Functional Analysis,
Approximation and
Computation
5:1 (2013), 57–72

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Stability of reciprocal difference and adjoint functional equations in paranormed spaces: Direct and fixed point methods

K. Ravi[a], J.M. Rassias[b], B.V. Senthil Kumar[c]

[a] *PG & Research Department of Mathematics, Sacred Heart College, Tirupattur - 635 601 TamilNadu, India*
[b] *Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Athens, Attikis 15342, GREECE*
[c] *Department of Mathematics, C. Abdul Hakeem College of Engg. and Tech., Melvisharam - 632 509, TamilNadu, India*

Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of reciprocal difference and adjoint functional equations in paranormed spaces by direct and fixed point methods. We also provide examples for nonstability.

1. Introduction

An interesting and famous talk presented by S.M. Ulam [39] in 1940, triggered the study of stability problems for various functional equations. In 1941, D.H. Hyers [10] was the first mathematician to present an affirmative partial answer to the question of S.M. Ulam. In 1950, T. Aoki [1] generalized the Hyers' theorem for additive mappings. In 1978, Th.M. Rassias [37] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. In 1982, J.M. Rassias [28] gave a further generalization of the result of D.H. Hyers and proved by theorem using weaker conditions controlled by a product of different powers of norms. P. Gavruta [9] obtained generalized result of Th.M. Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function. In 2008, J.M. Rassias [29] introduced mixed type product-sum of powers of norms. Beginning around the year 1980, the stability problems of a wide class of functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3], [5], [9], [6], [10], [11], [12], [38]).

In 1996, Isac and Th.M. Rassias [12] were the first to provide applications of stability theorem of functional equations for the proof of new fixed point theorems with applications.

Usually, the stability problem for functional equations is solved by direct method, in which the exact solution of the functional equation is explicitly constructed as a limit of a (Hyers) sequence, starting from

2010 *Mathematics Subject Classification*. Primary 39B82 ; Secondary 39B72)
Keywords. (Reciprocal type functional equation, Reciprocal Difference functional equation, Reciprocal Adjoint functional equation, Generalized Hyers-Ulam stability)
Received: 26 November 2012; Accepted: 12 April 2013
Communicated by Hari M. Srivastava
Email addresses: shckravi@yahoo.co.in (K. Ravi),
jrassias@primedu.uoa.gr;jrass@otenet.gr;Ioannis.Rassias@primedu.uoa.gr (J.M. Rassias), bvssree@yahoo.co.in (B.V. Senthil Kumar)

the given approximate solution ([2], [4], [11], [13], [14]).

K. Ravi and B.V. Senthil Kumar [30] investigated the Hyers-Ulam stability for the reciprocal functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)} \tag{1}$$

where $f : X \to Y$ is a mapping in the space of non-zero real numbers. The reciprocal function $f(x) = \frac{1}{x}$ is a solution of the functional equation (1).

K. Ravi, J.M. Rassias and B.V. Senthil Kumar ([31], [32]) investigated the generalized Hyers-Ulam stability of the reciprocal difference functional equation

$$f\left(\frac{x+y}{2}\right) - f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)} \tag{2}$$

and the reciprocal adjoint functional equation

$$f\left(\frac{x+y}{2}\right) + f(x+y) = \frac{3f(x)g(y)}{f(x)+f(y)} \tag{3}$$

in the spaces of non-zero real numbers using fixed point method and direct method.

K. Ravi, J.M. Rassias and B.V. Senthil Kumar [33] obtained the generalized Hyers-Ulam stability for the generalized reciprocal functional equation

$$f\left(\sum_{i=1}^{m} \alpha_i x_i\right) = \frac{\prod_{i=1}^{m} f(x_i)}{\sum_{i=1}^{m}\left[\alpha_i\left(\prod_{j=1, j\neq i}^{m} f(x_j)\right)\right]} \tag{4}$$

for arbitrary but fixed real numbers $(\alpha_1, \alpha_2, \ldots, \alpha_m) \neq (0, 0, \ldots, 0)$ so that $0 < \alpha = \sum_{i=1}^{m} \alpha_i \neq 1$ and $f : X \to Y$ with X and Y are the sets of non-zero real numbers.

