Functional Analysis, Approximation and Computation
5:1 (2013), 43–56

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Weak and strong convergence theorems for two generalized asymptotically quasi-nonexpansive mappings in Banach spaces

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Abstract. In this paper, we establish some weak and strong convergence theorems of modified two-step iteration process for two generalized asymptotically quasi-nonexpansive mappings to converge to common fixed points in the setting of real Banach spaces. The results presented in this paper extend, improve and generalize some previous results from the existing literature.

1. Introduction

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T\colon K \to K$ is said to be:

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in K$.

(2) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\| \tag{1.2}$$

for all $x \in K$, $p \in F(T)$.

(3) asymptotically nonexpansive [6] if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.3}$$

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 47H09, 47J25.
Keywords. Generalized asymptotically quasi-nonexpansive mapping; modified two-step iteration process; common fixed point; strong convergence; weak convergence; Banach space.
Received: 1 May 2012; 10 April 2013
Communicated by Dragan S. Djordjević
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for all $x, y \in K$ and $n \geq 1$.

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T^n x - p\right\| \leq k_n \left\|x - p\right\| \tag{1.4}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

(5) generalized asymptotically quasi-nonexpansive [7] if $F(T) \neq \emptyset$ and there exist two positive sequences $\{k_n\}$ in $[1, \infty)$ with $k_n \to 1$ and $\{s_n\}$ in $[0, \infty)$ with $s_n \to 0$ as $n \to \infty$ such that

$$\left\|T^n x - y\right\| \leq k_n \left\|x - y\right\| + s_n \tag{1.5}$$

for all $x \in K$, $y \in F(T)$ and $n \geq 1$.

(6) uniformly L-Lipschitzian if there exists a positive constant L such that

$$\left\|T^n x - T^n y\right\| \leq L \left\|x - y\right\| \tag{1.6}$$

for all $x, y \in K$ and $n \geq 1$.

If in definition (5), $s_n = 0$ for all $n \geq 1$, then T becomes asymptotically quasi-nonexpansive, and hence the class of generalized asymptotically quasi-nonexpansive maps includes the class of asymptotically quasi-nonexpansive maps.

In 1991, Schu [13] introduced the following Mann-type iterative process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T^n x_n, \end{aligned} \tag{1.7}$$

where $T\colon K \to K$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the condition $\delta \leq \alpha_n \leq 1 - \delta$ for all $n \geq 1$ and for some $\delta > 0$. Hence conclude that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Since 1972, many authors have studied weak and strong convergence problem of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see, for example, [6, 8, 10–14, 19] and references therein).

In 2007, Agarwal et al. [1] introduced the following iteration process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1 - \alpha_n) T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T^n x_n, \ n \geq 1, \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. They showed that this process converge at a rate same as that of Picard iteration and faster than Mann for contractions.

The above process deals with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debata [3] gave and studied a two mappings process. Later on, many authors, for example Khan and Takahashi [10], Shahzad and Udomene [15] and Takahashi and Tamura [18] have studied the two mappings case of iterative schemes for different types of mappings.

Recently, Khan et al. [9] modified the iteration process (1.2) to the case of two mappings as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \ n \geq 1, \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They established weak and strong convergence theorems in the setting of real Banach spaces.

Remark 1.1. *(i) Note that (1.9) reduces to (1.8) when $S = T$. Similarly, the process (1.9) reduces to (1.7) when $T = I$.*

(ii) The process (1.8) does not reduce to (1.7) but (1.9) does. Thus (1.9) not only covers the results proved by (1.8) but also by (1.7) which are not covered by (1.8).

In this paper, we prove some weak and strong convergence theorems for two generalized asymptotically quasi-nonexpansive mappings using iteration process (1.9) in the framework of real Banach spaces. The results presented in this paper extend, improve and generalize several known results given in the existing literature.

