \|T_{n+i} x_{n+i} - T_{n+i} x_n\| \\ &\leq (1 + L)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| \to 0\,,\end{aligned}$$

as $n \to \infty$. This implies that the sequence

$$\bigcup_{i=1}^{N}\{\|x_n - T_{n+i} x_n\|\}_{n=1}^{\infty} \to 0\,, \quad \text{as } n \to \infty\,.$$

Since for each $l = 1, 2, \cdots, N$, $\{\|x_n - T_l x_n\|\}$ is a subsequence of $\bigcup_{i=1}^{N}\{\|x_n - T_{n+i} x_n\|\}$, therefore, we have

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0\,, \quad \forall\, l \in \{1, 2, \ldots, N\}\,.$$

Set $l = n$, then we have,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{15}$$

Let $\{n_l\}_{l\in\mathbb{N}} \subset \mathbb{N}$ be such that $T_{n_l} = T_1$ for all $n \in \mathbb{N}$. Then from (15) we obtain that

$$\lim_{l\to\infty} \|T_1 x_{n_l} - x_{n_l}\| = 0.$$

Since $x_n \to x^*$ and the continuity of T_1 implies that $x^* = T_1 x^*$ and hence $x^* \in F(T_1)$. Similarly, we obtain that $x^* \in F(T_i)$, for $i = 2, 3, ..., N$ and hence $x^* \in \cap_{i=1}^{N} F(T_i)$. This completes the proof. □

Remark 3.2. *It is well known that every nonexpansive mapping is asymptotically pseudocontractive and uniformly L-Lipschitzian (see definition). Therefore for the sake of simplicity we are giving the example of nonexpansive mapping with the interior of the common fixed points nonempty.*

Example 3.3. *Suppose that* $X =: \mathbb{R}$ *and* $C =: [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$. *Let* $T_1, T_2 : C \to C$ *be defined by*

$$T_1(x) = \begin{cases} -x, & x \in [-\frac{1}{2}, 0] \\ x, & x \in (0, \frac{1}{2}], \end{cases}$$

and

$$T_2(x) = \begin{cases} x^2, & x \in [-\frac{1}{2}, 0) \\ x, & x \in [0, \frac{1}{2}]. \end{cases}$$

Then we observe that $F(T_1) = [0, \frac{1}{2}]$ *and* $F(T_2) = [0, \frac{1}{2}]$ *and hence the interior of* $F(T_1) \cap F(T_2) = (0, \frac{1}{2})$, *which is nonempty.*
Now, we show that T_1 *is nonexpansive. Suppose that* $A_1 = [-\frac{1}{2}, 0]$ *and* $A_2 = (0, \frac{1}{2}]$. *Then, if* $x, y \in A_1$, *we have that*

$$|T_1(x) - T_1(y)| = |-x + y| = |x - y|.$$

If $x, y \in A_2$, *then*

$$|T_1(x) - T_1(y)| = |x - y|.$$

And if $x \in A_1, y \in A_2$, *then*

$$|T_1(x) - T_1(y)| = |x + y| \leq |x - y|.$$

Hence we obtain that T_1 *is nonexpansive.*
Next, we show that T_2 *is nonexpansive. Suppose that* $B_1 = [-\frac{1}{2}, 0)$ *and* $B_2 = [0, \frac{1}{2})$. *Then, if* $x, y \in B_1$, *we have that*

$$|T_2(x) - T_2(y)| = |x^2 - y^2| = |x - y|\,|x + y| \leq |x - y|.$$

If $x, y \in B_2$, *then*

$$|T_2(x) - T_2(y)| = |x - y|.$$

And if $x \in B_1, y \in B_2$, *then*

$$|T_2(x) - T_2(y)| = |x^2 - y| \leq |x - y|.$$

Therefore we say that T_2 *is nonexpansive.*

Remark 3.4. *If in Theorem 3.1, we consider a single continuous uniformly L-Lipschitzian asymptotically pseudocontractive mapping, we have the following result:*

Corollary 3.5. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* $T : C \to C$ *be a continuous uniformly L-Lipschitzian asymptotically pseudocontractive mapping with the sequence* $\{k_n\} \subset [1, \infty)$ *satisfying* $\lim_{n\to\infty} k_n = 1$ *and Lipschitz constant L. Assume that the interior of* $F(T)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_0 \in C$ *given by* (2) *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, *where* $q_n = 2k_n - 1$ *for each* $n \geq 1$;

(ii) $a \leq \alpha_n \leq \beta_n \leq b$ *for some* $a > 0$ *and some* $b \in \left(0, \frac{\sqrt{1+L^2}-1}{L^2}\right)$.