K. Ravi, E. Thandapani and B.V. Senthil Kumar [34] proved the generalized Hyers-Ulam stability for the reciprocal type functional equations

$$f((k_1-k_2)x + (k_1-k_2)y) = \frac{f(k_1x-k_2y)f(k_1y-k_2x)}{f(k_1x-k_2y)+f(k_1y-k_2x)} \tag{5}$$

where k_1 and k_2 are any integers with $k_1 \neq k_2$ and

$$f((k_1+k_2)x + (k_1+k_2)y) = \frac{f(k_1x+k_2y)f(k_1y+k_2x)}{f(k_1x+k_2y)+f(k_1y+k_2x)} \tag{6}$$

where k_1 and k_2 are any integers with $k_1 \neq -k_2$.

Recently, Ch. Park and D.Y. Shin [26] proved the Hyers-Ulam stability of the Cauchy additive functional equation

$$f(x+y) = f(x) + f(y), \tag{7}$$

the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{8}$$

the cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{9}$$

and the quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{10}$$

in paranormed spaces.

C. Park [27] proved the Hyers-Ulam stability of an additive-quadratic-cubic-quartic functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \tag{11}$$

in paranormed spaces using the fixed point method and direct method.

The stability problem for the Jensen's functional equation using direct method and fixed point technique, Pexiderized quadratic functional equation, fuzzy stability of cubic functional equation and stability of cubic functional equation were considered in ([19], [20], [21], [22] and [25]), respectively, in the intuitionistic fuzzy normed spaces.

For notational convenience, let us define

$$R_1f(x,y)=f\left(\frac{x+y}{2}\right)-f(x+y)-\frac{f(x)f(y)}{f(x)+f(y)}$$

and

$$R_2f(x,y)=f\left(\frac{x+y}{2}\right)+f(x+y)-\frac{3f(x)f(y)}{f(x)+f(y)}.$$

In this paper, we apply direct method and fixed point method to investigate the generalized Hyers-Ulam stability of the functional equations

$$R_jf(x,y)=0 \quad \text{for} \quad j=1,2 \tag{12}$$

in paranormed spaces. We also provide counter-examples for non-stability.

2. Preliminaries

In this section, we recall basic facts concerning **Fréchet spaces** and fundamental results of fixed point theory.

The concept of statistical convergence for sequences of real numbers was introduced by Fast [7] and Steinhaus [36] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [8], [15], [23], [24], [35]). This notion was defined in normed spaces by Kolk [16].

Definition 2.1. *[40] Let X be a vector space. A paranorm $P: X \to [0,\infty)$ is a function on X such that*

(1) $P(0)=0$;
(2) $P(-x)=P(x)$;
(3) $P(x+y)\le P(x)+P(y)$ *(triangle inequality);*
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\}\subset X$ with $P(x_n\text{-}x)\to 0$, then $P(t_nx_n\text{-}tx)\to 0$ (continuity of multiplication);
The pair (X,P) is called a paranormed space if P is a paranorm on X.
(5) $P(x)=0$ *implies* $x=0$.

A **Fréchet space** *is a total and complete paranormed space.*

Definition 2.2. *Let A be a set. A function $d: A\times A \to [0,\infty]$ is called a generalized metric on A if d satisfies the following conditions:*

1. $d(x,y)=0$ *if and only if* $x=y$;

2. $d(x,y) = d(y,x)$ *for all* $x, y \in A$;
3. $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x, y, z \in A$.

We note that the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include infinity.

The following theorem is very useful for proving our main results which is due to Margolis and Diaz [17].

Theorem 2.3. *[17] Let* (X, d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then for each given element* $X \in X$, *either*

$$d\left(J^n x, J^{n+1} x\right) = \infty$$

for all non-negative integers n or there exists a positive integer n_0 *such that*

1. $d\left(J^n x, J^{n+1} x\right) < \infty$ *for all* $n \geq n_0$;
2. *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
3. y^* *is the unique fixed point of* J *in the set*

$$Y = \{y \in X | d\left(J^{n_0} x, y\right) < \infty\};$$

4. $d\left(y, y^*\right) < \frac{1}{1-L} d(y, Jy)$, *for all* $y \in Y$.