2. Preliminaries

For the sake of convenience, we restate the following concepts.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \Big\|\frac{1}{2}(x+y)\Big\| : \|x\| = 1, \ \|y\| = 1, \ \varepsilon = \|x-y\|\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Two mappings $S, T\colon K \to K$, where K is a subset of a normed space E, are said to satisfy the condition (A') [5] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \geq f(d(x,F))$ or $\|x - Tx\| \geq f(d(x,F))$ for all $x \in K$, where $d(x,F) = \inf\{\|x-p\| : p \in F\}$.

A mapping $T\colon K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.

A mapping $T\colon K \to K$ is said to be semi-compact [2] if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.

A Banach space E has the Kadec-Klee property [16] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Now, we state the following useful lemmas to prove our main results:

Lemma 2.1. *(See [13]) Let E be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.2. *(See [17]) Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty}\beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty}\alpha_n$ exists. In particular, $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.3. *(See [20]) Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g\colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that $\left\|\lambda x + (1-\lambda)y\right\|^p \leq \lambda \|x\|^p + (1-\lambda)\left\|y\right\|^p - W_p(\lambda)g(\left\|x-y\right\|)$ for all $x, y \in B_R(0) = \{x \in E : \|x\| \leq R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.*

Lemma 2.4. *(See [16]) Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $p, q \in w_w(x_n)$ (where $w_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty}\left\|tx_n + (1-t)p - q\right\|$ exists for all $t \in [0,1]$. Then $p = q$.*

3. Main Results

In this section, we prove some strong convergence theorems of the iteration scheme (1.9) for two generalized asymptotically quasi-nonexpansive mappings in the framework of real Banach spaces. In the sequel, we need the following lemma in order to prove our main theorems.

Lemma 3.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two generalized asymptotically quasi-nonexpanive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ and sequences $\{s_n\}$, $\{t_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F = F(S)\cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.9). Then $\lim_{n\to\infty}\left\|x_n - q\right\|$ exists for all $q \in F$.*

Proof. Let $q \in F$. Then from (1.9), we have

$$\begin{aligned}
\left\|y_n - q\right\| &= \left\|(1-\beta_n)x_n + \beta_n T^n x_n - q\right\| \\
&\leq (1-\beta_n)\left\|x_n - q\right\| + \beta_n\left\|T^n x_n - q\right\| \\
&\leq (1-\beta_n)\left\|x_n - q\right\| + \beta_n[l_n\left\|x_n - q\right\| + t_n] \\
&\leq (1-\beta_n)l_n\left\|x_n - q\right\| + \beta_n l_n\left\|x_n - q\right\| + \beta_n t_n \\
&\leq l_n\left\|x_n - q\right\| + t_n. \qquad (3.1)
\end{aligned}$$

Again using (1.9) and (3.1), we obtain

$$\begin{aligned}
\left\|x_{n+1} - q\right\| &= \left\|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - q\right\| \\
&\leq (1-\alpha_n)\left\|T^n x_n - q\right\| + \alpha_n\left\|S^n y_n - q\right\| \\
&\leq (1-\alpha_n)l_n\left\|x_n - q\right\| + \alpha_n[k_n\left\|y_n - q\right\| + s_n] \\
&\leq (1-\alpha_n)l_n\left\|x_n - q\right\| + \alpha_n k_n\left\|y_n - q\right\| + \alpha_n s_n \\
&\leq (1-\alpha_n)l_n\left\|x_n - q\right\| + \alpha_n k_n[l_n\left\|x_n - q\right\| + t_n] + \alpha_n s_n \\
&\leq (1-\alpha_n)k_n l_n\left\|x_n - q\right\| + \alpha_n k_n l_n\left\|x_n - q\right\| + \alpha_n k_n t_n + \alpha_n s_n \\
&\leq k_n l_n\left\|x_n - q\right\| + k_n t_n + s_n \\
&= [1 + (k_n l_n - 1)]\left\|x_n - q\right\| + A_n. \qquad (3.2)
\end{aligned}$$

where $A_n = k_n t_n + s_n$. Since by hypothesis $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$ and since $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$, by Lemma 2.2 we know that the limit $\lim_{n\to\infty}\left\|x_n - q\right\|$ exists. This completes the proof. □

Theorem 3.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ and sequences $\{s_n\}$, $\{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.9). Then $\{x_n\}$ converges strongly to a common fixed point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.*

Proof. The necessity of the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ is obvious. Let us prove the sufficiency part of the theorem. Since $S, T\colon K \to K$ are uniformly L-Lipschitzian mappings, so S and T are continuous mappings. Therefore the sets $F(S)$ and $F(T)$ are closed. Hence $F = F(S) \cap F(T)$ is a nonempty closed set.