Then $\{x_n\}$ *converges strongly to a fixed point of* T.

Remark 3.6. *Theorem 3.1 and Corollary 3.5 are also valid for continuous uniformly* $L-$*Lipschitzian asymptotically hemicontractive mapping. A mapping* $T : C \to H$ *is said to be asymptotically hemicontractive if*

$$\langle T^n x - p, x - p\rangle \leq k_n\|x - p\|^2, \ \forall\, x \in C \text{ and } p \in F(T).$$

We now prove a convergence theorem for a finite family of modified monotone mappings. But, firstly we should know about monotone mappings.

Definition 3.7. *Let H be a real Hilbert space. Then a mapping $A : H \to H$ is said to be monotone if*

$$\langle Ax - Ay, x - y\rangle \geq 0\,,\ \forall\, x, y \in H.$$

For modified monotone mapping, we replace the operator A by its n-th iterate A^n.

Remark 3.8. *Every modified monotone mapping is asymptotically pseudocontractive for $A^n =: I - T^n$. Since for all $x, y \in H$, we have*

$$\langle A^n x - A^n y, x - y\rangle \geq 0$$
$$\Rightarrow \langle (I - T^n)x - (I - T^n)y, x - y\rangle \geq 0.$$

Therefore

$$\langle T^n x - T^n y, x - y\rangle \leq k_n \|x - y\|^2.$$

Theorem 3.9. *Let H be a real Hilbert space and $A_i : H \to H$, $i = 1, 2, ..., N$, be a finite family of continuous uniformly L-Lipschitzian modified monotone mappings with sequence $\{k_n^{(i)}\} \subseteq [1, \infty)$ satisfying $\lim_{n\to\infty} k_n^{(i)} = 1$ and Lipschitz constants L_i for $i = 1, 2, ..., N$, respectively. Assume that the interior of $F = \bigcap_{i=1}^N N(A_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in H$ by*

$$\begin{cases} y_n = x_n - \beta_n A_i^k x_n; \\ x_{n+1} = x_n - \alpha_n A_i^k y_n \end{cases} \tag{16}$$

where $n = (k-1)N + i$, for each $n \geq 1$, $\quad i = 1, 2, \ldots, N$, $\ k \geq 1$ is a positive integer with $k \to \infty$ as $n \to \infty$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, where $q_n = 2k_n - 1$ for each $n \geq 1$, and $k_n = \max\{k_n^{(i)} : i = 1, 2, \ldots, N\}$;

(ii) $a \leq \alpha_n \leq \beta_n \leq b$ for some $a > 0$ and some $b \in \left(0, \frac{\sqrt{1+L^2}-1}{L^2}\right)$ for $L =: \max\{L_i : i = 1, 2, ..., N\}$.

Then $\{x_n\}$ converges strongly to a common zero point of $\{A_1, A_2, ..., A_N\}$.

Proof. Suppose that $T_i^k(x) =: (I - A_i^k)x$ for $i = 1, 2, ..., N$. Then we get that for every $i = 1, 2, ..., N$, T_i is a continuous uniformly L-Lipschitzian modified monotone mapping with $\bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N N(A_i) \neq \emptyset$. Moreover, when A_i^k is replaced by $(I - T_i^k)$, condition (16) reduced to (6) and hence the conclusion follows from Theorem 3.1. □

If in Theorem 3.9, we consider a single continuous uniformly L-Lipschitzian modified monotone mapping, then we obtain the following corollary:

Corollary 3.10. *Let H be real Hilbert space and $A : H \to H$ be a continuous uniformly L-Lipschitzian modified monotone mapping with the sequence $\{k_n\} \subset [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ and Lipschitz constant L. Assume that the interior of $N(A)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in H$ by*

$$y_n = x_n - \beta_n A^n x_n;$$
$$x_{n+1} = x_n - \alpha_n A^n y_n$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, where $q_n = 2k_n - 1$ for each $n \geq 1$;

(ii) $a \leq \alpha_n \leq \beta_n \leq b$ for some $a > 0$ and some $b \in \left(0, \frac{\sqrt{1+L^2}-1}{L^2}\right)$.