Throughout this paper, let (X, P) be a Fréchet space and that $(Y, \|.\|)$ be a Banach space. In the following results, we assume that $x \neq 0$, $y \neq 0$, $y \neq -x$, $f(x) + f(y) \neq 0$.

3. Generalized Hyers-Ulam stability of functional equations (12): Direct Method

In this section, we investigate the Gavruta stability (see [9]), for the reciprocal difference and adjoint functional equations (12), for $j = 1, 2$ and present the Th.M. Rassias stability (sum of powers of norms) (see [37]), J.M. Rassias stabilities (product of powers of norms and mixed-type product-sum of powers of norms) ([28], [29]) in the consequent Corollaries.

Theorem 3.1. *Let* $\phi : Y \times Y \to [0, \infty)$ *be a function satisfying*

$$\sum_{i=0}^{\infty} 2^i \phi\left(2^i x, 2^i y\right) < +\infty \tag{13}$$

for all $x, y \in Y$. *If a function* $f : Y \to X$ *satisfies the functional inequality*

$$P\left(R_j f(x,y)\right) \leq \phi(x,y) \tag{14}$$

for all $x, y \in Y$, $j = 1, 2$, *then there exists a unique reciprocal mapping* $r : Y \to X$ *which satisfies (12), for* $j = 1, 2$ *and the inequality*

$$P(f(x) - r(x)) \leq 2 \sum_{i=0}^{\infty} 2^i \phi\left(2^i x, 2^i x\right) \tag{15}$$

for all $x \in Y$.

Proof. Replacing (x, y) by (x, x) in (14) and multiplying by 2, we get

$$P(f(x) - 2f(2x)) \leq 2\phi(x, x) \tag{16}$$

for all $x \in Y$. Now, replacing x by $2x$ in (16), multiplying by 2 and summing the resulting inequality with (16), we obtain

$$P\left(f(x) - 2^2 f\left(2^2 x\right)\right) \leq 2 \sum_{i=0}^{1} 2^i \phi\left(2^i x, 2^i x\right)$$

for all $x \in Y$. Proceeding further and using induction arguments on a positive integer n, we arrive

$$P\left(f(x) - 2^n f\left(2^n x\right)\right) \leq 2 \sum_{i=0}^{n-1} 2^i \phi\left(2^i x, 2^i x\right) \tag{17}$$

for all $x \in Y$. Hence for any non-negative integers l, k with $l > k$, we obtain by using the triangle inequality

$$\begin{aligned} P\left(2^l f\left(2^l x\right) - 2^k f\left(2^k x\right)\right) &\leq P\left(2^l f\left(2^l x\right) - f(x)\right) + P\left(f(x) - 2^k f\left(2^k x\right)\right) \\ &\leq \sum_{i=0}^{l-1} 2^i \phi\left(2^i x, 2^i x\right) + \sum_{i=0}^{k-1} 2^i \phi\left(2^i x, 2^i x\right) \\ &\leq \sum_{i=k}^{l-1} 2^i \phi\left(2^i x, 2^i x\right) \end{aligned} \tag{18}$$

for all $x \in Y$. Taking the limit as $k \to +\infty$ in (18) and considering (13), it follows that the sequence $r_n(x) = \{2^n f\left(2^n x\right)\}$ is a Cauchy sequence for each $x \in Y$. Since X is complete, we can define $r : Y \to X$ by

$$r(x) = \lim_{n \to \infty} 2^n f\left(2^n x\right). \tag{19}$$

To show that r satisfies (12), for $j = 1, 2$, replacing (x, y) by $(2^n x, 2^n y)$ in (14) and multiplying by 2^n, we obtain

$$P\left(2^n R_j f\left(2^n x, 2^n y\right)\right) \leq 2^n \phi\left(2^n x, 2^n y\right) \tag{20}$$