For any given $q \in F$, form (3.2), we have

$$\|x_{n+1} - q\| \leq (1 + d_n)\|x_n - q\| + A_n. \tag{3.3}$$

where $d_n = (k_n l_n - 1)$ with $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} A_n < \infty$. Hence, we have

$$d(x_{n+1}, F) \leq (1 + d_n) d(x_n, F) + A_n. \tag{3.4}$$

From (3.4) and Lemma 2.2, we obtain that $\lim_{n\to\infty} d(x_n, F)$ exists. By condition $\liminf_{n\to\infty} d(x_n, F) = 0$, we get

$$\lim_{n\to\infty} d(x_n, F) = \liminf_{n\to\infty} d(x_n, F) = 0. \tag{3.5}$$

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. In fact, due to $1 + x \leq exp(x)$ for all $x > 0$, and from (3.3), we obtain

$$\|x_{n+1} - q\| \leq \exp(d_n)\|x_n - q\| + A_n. \tag{3.6}$$

Hence for any positive integers m, n and from (3.6) with $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} A_n < \infty$, we have

$$\begin{aligned}
\|x_{n+m} - q\| &\leq \exp(d_{n+m-1})\|x_{n+m-1} - q\| + A_{n+m-1} \\
&\leq \exp(d_{n+m-1})[\exp(d_{n+m-2})\|x_{n+m-2} - q\| \\
&\quad + A_{n+m-2}] + A_{n+m-1} \\
&\leq \exp(d_{n+m-1} + d_{n+m-2})\|x_{n+m-2} - q\| \\
&\quad + \exp(d_{n+m-1})[A_{n+m-1} + A_{n+m-2}] \\
&\leq \dots \\
&\leq \exp\Big(\sum_{k=n}^{n+m-1} d_k\Big)\|x_n - q\| + \exp\Big(\sum_{k=n+1}^{n+m-1} d_k\Big)\sum_{k=n}^{n+m-1} A_k \\
&\leq \exp\Big(\sum_{k=1}^{\infty} d_k\Big)\|x_n - q\| + \exp\Big(\sum_{k=1}^{\infty} d_k\Big)\sum_{k=n}^{n+m-1} A_k,
\end{aligned} \tag{3.7}$$

which implies that

$$\|x_{n+m} - q\| \leq W\|x_n - q\| + W\sum_{k=n}^{n+m-1} A_k \tag{3.8}$$

for all $q \in F$, where $W = \exp\Big(\sum_{k=1}^{\infty} d_k\Big) < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$, then for any given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$d(x_{n_0}, F) < \frac{\varepsilon}{2(W+1)}, \tag{3.9}$$

and

$$\sum_{k=n_0}^{n+m-1} A_k < \frac{\varepsilon}{2W}, \tag{3.10}$$

Therefore there exists a $q_1 \in F$ such that

$$\left\|x_{n_0} - q_1\right\| < \frac{\varepsilon}{2(W+1)}. \tag{3.11}$$

Consequently, for all $n \geq n_0$ and $m, n \geq 1$ from (3.8), we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \left\|x_{n+m} - q_1\right\| + \left\|x_n - q_1\right\| \\ &\leq W \left\|x_{n_0} - q_1\right\| + W \sum_{k=n_0}^{n+m-1} A_k + \left\|x_{n_0} - q_1\right\| \\ &= (W+1) \left\|x_{n_0} - q_1\right\| + W \sum_{k=n_0}^{n+m-1} A_k \\ &< (W+1). \frac{\varepsilon}{2(W+1)} + W. \frac{\varepsilon}{2W} = \varepsilon. \end{aligned} \tag{3.12}$$

