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Some common fixed point theorems of generalized contractive mappings in cone metric spaces

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Abstract. In this paper we prove some common fixed point theorems for a sequence of self maps satisfying generalized contractive condition for a cone metric space which is not necessarily normal. The results presented in this paper generalize the corresponding results of [10, 13, 19] and many others from the current literature.

1. Introduction and Preliminaries

The Well-known Banach contraction principle and its several generalization in the setting of metric spaces play a central role for solving many problems of nonlinear analysis. For example, see [2, 5, 6, 15, 16].

Huang and Zhang [8] the concept of cone metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [1, 9, 19, 21] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space.

Recently, Rezapour and Hamlbarani [19] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. In [11] the authors introduced the concept of a compatible pair of self maps in a cone metric space and established a basic result for a non-normal cone metric space with an example, while in [12] weakly compatible maps have been studied. In this paper we prove a common fixed point theorem for a sequence of self maps satisfying a generalized contractive condition for a non-normal cone metric space.

Definition 1.1. *(See [8]) Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:*

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(C_1) P is closed, nonempty and $P \neq \{0\}$;

(C_2) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;

(C_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P. If $P^0 \neq \emptyset$ then P is called a solid cone (see [20]).

There exist two kinds of cones- normal (with the normal constant K) and non-normal ones [6]).

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \quad \text{imply} \quad \|x\| \leq K \left\|y\right\|, \tag{1.1}$$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \quad \text{imply} \quad \lim_{n\to\infty} y_n = x. \tag{1.2}$$

The least positive number K satisfying (1.1) is called the normal constant of P.

Example 1.2. *(See [20]) Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \leq x_n \leq y_n$, and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (1.2) that P is a non-normal cone.*

Proposition 1.3. *(See [13]) Let P be a cone in a real Banach space E. If for $a \in P$ and $a \leq ka$ for some $k \in [0, 1)$, then $a = 0$.*

Proposition 1.4. *(See [10]) Let P be a cone in a real Banach space E with non-empty interior. If for $a \in E$ and $a \ll c$ for all $c \in P^0$, then $a = 0$.*

Remark 1.5. *(See [19]) $\lambda P^0 \subseteq P^0$ for $\lambda > 0$ and $P^0 + P^0 \subseteq P^0$.*

Definition 1.6. *(See [8, 22]) Let X be a nonempty set. Suppose that the mapping $d \colon X \times X \to E$ satisfies:*

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ $x, y, z \in X$.

Then d is called a cone metric [8] or K-metric [22] on X and (X, d) is called a cone metric [8] or K-metric space [22] (we shall use the first term).

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 1.7. *(See [8]) Let* $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ *and* $d \colon X \times X \to E$ *defined by* $d(x, y) = (|x - y|, \alpha|x - y|)$, *where* $\alpha \geq 0$ *is a constant. Then* (X, d) *is a cone metric space with normal cone* P *where* $K = 1$.

Example 1.8. *(See [18]) Let* $E = \ell^2$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{for all } n\}$, (X, ρ) *a metric space, and* $d \colon X \times X \to E$ *defined by* $d(x, y) = \{\rho(x, y)/2^n\}_{n\geq 1}$. *Then* (X, d) *is a cone metric space.*

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 1.9. *(See [8]) Let* (X, d) *be a cone metric space. We say that* $\{x_n\}$ *is:*

(i) *a Cauchy sequence if for every* ε *in* E *with* $0 \ll \varepsilon$, *then there is an* N *such that for all* $n, m > N$, $d(x_n, x_m) \ll \varepsilon$;

(ii) *a convergent sequence if for every* ε *in* E *with* $0 \ll \varepsilon$, *then there is an* N *such that for all* $n > N$, $d(x_n, x) \ll \varepsilon$ *for some fixed* x *in* X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

In the following (X, d) will stands for a cone metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and $\leq$ is partial ordering in E with respect to P.

Remark 1.10. *It follows from above definition that if* $\{x_{2n}\}$ *is a subsequence of a Cauchy sequence* $\{x_n\}$ *in a cone metric space* (X, d) *and* $x_{2n} \to u$ *as* $n \to \infty$ *then* $x_n \to u$ *as* $n \to \infty$.

Proposition 1.11. *(See [13]) Let* (X, d) *be a cone metric space and* P *be a cone in a real Banach space* E. *If* $u \leq v$, $v \ll w$, *then* $u \ll w$.