for all $x, y \in Y$, for all positive integer n and for $j = 1, 2$. Using (13) and (17) in (20), we see that r satisfies (12), for all $x, y \in Y$, for $j = 1, 2$. Taking limit $n \to \infty$ in (17), we arrive (15). Now, it remains to show that r is uniquely defined. Let $R : Y \to X$ be another reciprocal mapping which satisfies (1.12) for $j = 1, 2$ and the inequality (15). Then we have

$$\begin{aligned} P(r(x) - R(x)) &= P\left(2^n r\left(2^n x\right) - 2^n R\left(2^n x\right)\right) \\ &\leq P\left(2^n r\left(2^n x\right) - 2^n f\left(2^n x\right)\right) + P\left(2^n f\left(2^n x\right) - 2^n R\left(2^n x\right)\right) \\ &\leq 4 \sum_{i=0}^{\infty} 2^{n+i} \phi\left(2^{n+i} x, 2^{n+i} x\right) \\ &\leq 4 \sum_{i=n}^{\infty} 2^i \phi\left(2^i x, 2^i x\right) \end{aligned} \tag{21}$$

for all $x \in Y$. Allowing $n \to \infty$ in (21), we see that r is unique, which completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $\phi : Y \times Y \to [0, \infty)$ be a function satifying*

$$\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \phi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) < +\infty \tag{22}$$

for all $x, y \in Y$. If a function $f : Y \to X$ satisfies the functional inequality (14), for all $x, y \in Y$, for $j = 1, 2$, then there exists a unique reciprocal mapping $r : Y \to X$ which satisfies (1.12), for $j = 1, 2$ and the inquality

$$P(f(x) - r(x)) \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \phi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) \tag{23}$$

for all $x \in Y$.

Proof. The proof is obtained by replacing (x, y) by $\left(\frac{x}{2}, \frac{x}{2}\right)$ in (14), for $j = 1, 2$ and proceeding by similar arguments as in Theorem 3.1. □

Corollary 3.3. *Let $f : Y \to X$ be a mapping and let there exist real numbers $\alpha \neq -1$ and $c_1 \geq 0$ such that*

$$P\left(R_j f(x, y)\right) \leq c_1 \left(||x||^\alpha + ||y||^\alpha\right) \tag{24}$$

for all $x, y \in Y$, for $j = 1, 2$. Then there exists a unique reciprocal mapping $r : Y \to X$ satisfying (12), for $j = 1, 2$ and

$$P(f(x) - r(x)) \leq \begin{cases} \frac{4c_1}{1-2^{\alpha+1}} ||x||^\alpha & \text{for } \alpha < -1 \\ \frac{4c_1}{2^{\alpha+1}-1} ||x||^\alpha & \text{for } \alpha > -1 \end{cases} \tag{25}$$

for every $x \in Y$.

Proof. **Case (i).** Let $\alpha < -1$.
Replacing (x, y) by (x, x) in (24) and multiplying by 2, we get

$$P(f(x) - 2f(2x)) \leq 4c_1 ||x||^\alpha \tag{26}$$

for all $x \in Y$. Now, replacing x by $2x$ in (26), multiplying by 2 and summing the resulting inequality with (26), we obtain

$$P\left(f(x) - 2^2 f\left(2^2 x\right)\right) \leq 4c_1 \sum_{i=0}^{1} 2^{i(\alpha+1)} ||x||^\alpha$$

for all $x \in Y$. Proceeding further and using induction arguments on a positive integer n, we arrive

$$\begin{aligned} P\left(f(x) - 2^n f\left(2^n x\right)\right) &\leq 4c_1 \sum_{i=0}^{n-1} 2^{i(\alpha+1)} ||x||^\alpha \\ &\leq 4c_1 \sum_{i=0}^{\infty} 2^{i(\alpha+1)} ||x||^\alpha \\ &\leq \frac{4c_1}{1 - 2^{\alpha+1}} ||x||^\alpha \end{aligned} \tag{27}$$

for all $x \in Y$. In order to prove the convergence of the sequence $\{2^n f(2^n x)\}$, we have if $n > m > 0$, then

$$P\left(2^n f(2^n x) - 2^m f(2^m x)\right) = 2^m P\left(2^{n-m} f(2^n x) - f(2^m x)\right)$$