This implies that $\{x_n\}$ is a Cauchy sequence in E and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F$. Indeed, we know that the set $F = F(S) \cap F(T)$ is closed. From the continuity of $d(x, F) = 0$ with $\lim_{n\to\infty} d(x_n, F) = 0$ and $\lim_{n\to\infty} x_n = q^*$, we get $d(q^*, F) = 0$, and so $q^* \in F$, that is, q^* is a common fixed point of S and T. This completes the proof. □

Theorem 3.3. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ and sequences $\{s_n\}$, $\{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (1.9). Then* $\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \left\|x_n - q\right\|$ exists for all $q \in F$. Assume that $\lim_{n\to\infty} \left\|x_n - q\right\| = r$. If $r = 0$, the conclusion is obvious. Now suppose $r > 0$. We claim $\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $x_n - q,\ y_n - q \in B_R(0)$ for all $n \geq 1$. Using (1.9) and Lemma 2.3, we have

$$\begin{aligned} \left\|y_n - q\right\|^2 &= \left\|(1-\beta_n)T^n x_n + \beta_n x_n - q\right\|^2 \\ &\leq (1-\beta_n) \left\|T^n x_n - q\right\|^2 + \beta_n \left\|x_n - q\right\|^2 \\ &\quad -W_2(\beta_n) g(\|T^n x_n - x_n\|) \\ &\leq (1-\beta_n) \left\|T^n x_n - q\right\|^2 + \beta_n \left\|x_n - q\right\|^2 \\ &\leq (1-\beta_n)[l_n \left\|x_n - q\right\| + t_n]^2 + \beta_n \left\|x_n - q\right\|^2 \\ &\leq (1-\beta_n)[l_n^2 \left\|x_n - q\right\|^2 + \theta_n] + \beta_n \left\|x_n - q\right\|^2 \\ &\leq (1-\beta_n)[l_n^2 \left\|x_n - q\right\|^2 + \theta_n] + \beta_n l_n^2 \left\|x_n - q\right\|^2 \\ &\leq l_n^2 \left\|x_n - q\right\|^2 + \theta_n \end{aligned} \tag{3.13}$$

where $\theta_n = 2l_n t_n \left\|x_n - q\right\| + t_n^2$ with $\sum_{n=1}^{\infty} \theta_n < \infty$. Again using (1.9), (3.13) and Lemma 2.3, we have

$$\begin{aligned}
\left\|x_{n+1} - q\right\|^2 &= \left\|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - q\right\|^2 \\
&\leq (1-\alpha_n)\left\|T^n x_n - q\right\|^2 + \alpha_n \left\|S^n y_n - q\right\|^2 \\
&\quad -W_2(\alpha_n) g\Big(\left\|T^n x_n - S^n y_n\right\|\Big) \\
&\leq (1-\alpha_n)[l_n \left\|x_n - q\right\| + t_n]^2 + \alpha_n [k_n \left\|y_n - q\right\| + s_n]^2 \\
&\quad -W_2(\alpha_n) g\Big(\left\|T^n x_n - S^n y_n\right\|\Big) \\
&\leq (1-\alpha_n)[l_n^2 \left\|x_n - q\right\|^2 + \theta_n] + \alpha_n [k_n^2 \left\|y_n - q\right\|^2 + \lambda_n] \\
&\quad -W_2(\alpha_n) g\Big(\left\|T^n x_n - S^n y_n\right\|\Big) \\
&\leq (1-\alpha_n)[l_n^2 \left\|x_n - q\right\|^2 + \theta_n] + \alpha_n k_n^2 [l_n^2 \left\|x_n - q\right\|^2 + \theta_n] \\
&\quad +\alpha_n \lambda_n - W_2(\alpha_n) g\Big(\left\|T^n x_n - S^n y_n\right\|\Big) \\
&\leq (1-\alpha_n) k_n^2 l_n^2 \left\|x_n - q\right\|^2 + (1-\alpha_n) k_n^2 l_n^2 \theta_n + \alpha_n k_n^2 [l_n^2 \left\|x_n - q\right\|^2 \\
&\quad +l_n^2 \theta_n] + \alpha_n \lambda_n - W_2(\alpha_n) g\Big(\left\|T^n x_n - S^n y_n\right\|\Big) \\
&\leq k_n^2 l_n^2 \left\|x_n - q\right\|^2 + k_n^2 l_n^2 \theta_n + \lambda_n - W_2(\alpha_n) g\Big(\left\|T^n x_n - S^n y_n\right\|\Big) \\
&\leq \left\|x_n - q\right\|^2 + M_n R^2 + (1+M_n)\theta_n + \lambda_n - W_2(\alpha_n) g\Big(\left\|T^n x_n - S^n y_n\right\|\Big)
\end{aligned} \tag{3.14}$$