Lemma 1.12. *(See [13]) Let* (X, d) *be a cone metric space and* P *be a cone in a real Banach space* E *and* $l, l_1, l_2 > 0$ *are some fixed real numbers. If* $x_n \to x$, $y_n \to y$ *in* X *and for some* $a \in P$

$$la \leq l_1\, d(x_n, x) + l_2\, d(y_n, y),$$

for all $n > N$, *for some integer* N, *then* $a = 0$.

2. Generalized Contraction Mapping

Let X be a cone metric space and $T \colon X \to X$ be a mapping. Then T is called generalized contractive mapping if it satisfies the following condition:

$$d(Tx, Ty) \leq a\, d(x, y) + b\,[d(x, Tx) + d(y, Ty)] + c\,[d(x, Ty) + d(y, Tx)] \tag{2.1}$$

for all $x, y \in X$ and $a, b, c \in [0, 1)$ are constants such that $a + 2b + 2c < 1$.

Remark 2.1. *(1) If* $b = c = 0$ *and* $a \in [0, 1)$, *then (2.1) reduces to contraction mapping defined by Banach [3].*

(2) If $a = c = 0$ *and* $b \in [0, 1/2]$, *then (2.1) reduces to contraction mapping defined by Kannan [14].*

(3) If $c = 0$ *and* $a, b \in [0, 1/2]$, *then (2.1) reduces to contraction mapping defined by Fisher [7].*

(4) If $a, b = 0$ *and* $c \in [0, 1/2]$, *then (2.1) reduces to contraction mapping defined by Chaterjee [4].*

(5) If $b = 0$ *and* $a, c \in [0, 1)$, *then (2.1) reduces to contraction mapping defined by Reich [17].*

3. Main Results

In this section we shall prove some fixed point theorems of generalized contractive mapping.

Theorem 3.1. *Let (X, d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Let $\{T_n\}$ be a sequence of self maps on X satisfying generalized contractive condition (2.1) with $a + 2b + 2c < 1$ for some $a, b, c \in [0, 1)$. For $x_0 \in X$, let $x_n = T_n x_{n-1}$ for all n. Then the sequence $\{x_n\}$ converges in X and its limit v is a common fixed point of all the maps of the sequence $\{T_n\}$. This common fixed point is unique if $a + 2c < 1$.*

Proof. Taking $x = x_{n-1}$, $y = x_n$, $T = T_n$ and $T = T_{n+1}$ in (2.1), we have

$$\begin{aligned} d(T_n x_{n-1}, T_{n+1} x_n) &\leq a\, d(x_{n-1}, x_n) + b\,[d(x_{n-1}, T_n x_{n-1}) + d(x_n, T_{n+1} x_n)] \\ &\quad + c\,[d(x_{n-1}, T_{n+1} x_n) + d(x_n, T_n x_{n-1})]. \end{aligned}$$

As $x_n = T_n x_{n-1}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq a\, d(x_{n-1}, x_n) + b\,[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + c\,[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &\leq a\, d(x_{n-1}, x_n) + b\,[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + c\,[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

Writing $d(x_n, x_{n+1}) = \rho_n$, we have

$$\rho_n \leq (a + b + c)\, \rho_{n-1} + (b + c)\, \rho_n,$$

$$(1 - b - c)\, \rho_n \leq (a + b + c)\, \rho_{n-1},$$

which implies that

$$\rho_n \leq t\, \rho_{n-1}, \tag{3.1}$$

where

$$t = \frac{a + b + c}{1 - b - c}.$$

As $a + 2b + 2c < 1$, we obtain that $t < 1$.

Now

$$\rho_n \leq t\, \rho_{n-1} \leq t^2\, \rho_{n-2} \leq \cdots \leq t^n\, \rho_0,$$

where $\rho_0 = d(x_0, x_1)$. Also for $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq (t^{n-1} + t^{n-2} + \cdots + t^m) d(x_1, x_0) \leq \frac{t^m}{1 - t}\, d(x_1, x_0) = \frac{t^m}{1 - t}\, \rho_0. \end{aligned}$$

As $t < 1$ and P is closed, thus we obtain that

$$d(x_n, x_m) \leq \frac{t^m}{1 - t}\, \rho_0. \tag{3.2}$$

Now for $\varepsilon \in P^0$, there exists $r > 0$ such that $\varepsilon - y \in P^0$, if $\|y\| < r$. Choose a positive integer N_ε such that for all $n \geq N_\varepsilon$, $\left\|\frac{t^m}{1-t}\, \rho_0\right\| < r$, which implies $\varepsilon - \frac{t^m}{1-t}\, \rho_0 \in P^0$ and $\frac{t^m}{1-t}\, \rho_0 - d(x_n, x_m) \in P$ by using (3.2).