holds for all $x \in Y$ and $n, m \in \mathbb{N}$. Setting $2^m x = y$ in this relation and using (27), we obtain

$$\begin{aligned} P\left(2^n f(2^n x) - 2^m f(2^m x)\right) &= 2^m P\left(2^{n-m} f(2^{n-m} y) - f(y)\right) \\ &\leq 2^m \frac{4c_1}{1 - 2^{\alpha+1}} ||y||^\alpha \\ &\leq 2^{m(\alpha+1)} \frac{4c_1}{1 - 2^{\alpha+1}} ||x||^\alpha \end{aligned} \tag{28}$$

for all $x \in Y$. As $\alpha < -1$, the right-hand side of (28) tends to 0 as $m \to \infty$. This shows that the sequence $r_n(x) = \{2^n f(2^n x)\}$ is a Cauchy sequence for each $x \in Y$. Since X is complete, we can define $r : Y \to X$ by $r(x) = \lim_{n\to\infty} 2^n f(2^n x)$. Letting $n \to \infty$ in (27), we arrive

$$P(f(x) - r(x)) \leq \frac{4c_1}{1 - 2^{\alpha+1}} \|x\|^\alpha \tag{29}$$

for all $x \in Y$. To show that r satisfies (12), for $j = 1, 2$, replacing (x, y) by $(2^n x, 2^n y)$ in (24) and multiplying by 2^n, we obtain

$$\begin{aligned} P(2^n R_j f(2^n x, 2^n y)) &\leq 2^n c_1 \left(\|2^n x\|^\alpha + \|2^n y\|^\alpha\right) \\ &\leq 2^{n(\alpha+1)} c_1 \left(\|x\|^\alpha + \|y\|^\alpha\right) \end{aligned} \tag{30}$$

for all $x, y \in Y$. Allowing $n \to \infty$ in (30), we see that r satisfies (12), for all $x, y \in Y$, for $j = 1, 2$. To prove r is a unique reciprocal functionl satisfying (12) subject to (29), let us consider another reciprocal function $R : Y \to X$ which satisfies (12) for $j = 1, 2$ and the inequality (29). Then we have

$$\begin{aligned} P(r(x) - R(x)) &= P(2^n r(2^n x) - 2^n R(2^n x)) \\ &\leq P(2^n r(2^n x) - 2^n f(2^n x)) + P(2^n f(2^n x) - 2^n R(2^n x)) \\ &\leq 2^n \left(P(r(2^n x) - f(2^n x)) + P(f(2^n x) - R(2^n x))\right) \\ &\leq 2^{n(\alpha+1)} \frac{8c_1}{1 - 2^{\alpha+1}} \|x\|^\alpha \end{aligned} \tag{31}$$

which tends to zero as $n \to \infty$ for all $x \in Y$. Allowing $n \to \infty$ in (31), we find that r is unique reciprocal mapping satifying (12).

Case (ii). Let $\alpha > -1$.

Replacing (x, y) by $\left(\frac{x}{2}, \frac{x}{2}\right)$ in (24), we get

$$P\left(\frac{1}{2} f\left(\frac{x}{2}\right) - f(x)\right) \leq \frac{2c_1}{2^\alpha} \|x\|^\alpha \tag{32}$$

for all $x \in Y$. Now, replacing x by $\frac{x}{2}$ in (32), dividing by 2 and summing the resulting inequality with (32), we obtain

$$P\left(\frac{1}{2^2} f\left(\frac{x}{2^2}\right) - f(x)\right) \leq \frac{2c_1}{2^\alpha} \sum_{i=0}^{1} \frac{1}{2^{i(\alpha+1)}} \|x\|^\alpha$$

for all $x \in Y$. Proceeding further and using induction arguments on a positive integer n, we arrive