where $\lambda_n = 2k_n s_n \left\|y_n - q\right\| + s_n^2$ with $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $M_n = (k_n^2 l_n^2 - 1)$ with $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} (k_n^2 l_n^2 - 1) = (k_n l_n + 1) \sum_{n=1}^{\infty} (k_n l_n - 1) < \infty$.

Observe that $W_2(\alpha_n) \geq \delta^2$. Now (3.14) implies that

$$\begin{aligned}
\delta^2 \sum_{n=1}^{\infty} g\Big(\left\|T^n x_n - S^n y_n\right\|\Big) &< \left\|x_1 - q\right\|^2 + R^2 \sum_{n=1}^{\infty} M_n + \Big(1 + \sum_{n=1}^{\infty} M_n\Big) \sum_{n=1}^{\infty} \theta_n \\
&\quad + \sum_{n=1}^{\infty} \lambda_n < \infty.
\end{aligned} \tag{3.15}$$

Therefore, we have $\lim_{n\to\infty} g\Big(\left\|T^n x_n - S^n y_n\right\|\Big) = 0$. Since g is strictly increasing and continuous at 0, it follows that

$$\lim_{n\to\infty} \left\|T^n x_n - S^n y_n\right\| = 0. \tag{3.16}$$

Now taking limsup on both the sides of (3.1), we obtain

$$\limsup_{n\to\infty} \left\|y_n - q\right\| \leq r. \tag{3.17}$$

Since T is generalized asymptotically quasi-nonexpansive, we can get that

$$\lim_{n\to\infty} \left\|T^n x_n - q\right\| \leq l_n \left\|x_n - q\right\| + t_n. \tag{3.18}$$

for all $n \geq 1$. Taking limsup on both the sides of (3.18), we obtain

$$\limsup_{n\to\infty} \left\|T^n x_n - q\right\| \leq r. \tag{3.19}$$

Now

$$\begin{aligned} \left\|x_{n+1} - q\right\| &= \left\|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - q\right\| \\ &= \left\|(T^n x_n - q) + \alpha_n (S^n y_n - T^n x_n)\right\| \\ &\leq \left\|T^n x_n - q\right\| + \alpha_n \left\|S^n y_n - T^n x_n\right\| \end{aligned}$$

yields that

$$r \leq \liminf_{n\to\infty} \left\|T^n x_n - q\right\|. \tag{3.20}$$

So that (3.19) gives $\lim_{n\to\infty} \left\|T^n x_n - q\right\| = r$.

On the other hand, since S is generalized asymptotically quasi-nonexpansive, we have

$$\begin{aligned} \left\|T^n x_n - q\right\| &\leq \left\|T^n x_n - S^n y_n\right\| + \left\|S^n y_n - q\right\| \\ &\leq \left\|T^n x_n - S^n y_n\right\| + k_n \left\|y_n - q\right\| + s_n, \end{aligned}$$

so we have

$$r \leq \liminf_{n\to\infty} \left\|y_n - q\right\|. \tag{3.21}$$

By using (3.17) and (3.21), we obtain

$$\lim_{n\to\infty} \left\|y_n - q\right\| = r. \tag{3.22}$$

Thus $r = \lim_{n\to\infty} \left\|y_n - q\right\| = \lim_{n\to\infty} \left\|(1-\beta_n)(x_n - q) + \beta_n (T^n x_n - q)\right\|$ gives by Lemma 2.1 that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.23}$$