So we have $\varepsilon - d(x_n, x_m) \in P^0$ for all $n > N_\varepsilon$ and for all m by proposition 1.11. This implies $d(x_n, x_m) \ll \varepsilon$ for all $n > N_\varepsilon$ and for all m. Hence $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. For an arbitrary fixed m we show that $T_m z = z$. Now

$$\begin{aligned} d(T_m z, z) &\leq d(T_m z, T_n x_{n-1}) + d(T_n x_{n-1}, z) \\ &= d(x_n, z) + d(T_m z, T_n x_{n-1}). \end{aligned}$$

Using (2.1), we have

$$\begin{aligned}
d(T_m z, z) &\leq d(T_m z, T_n x_{n-1}) + d(T_n x_{n-1}, z) \\
&= d(x_n, z) + d(T_m z, T_n x_{n-1}) \\
&\leq d(x_n, z) + a\, d(z, x_{n-1}) + b\, [d(z, T_m z) + d(x_{n-1}, T_n x_{n-1}] \\
&\quad + c\, [d(z, T_n x_{n-1}) + d(x_{n-1}, T_m z)] \\
&= d(x_n, z) + a\, d(z, x_{n-1}) + b\, [d(z, T_m z) + d(x_{n-1}, x_n] \\
&\quad + c\, [d(z, x_n) + d(x_{n-1}, T_m z)] \\
&\leq d(x_n, z) + a\, d(z, x_{n-1}) + b\, [d(z, T_m z) + d(x_{n-1}, z) \\
&\quad + d(z, x_n)] + c\, [d(z, x_n) + d(x_{n-1}, z) + d(z, T_m z)] \\
&= (1 + b + c)\, d(x_n, z) + (a + b + c)\, d(z, x_{n-1}) \\
&\quad + (b + c)\, d(T_m z, z).
\end{aligned}$$

So, we have

$$(1 - b - c) d(T_m z, z) \leq (1 + b + c)\, d(x_n, z) + (a + b + c)\, d(z, x_{n-1}).$$

As $x_n \to z$, $x_{n-1} \to z$ $(n \to \infty)$, and $1 - b - c > 0$, using Lemma 1.12, we have $d(T_m z, z) = 0$, and we get $T_m z = z$. Thus z is a common fixed point of all the maps of the sequence $\{T_n\}$.

Uniqueness

Let $T_n v = v$ for all n be another common fixed point of all the maps of the sequence $\{T_n\}$. Now

$$\begin{aligned}
d(v, z) &= d(T_n v, T_n z) \\
&\leq a\, d(v, z) + b\, [d(v, T_n v) + d(z, T_n z)] + c\, [d(v, T_n z) + d(z, T_n v)]
\end{aligned}$$

which gives

$$d(v, z) \leq (a + 2c)\, d(v, z).$$

As $a + 2c < 1$, using proposition 1.3, we have $d(v, z) = 0$, i.e. $v = z$. Thus v is the unique common fixed point of all the maps of the sequence $\{T_n\}$. □

Theorem 3.2. *Let (X, d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Let $\{S_n\}$ be a sequence of self maps in X satisfying: for some $a_n, b_n, c_n \in [0, 1)$ with $a_n + 2b_n + 2c_n < 1$ and $a_n + 2c_n < 1$, there exists positive integer m_i for each i such that for all $x, y \in X$*

$$d(S_i^{m_i} x, S_j^{m_j} y) \leq a_n\, d(x, y) + b_n\, [d(x, S_i^{m_i} x) + d(y, S_j^{m_j} y)] + c_n\, [d(x, S_j^{m_j} y) + d(y, S_i^{m_i} x)]. \tag{3.3}$$

Then all the maps of the sequence $\{S_n\}$ have a unique common fixed point in X.