$$\begin{aligned} P\left(\frac{1}{2^n} f\left(\frac{x}{2^n}\right) - f(x)\right) &\leq \frac{2c_1}{2^\alpha} \sum_{i=0}^{n-1} \frac{1}{2^{i(\alpha+1)}} \|x\|^\alpha \\ &\leq \frac{2c_1}{2^\alpha} \sum_{i=0}^{\infty} \frac{1}{2^{i(\alpha+1)}} \|x\|^\alpha \\ &\leq \frac{4c_1}{2^{\alpha+1} - 1} \|x\|^\alpha \end{aligned} \tag{33}$$

for all $x \in Y$. In order to prove the convergence of the sequence $\left\{\frac{1}{2^n} f\left(\frac{x}{2^n}\right)\right\}$, we have if $n > m > 0$, then

$$P\left(\frac{1}{2^n} f\left(\frac{x}{2^n}\right) - \frac{1}{2^m} f\left(\frac{x}{2^m}\right)\right) = \frac{1}{2^m} P\left(\frac{1}{2^{n-m}} f\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^m}\right)\right)$$

holds for all $x \in Y$ and $n, m \in \mathbb{N}$. Setting $\frac{x}{2^m} = y$ in this relation and using (33), we obtain

$$p\left(\frac{1}{2^n} f\left(\frac{x}{2^n}\right) - \frac{1}{2^m} f\left(\frac{x}{2^m}\right)\right) = \frac{1}{2^m} P\left(\frac{1}{2^{n-m}} f\left(\frac{y}{2^{n-m}}\right) - f(y)\right)$$
$$\leq \frac{1}{2^m}\left(\frac{4c_1}{2^{\alpha+1}-1}\right)||y||^{\alpha}$$
$$\leq \frac{1}{2^{m(\alpha+1)}}\left(\frac{4c_1}{2^{\alpha+1}-1}\right)||x||^{\alpha} \tag{34}$$

for all $x \in Y$. As $\alpha > -1$, the right-hand side of (34) tends to 0 as $m \to \infty$. This shows that the sequence $r_n(x) = \{\frac{1}{2^n} f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for each $x \in Y$. Since X is complete, we can define $r : Y \to X$ by $r(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(\frac{x}{2^n}\right)$. Letting $n \to \infty$ in (33), we arrive

$$P(r(x) - f(x)) \leq \frac{4c_1}{2^{\alpha+1}-1} ||x||^{\alpha} \tag{35}$$

for all $x \in Y$. To show that r satisfies (12), for $j = 1, 2$, replacing (x, y) by $\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$ in (24) and dividing by 2^n, we obtain

$$P\left(\frac{1}{2^n} R_j f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) \leq c_1\left(\left\|\frac{x}{2^n}\right\|^{\alpha} + \left\|\frac{y}{2^n}\right\|^{\alpha}\right)$$
$$\leq \frac{1}{2^{n(\alpha+1)}} c_1\left(||x||^{\alpha} + ||y||^{\alpha}\right) \tag{36}$$

for all $x, y \in Y$. Allowing $n \to \infty$ in (36), we see that r satisfies (12), for all $x, y \in Y$, for $j = 1, 2$. To prove r is a unique reciprocal function satisfying (12) subject to (35), let us consider another reciprocal function $R : Y \to X$ which satisfies (12) for $j = 1, 2$ and the inequality (35). Then we have

$$P(r(x) - f(x)) = P\left(\frac{1}{2^n} r\left(\frac{x}{2^n}\right) - \frac{1}{2^n} R\left(\frac{x}{2^n}\right)\right)$$
$$\leq P\left(\frac{1}{2^n} r\left(\frac{x}{2^n}\right) - \frac{1}{2^n} f\left(\frac{x}{2^n}\right)\right) + P\left(\frac{1}{2^n} f\left(\frac{x}{2^n}\right) - \frac{1}{2^n} R\left(\frac{x}{2^n}\right)\right)$$
$$\leq \frac{1}{2^n}\left(\frac{8c_1}{2^{\alpha+1}-1}\right)\left\|\frac{x}{2^n}\right\|^{\alpha}$$
$$\leq \frac{1}{2^{n(\alpha+1)}}\left(\frac{8c_1}{2^{\alpha+1}-1}\right)||x||^{\alpha} \tag{37}$$

which tends to zero as $n \to \infty$ for all $x \in Y$. Allowing $n \to \infty$ in (37), we find that r is unique reciprocal mapping satisfying (12), which completes the proof of Corollary 3.3. □