Now

$$\left\|y_n - x_n\right\| = \beta_n \|T^n x_n - x_n\|.$$

Hence by (3.23), we obtain

$$\lim_{n\to\infty} \left\|y_n - x_n\right\| = 0. \tag{3.24}$$

Also note that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \left\|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - x_n\right\| \\ &\leq \|T^n x_n - x_n\| + \alpha_n \left\|T^n x_n - S^n y_n\right\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{3.25}$$

so that

$$\begin{aligned}\left\|x_{n+1}-y_n\right\| &\le \|x_{n+1}-x_n\|+\left\|y_n-x_n\right\| \\ &\to 0 \text{ as } n\to\infty. \end{aligned} \tag{3.26}$$

Furthermore, from

$$\begin{aligned}\left\|x_{n+1}-S^n y_n\right\| &\le \|x_{n+1}-x_n\|+\|x_n-T^n x_n\| \\ &\quad +\left\|T^n x_n-S^n y_n\right\|\end{aligned}$$

using (3.16), (3.23) and (3.25), we find that

$$\lim_{n\to\infty}\left\|x_{n+1}-S^n y_n\right\| = 0. \tag{3.27}$$

Then

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\| &\le \left\|x_{n+1}-T^{n+1}x_{n+1}\right\|+\left\|T^{n+1}x_{n+1}-T^{n+1}x_n\right\| \\ &\quad +\left\|T^{n+1}x_n-Tx_{n+1}\right\| \\ &\le \left\|x_{n+1}-T^{n+1}x_{n+1}\right\|+L\|x_{n+1}-x_n\| \\ &\quad +L\|T^n x_n-x_{n+1}\| \\ &= \left\|x_{n+1}-T^{n+1}x_{n+1}\right\|+L\|x_{n+1}-x_n\| \\ &\quad +L\alpha_n\left\|T^n x_n-S^n y_n\right\|\end{aligned}$$

yields

$$\lim_{n\to\infty}\|x_n-Tx_n\| = 0. \tag{3.28}$$

Now

$$\begin{aligned}\|x_n-S^n x_n\| &\le \|x_n-x_{n+1}\|+\left\|x_{n+1}-S^n y_n\right\| \\ &\quad +\left\|S^n y_n-S^n x_n\right\| \\ &\le \|x_n-x_{n+1}\|+\left\|x_{n+1}-S^n y_n\right\| \\ &\quad +L\left\|y_n-x_n\right\| \\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus

$$\begin{aligned}\|x_{n+1}-Sx_{n+1}\| &\le \left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+\left\|S^{n+1}x_{n+1}-Sx_{n+1}\right\| \\ &\le \left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+L\|S^n x_{n+1}-x_{n+1}\| \\ &\le \left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+L\Big(\left\|S^n x_{n+1}-S^n y_n\right\| \\ &\quad +\left\|S^n y_n-x_{n+1}\right\|\Big) \\ &\le \left\|x_{n+1}-S^{n+1}x_{n+1}\right\|+L^2\left\|x_{n+1}-y_n\right\| \\ &\quad +L\left\|S^n y_n-x_{n+1}\right\|\end{aligned}$$

implies

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.29}$$

This completes the proof. □

Theorem 3.4. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ and sequences $\{s_n\}, \{t_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (1.9). If at least one of the mappings S and T is semi-compact, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. Without loss of generality, we may assume that T is semi-compact. This with (3.28) means that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ as $n_k \to \infty$. Since S and T are continuous, then from (3.28) and (3.29), we find

$$\|x^* - Tx^*\| = \lim_{n_k\to\infty} \left\|x_{n_k} - Tx_{n_k}\right\| = 0, \tag{3.30}$$

and

$$\|x^* - Sx^*\| = \lim_{n_k\to\infty} \left\|x_{n_k} - Sx_{n_k}\right\| = 0. \tag{3.31}$$

This shows that $x^* \in F = F(S) \cap F(T)$. According to Lemma 3.1 the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Then