Proof. From Theorem 3.1 all the maps of the sequence $\{S_i^{m_i}\}$ have a unique common fixed point, say z. Hence $S_i^{m_i} z = z$ for all i. Now $S_1^{m_1} z = z$ implies $S_1^{m_1} S_1 z = S_1 z$. Taking $x = S_1 z$, $y = z$, $i = 1$ and $j = 2$ in (3.3), we have $S_1 z = z$. Continuing in similar way it follows that $S_i z = z$ for all i. Thus z is a common fixed point of all the maps of the sequence $\{S_i\}$. Its uniqueness follows from the fact that $S_i z = z$ implies $S_i^{m_i} z = z$ for all i. □

In Theorem 3.1 taking $T_1 = T_2 = T_3 = \cdots = T_n = \cdots = T$, we get the following general form of Banach contraction principle in a cone metric space which is not necessarily normal.

Theorem 3.3. *Let (X, d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Let T be a self map in X satisfying generalized contractive condition (2.1) with $a+2b+2c < 1$ for some $a, b, c \in [0, 1)$. Then for each $x \in X$ the sequence $\{T^n x\}$ converges in X and its limit u is a fixed point T. This fixed point is unique if $a + 2c < 1$.*

Theorem 3.4. *Let (X, d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Suppose the mapping $T\colon X \to X$ satisfies for some positive integer n:*

$$d(T^n x, T^n y) \leq a_n\, d(x, y) + b_n\, [d(x, T^n x) + d(y, T^n y)] + c_n\, [d(x, T^n y) + d(y, T^n x)]$$

for all $x, y \in X$ and $a_n, b_n, c_n \in [0, 1)$ are constants such that $a_n + 2b_n + 2c_n < 1$. Then T has a unique fixed point in X.

Proof. From Theorem 3.3, T^n has a unique fixed point u. But $T^n(Tu) = T(T^n u) = Tu$, so Tu is also a fixed point of T^n. Hence $Tu = u$, u is a fixed of T. Since the fixed point of T is also a fixed point of T^n, the fixed point of T is unique. □

Corollary 3.5. *Let (X, d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Suppose the mapping $T\colon X \to X$ satisfies for some positive integer m, n:*

$$d(T^m x, T^n y) \leq a_n\, d(x, y) + b_n\, [d(x, T^m x) + d(y, T^n y)] + c_n\, [d(x, T^n y) + d(y, T^m x)]$$

for all $x, y \in X$ and $a_n, b_n, c_n \in [0, 1)$ are constants such that $a_n + 2b_n + 2c_n < 1$ and $b_n = c_n$. Then T has a unique fixed point in X.

Proof. By Theorem 3.4, we get $x \in X$ such that $T^m x = T^n x = x$. The result then follows from the fact that

$$\begin{aligned} d(Tx, x) &= d(TT^m x, T^n x) = d(T^m Tx, T^n x) \\ &\leq a_n\, d(Tx, x) + b_n\, [d(Tx, T^m Tx) + d(x, T^n x)] + c_n\, [d(Tx, T^n x) + d(x, T^m Tx)] \\ &\leq a_n\, d(Tx, x) + b_n\, [d(Tx, Tx) + d(x, x)] + c_n\, [d(Tx, x) + d(x, Tx)] \\ &= (a_n + 2c_n) d(Tx, x), \end{aligned}$$

which implies $Tx = x$. □

Example 3.6. *(Applications)* $X = C([1, 3], \mathbb{R})$, $E = \mathbb{R}^2$, $\alpha > 0$ *and*

$$d(x, y) = \Big(\sup_{t \in [1,3]} |x(t) - y(t)|,\ \alpha \sup_{t \in [1,3]} |x(t) - y(t)| \Big)$$

for every $x, y \in X$, and $P = \Big\{(u, v) \in \mathbb{R}^2 : u, v \geq 0\Big\}$. It is easily seen that (X, d) is a complete cone metric space. Define $T\colon X \to X$ by

$$T(x(t)) = 4 + \int_1^t \Big(x(u) + u^2 \Big) e^{u-1} du.$$

For $x, y \in X$

$$\begin{aligned} d(Tx, Ty) &= \Big(\sup_{t \in [1,3]} |T(x(t)) - T(y(t))|,\ \alpha \sup_{t \in [1,3]} |T(x(t)) - T(y(t))| \Big) \\ &\leq \Big(\int_1^3 |(x(u) - y(u))| e^2 du,\ \alpha \int_1^3 |(x(u) - y(u))| e^2 du \Big) \\ &= 2e^2 d(x, y). \end{aligned}$$