Corollary 3.4. *Let $f : Y \to X$ be a mapping and let there exist real numbers a, b such that $\rho = a + b \neq -1$. Let there exist $c_2 \geq 0$ such that*

$$P\left(R_j f(x, y)\right) \leq c_2 ||x||^a ||y||^b \tag{38}$$

for all $x, y \in Y$, for $j = 1, 2$. Then there exists a unique reciprocal mapping $r : Y \to X$ satisfying (12), for $j = 1, 2$ and

$$P(f(x) - r(x)) \leq \begin{cases} \frac{2c_2}{1-2^{\rho+1}} ||x||^{\rho} & \text{for } \rho < -1 \\ \frac{2c_2}{2^{\rho+1}-1} ||x||^{\rho} & \text{for } \rho > -1 \end{cases} \tag{39}$$

for every $x \in Y$.

Proof. The required results in Corollary 3.4 can be easily derived by considering $\phi(x, y) = c_2 ||x||^a ||y||^b$, for all $x, y \in Y$ in Theorems 3.1 and 3.2 respectively and by similar arguments as in Corollary 3.3. □

Corollary 3.5. *Let $c_3 \geq 0$ and p, q be real numbers such that $\lambda = p+q \neq -1$, and $f : Y \to X$ be a mapping satisfying the functional inequality*

$$P\left(R_j f(x,y)\right) \leq c_3\left(||x||^p ||y||^q + (||x||^{p+q} + ||y||^{p+q})\right) \tag{40}$$

for all $x, y \in Y$, for $j = 1,2$. Then there exists a unique reciprocal mapping $r : Y \to X$ satisfying (12), for $j = 1,2$ and

$$P(f(x) - r(x)) \leq \begin{cases} \frac{6c_3}{1-2^{\lambda+1}} ||x||^{\lambda} & \text{for } \lambda < -1 \\ \frac{6c_3}{2^{\lambda+1}-1} ||x||^{\lambda} & \text{for } \lambda > -1 \end{cases} \tag{41}$$

for every $x \in Y$.

Proof. By choosing $\phi(x,y) = c_3\left(||x||^p ||y||^q + + (||x||^{p+q} + ||y||^{p+q})\right)$, for all $x, y \in Y$ in Theorems 3.1 and 3.2 respectively and using similar arguments as in Corollary 3.3, the proof of Corollary 3.5 is complete. □

Theorem 3.6. *Let $\phi : X \times X \to [0,\infty)$ be a function satisfying*

$$\sum_{i=0}^{\infty} 2^i \phi\left(2^i x, 2^i x\right) < +\infty \tag{42}$$

for all $x, y \in X$. If a function $f : X \to Y$ satisfies the functional inequality

$$\left\|R_j f(x,y)\right\| \leq \phi(x,y) \tag{43}$$

for all $x, y \in X$, for $j = 1,2$, then there exists a unique reciprocal mapping $r : X \to Y$ which satisfies (12), for $j = 1,2$ and the inequality

$$\left\|f(x) - r(x)\right\| \leq 2\sum_{i=0}^{\infty} 2^i \phi\left(2^i x, 2^i x\right) \tag{44}$$

for all $x \in X$.

Proof. The proof is obtained by similar arguments as in Theorem 3.1. □

Theorem 3.7. *Let $\phi : X \times X \to [0,\infty)$ be a function satisfying*

$$\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \phi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) < +\infty \tag{45}$$

for all $x, y \in X$. If a function $f : X \to Y$ satisfies the functional inequality (43) for all $x, y \in X$, for $j = 1,2$, then there exists a unique reciprocal mapping $r : X \to Y$ which satisfies (12), for $j = 1,2$ and the inequality

$$\left\|f(x) - r(x)\right\| \leq 2\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \phi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) \tag{46}$$

for all $x \in X$.