Then $\{x_n\}$ converges strongly to a common zero point of A.

Remark 3.11. *Theorem 3.1 provides a convergence sequence to a common fixed point of finite family of continuous uniformly L-Lipschitzian asymptotically pseudocontractive mappings where as Theorem 3.9 provides a convergence sequence to a common zero point of finite family of modified monotone mappings in Hilbert spaces. No compactness assumption is imposed either on T or on C.*

Remark 3.12. *Theorems 3.1 and 3.9 improve on Theorems 2.1 and 2.3 respectively of Zegeye et al. [17] and also Corollaries 3.5 and 3.10 improve on Theorem 2.5 and Corollary 2.6 respectively of the same.*

References

[1] Y.Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in: Volume 178 pp. 15–50, Lecture Notes in Pure and Applied Mathematics Series, Marcel Dekker, New York, 1996.

[2] S.S.Chang, Y.J.Cho, J.K.Kim, Some results for uniformly L-Lipschitzian mappings in Banach spaces, Applied Mathematics Letters 22 (2009) 121–125.

[3] C.E. Chidume, S.A. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, Proceedings of the American Mathematical Society 129 (2001) 2359–2363.

[4] G. Das, J. P. Debata, Fixed points of quasi-nonexpansive mappings, Indian Journal of Pure and Applied Mathematics 17 (1986) 1263–1269.

[5] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proceedings of the American Mathematical Society 35 (1972) 171–174.

[6] S. Ishikawa, Fixed point by a new iteration method, Proceedings of the American Mathematical Society 44 (1974) 147–150.

[7] S. Kamimura, W. Takahashi, Strong convergence of proximal-type algorithm in a Banach space, SIAM Journal on Optimization 13 (2002) 938–945.

[8] W. R. Mann, Mean value methods in iteration, Proceedings of the American Mathematical Society 4 (1953) 506–510.

[9] E. U. Ofoedu, Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space, Journal of Mathematical Analysis and Applications 321 (2006) 722–728.

[10] M. O. Osilike and S. C. Aniagbosor, Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, Mathematical and Computer Modelling 32 (2000) 1181–1191.

[11] A. Rafiq, On iterations for families of asymptotically pseudocontractive mappings, Applied Mathematics Letters 24 (2011) 33–38.

[12] S. Reich, A weak convergence theorem for the alternating method with Bergman distance, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in: Volume 178 pp. 313–318, Lecture Notes in Pure and Applied Mathematics Series, Marcel Dekker, New York, 1996.

[13] B. E. Rhoades, A comparison of various definition of contractive mappings, Transactions of the American Mathematical Society 226 (1977) 257–290.

[14] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, Journal of Mathematical Analysis and Applications 158 (1991) 407–413.

[15] Y. C. Tang, J. G. Peng, L. W. Liu, Strong convergence theorem for pseudocontractive mappings in Hilbert spaces, Nonlinear Analysis 74 (2011) 380–385.

[16] Y. H. Yao, Y. C. Liou, G. Marino, A hybrid algorithm for pseudocontractive mappings, Nonlinear Analysis 71 (2009) 4997–5002.

[17] H. Zegeye, N. Shahzad and M. A. Alghamdi, Convergence of Ishikawa's iteration method for pseudocontractive mappings, Nonlinear Analysis 74 (2011) 7304–7311.

[18] H. Zhou, Convergence theorem of fixed points for Lipschitz pseudocontractions in Hilbert spaces, Journal of Mathematical Analysis and Applications 343 (2008) 546–556.