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n_k\to\infty} \left\|x_{n_k} - x^*\right\| = 0,$$

which means that $\{x_n\}$ converges to $x^* \in F$. This completes the proof. □

Applying Theorem 3.2, we obtain strong convergence of the process (1.9) under the condition (A') as follows:

Theorem 3.5. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ and sequences $\{s_n\}, \{t_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (1.9). Let S and T satisfy the condition (A'), then the sequence $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. We proved in Theorem 3.3 that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0, \ \lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.32}$$

From the condition (A') and (3.32), either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0,$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Hence

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

It follows, as in the proof of Theorem 3.2, that $\{x_n\}$ converges strongly to a common fixed point of the mappings S and T. This completes the proof.
□

4. Weak convergence theorem

In this section, we prove a weak convergence theorem of the iteration process (1.9) in the framework of real uniformly convex Banach spaces.

Lemma 4.1. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ and sequences $\{s_n\}$, $\{t_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_nl_n - 1) < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F = F(S)\cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (1.9). Then $\lim_{n\to\infty} \left\|tx_n + (1-t)p - q\right\|$ exists for all p, $q \in F$ and $t \in [0,1]$.*

Proof. By Lemma 3.1, we know that $\{x_n\}$ is bounded. Letting

$$a_n(t) = \left\|tx_n + (1-t)p - q\right\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \left\|p - q\right\|$ and $\lim_{n\to\infty} a_n(1) = \left\|x_n - q\right\|$ exists by Lemma 3.1. It, therefore, remains to prove the Lemma 4.1 for $t \in (0,1)$. For all $x \in K$, we define the mapping $T_n\colon K \to K$ by

$$T_n x = (1-\alpha_n)T^n x + \alpha_n S^n((1-\beta_n)x + \beta_n T^n x).$$

Then

$$\left\|T_nx - T_ny\right\| \leq \mu_n \left\|x - y\right\| + A_n, \tag{4.1}$$

for all $x, y \in K$, where $\mu_n = (1 + d_n)$ and $d_n = (k_nl_n - 1)$ with $\sum_{n=1}^{\infty} d_n < \infty$ and $\mu_n \to 1$ as $n \to \infty$. Setting

$$S_{n,m} = T_{n+m-1}T_{n+m-2}\dots T_n, \quad m \geq 1 \tag{4.2}$$

and

$$b_{n,m} = \left\|S_{n,m}(tx_n + (1-t)p) - (tS_{n,m}x_n + (1-t)S_{n,m}q)\right\|. \tag{4.3}$$

From (4.1) and (4.2), we have

$$\begin{aligned}\left\|S_{n,m}x - S_{n,m}y\right\| &\leq \mu_n\mu_{n+1}\ldots\mu_{n+m-1}\left\|x-y\right\| + \sum_{i=n}^{n+m-1} A_i \\ &\leq \Big(\prod_{i=n}^{n+m-1}\mu_i\Big)\left\|x-y\right\| + \sum_{i=n}^{n+m-1} A_i \\ &= H_n\left\|x-y\right\| + \sum_{i=n}^{n+m-1} A_i \qquad (4.4)\end{aligned}$$

for all $x, y \in K$, where $H_n = \prod_{i=n}^{n+m-1}\mu_i$, $H_n \to 1$ as $n \to \infty$ and $S_{n,m}x_n = x_{n+m}$, $S_{n,m}p = p$ for all $p \in F$. Thus

$$\begin{aligned}a_{n+m}(t) &= \left\|tx_{n+m} + (1-t)p - q\right\| \\ &\leq b_{n,m} + \left\|S_{n,m}(tx_n + (1-t)p) - q\right\| \\ &\leq b_{n,m} + H_n a_n(t) + \sum_{i=n}^{n+m-1} A_i. \qquad (4.5)\end{aligned}$$