Similarly,

$$d(T^n x, T^n y) \leq e^{2n} \frac{2^n}{n!} d(x, y).$$

Note that

$$e^{2n} \frac{2^n}{n!} = \begin{cases} 109 & \text{if } n = 2, \\ 1987 & \text{if } n = 4, \\ 1.37 & \text{if } n = 37, \\ 0.53 & \text{if } n = 38. \end{cases}$$

Thus for $a_n = 0.53$, $b_n = c_n = 0$, $m = n = 38$, *all conditions of Corollary 3.5 are satisfied and so T has a unique fixed point, which is the unique solution of the integral equation:*

$$x(t) = 4 + \int_1^t \Big(x(u) + u^2\Big) e^{u-1} du,$$

or the differential equation:

$$x'(t) = \Big(x(t) + t^2\Big) e^{t-1}, \ t \in [1, 3], \ x(1) = 4.$$

Hence, the use of Corollary 3.5 is a delightful way of showing the existence and uniqueness of solutions for the following class of integral equations:

$$q + \int_p^t K(x(u), u) du = x(t) \in C\Big([p, q], \mathbb{R}^n\Big).$$

In Huang and Zhang [8] and Rezapour and Hamlbarani [19] proved the following various form of Banach contraction principle in a normal cone metric space and in a cone metric space.

Theorem 1[8] and Theorem 2.3 [19]. Let (X, d) be a complete cone metric space. Suppose the mapping $T\colon X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq k\, d(x, y),$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 3[8] and Theorem 2.6 [19]. Let (X, d) be a complete cone metric space. Suppose the mapping $T\colon X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq k\, [d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$, where $k \in [0, 1/2)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 4[8] and Theorem 2.7 [19]. Let (X, d) be a complete cone metric space. Suppose the mapping $T\colon X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq k\, [d(y, Tx) + d(x, Ty)],$$

for all $x, y \in X$, where $k \in [0, 1/2)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Remark 3.7. *Above Theorems of [8] and [19] follows from Theorem 3.3 of this paper by taking:*

(i) $b = c = 0$ *and* $a = k$,

(ii) $a = c = 0$ *and* $b = k$,

(iii) $a = b = 0$ *and* $c = k$,

respectively in it.

Precisely, Theorem 3.3 synthesizes and generalizes all the results of [8] and [19] for a non-normal cone metric space. Theorem 3.2 is a generalized form of Banach contraction principle in a complete cone metric space which is not necessarily normal.

Remark 3.8. *Our results also generalize the corresponding results of Jain et al. [13].*

We conclude with an example.

Example 3.9. *(of Theorem 2.3) Let* $E = R^2$, *the Euclidean plane, and* $P = \{(x, y) \in R^2 : x, y \geq 0\}$ *a normal cone in* P. *Let* $X = \{(x, 0) \in R^2 : 0 \leq x \leq 1\} \cup \{(0, x) \in R^2 : 0 \leq x \leq 1\}$. *The mapping* $d \colon X \times X \to E$ *is defined by*

$$d((x,0),(y,0)) = \Big(\frac{5}{3}\,|x-y|, |x-y|\Big),$$

$$d((0,x),(0,y)) = \Big(|x-y|, \frac{2}{3}\,|x-y|\Big),$$

$$d((x,0),(0,y)) = d((0,y),(x,0)) = \Big(\frac{5}{3}\,x + y, x + \frac{2}{3}\,y\Big).$$

Then (X, d) *is a complete cone metric space.*

Let mapping $T \colon X \to X$ *with*

$$T((x,0)) = (0,x) \quad \text{and} \quad T((0,x)) = \Big(\frac{1}{2}\,x, 0\Big).$$

Then T *satisfies the generalized contractive condition*

$$\begin{aligned} d\Big(T((x,x')), T((y,y'))\Big) \quad &\leq \quad a\, d\Big((x,x'),(y,y')\Big) + b\Big[d\Big((x,x'),T((x,x'))\Big) + d\Big((y,y'),T((y,y'))\Big)\Big] \\ &\quad +c\Big[d\Big((x,x'),T((y,y'))\Big) + d\Big((y,y'),T((x,x'))\Big)\Big] \end{aligned}$$

for all (x, x'), $(y, y') \in X$ *with the constant* $\lambda = a + 2b + 2c < 1$, *where* a, b, c *are such that* $a = b = c = \frac{1}{6}$. *Then it is obvious that* T *has a unique fixed point* $(0, 0) \in X$, *where* $\lambda = \frac{5}{6} \in [0, 1)$.

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