By using [[4], Theorem 2.3], we have

$$\begin{aligned}b_{n,m} &\leq \phi^{-1}\Big(\left\|x_n - p\right\| - \left\|S_{n,m}x_n - S_{n,m}p\right\|\Big) \\ &\leq \phi^{-1}\Big(\left\|x_n - p\right\| - \left\|x_{n+m} - p + p - S_{n,m}p\right\|\Big) \\ &\leq \phi^{-1}\Big(\left\|x_n - p\right\| - \Big(\left\|x_{n+m} - p\right\| - \left\|S_{n,m}p - p\right\|\Big)\Big), \qquad (4.6)\end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges to 0 as $n \to \infty$ for all $m \geq 1$. Thus, fixing n and letting $m \to \infty$ in (4.6), we have

$$\begin{aligned}\limsup_{m\to\infty} a_{n+m}(t) &\leq \phi^{-1}\Big(\left\|x_n - p\right\| - \Big(\lim_{m\to\infty}\left\|x_m - p\right\| - \left\|S_{n,m}p - p\right\|\Big)\Big) \\ &\quad + H_n a_n(t) + \sum_{i=n}^{n+m-1} A_i, \qquad (4.7)\end{aligned}$$

and again letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} a_n(t) \leq \phi^{-1}(0) + \liminf_{n\to\infty} a_n(t) + 0 = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is,

$$\lim_{n\to\infty}\left\|tx_n + (1-t)p - q\right\|$$

exists for all $t \in [0, 1]$. This completes the proof. □

Theorem 4.2. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ and sequences $\{s_n\}$, $\{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (1.9). If the* *mappings $I - S$ and $I - T$, where I denotes the identity mapping, are demiclosed at zero. Then $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T.*

Proof. By Lemma 3.1, we know that $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in K$. By Theorem 3.3, we have

$$\lim_{n\to\infty} \left\|x_{n_j} - Sx_{n_j}\right\| = 0, \ \lim_{n\to\infty} \left\|x_{n_j} - Tx_{n_j}\right\| = 0.$$

Since the mappings $I - S$ and $I - T$ are demiclosed at zero, therefore $Sp = p$ and $Tp = p$, which means $p \in F$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in K$. By the same method as above, we have $q \in F$ and $p, q \in w_w(x_n)$. By Lemma 4.1, the limit

$$\lim_{n\to\infty} \left\|tx_n + (1-t)p - q\right\|$$

exists for all $t \in [0, 1]$ and so $p = q$ by Lemma 2.4. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof. □

Remark 4.3. *Theorems of this paper can also be proved with error terms.*

Remark 4.4. *Our results extend, improve and generalize many known results from the existing literature.*

Example 4.5. *Let $E = [-\pi, \pi]$ and let T be defined by*

$$Tx = x\cos x$$

for each $x \in E$. Clearly $F(T) = \{0\}$. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = |x||\cos x| \leq |x| = |x - z|,$$

and hence T is generalized asymptotically quasi-nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$ and $\{s_n\} = \{0\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$|Tx - Ty| = \left|\frac{\pi}{2}\cos\frac{\pi}{2} - \pi\cos\pi\right| = \pi,$$

whereas

$$|x - y| = \left|\frac{\pi}{2} - \pi\right| = \frac{\pi}{2}.$$

Example 4.6. *Let $E = \mathbb{R}$ and let T be defined by*

$$T(x) = \begin{cases} \frac{x}{2}\cos\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$ and $Tx = x$, then $x = \frac{x}{2}\cos\frac{1}{x}$. Thus $2 = \cos\frac{1}{x}$. This is not hold. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = \left|\frac{x}{2}\right|\left|\cos\frac{1}{x}\right| \leq \frac{|x|}{2} < |x| = |x - z|,$$

and hence T is generalized asymptotically quasi-nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$ and $\{s_n\} = \{0\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take

$x = \frac{2}{3\pi}$ *and* $y = \frac{1}{\pi}$, *then*

$$|Tx - Ty| = \left|\frac{1}{3\pi} cos \frac{3\pi}{2} - \frac{1}{2\pi} cos\, \pi\right| = \frac{1}{2\pi},$$

whereas

$$|x - y| = \left|\frac{2}{3\pi} - \frac{1}{\pi}\right| = \frac{1}{3\pi}.$$

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