Proof. The proof is analogous to the proof of Theorem 3.2. □

Corollary 3.8. *Let $f : X \to Y$ be a mapping and let there exist real numbers $\alpha < -1$ and $c_1 \geq 0$ such that*

$$\left\|R_j f(x,y)\right\| \leq c_1\left(P(x)^{\alpha} + P(y)^{\alpha}\right) \tag{47}$$

for all $x, y \in X$, for $j = 1,2$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying (12), for $j = 1,2$ and

$$\left\|f(x) - r(x)\right\| \leq \frac{4c_1}{1 - 2^{\alpha+1}} P(x)^{\alpha} \tag{48}$$

for every $x \in X$.

Proof. The proof follows immediately by taking $\phi(x, y) = c_1 (P(x)^\alpha + P(y)^\alpha)$, for all $x, y \in X$ in Theorem 3.6. □

Corollary 3.9. *Let $f : X \to Y$ be a mapping and let there exist real numbers a, b such that $\rho = a + b < -1$. Let there exist $c_2 \geq 0$ such that*

$$\left\|R_j f(x, y)\right\| \leq c_2 P(x)^a P(y)^b \tag{49}$$

for all $x, y \in X$, for $j = 1, 2$. Then there exists a unique reciprocal mapping $r :; X \to Y$ satisfying (12), for $j = 1, 2$ and

$$\left\|f(x) - r(x)\right\| \leq \frac{2c_2}{1 - 2^{\rho+1}} P(x)^\rho \tag{50}$$

for every $x \in X$.

Proof. The required results in Corollary 3.9 can be easily derived by considering $\phi(x, y) = c_2 P(x)^a P(y)^b$, for all $x, y \in X$ in Theorem 3.6. □

Corollary 3.10. *Let $c_3 \geq 0$ and p, q be real numbers such that $\lambda = p + q < -1$, and $f : X \to Y$ be a mapping satisfying the functional inequality*

$$\left\|R_j f(x, y)\right\| \leq c_3 \left(P(x)^p P(y)^q + (P(x)^{p+q} + P(y)^{p+q})\right) \tag{51}$$

for all $x, y \in X$, for $j = 1, 2$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying (12), for $j = 1, 2$ and

$$\left\|f(x) - r(x)\right\| \leq \frac{6c_3}{1 - 2^{\lambda+1}} P(x)^\lambda \tag{52}$$

for every $x \in X$.

Proof. By choosing $\phi(x, y) = c_3 \left(P(x)^p P(y)^q + (P(x)^{p+q} + P(y)^{p+q})\right)$, for all $x, y \in X$ in Theorem 3.6, the proof of Corollary 3.10 is complete. □

4. Generalized Hyers-Ulam stability of functional equations (12): Fixed Point Method

Theorem 4.1. *Suppose that the mapping $f : Y \to X$ satisfies the inequality*

$$P\left(R_j f(x, y)\right) \leq \psi(x, y) \tag{53}$$

for all $x, y \in Y$, for $j = 1, 2$, where $\psi : Y \times Y \to [0, \infty)$ is a given function. If there exists $L < 1$ such that

$$\psi(x, y) \leq \frac{1}{2} L \psi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{54}$$

for all $x, y \in Y$, then there exists a unique reciprocal mapping $r : Y \to X$ such that

$$P(r(x) - f(x)) \leq \frac{L}{1 - L} \psi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{55}$$

for all $x \in Y$.

Proof. Define a set S by

$$S = \{h : Y \to X | h \text{ is a function}\}$$

and introduce the generalized metric d on S as follows:

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : P(g(x) - h(x)) \leq C\psi(x, x), \text{ for all } x \in Y\} \tag{56}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [18], Lemma 2.1]).

Define a mapping $\sigma : S \to S$ by

$$\sigma h(x) = 2h(2x) \quad (x \in Y) \tag{57}$$

for all $h \in S$. We claim that σ is strictly contractive on S. For any given $g, h \in S$, let $C_{gh} \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C_{gh}$. Hence

$$\begin{aligned} d(g,h) < C_{gh} &\Rightarrow P(g(x) - h(x)) \leq C_{gh}\psi(x,x), \forall x \in Y \\ &\Rightarrow P(2g(2x) - 2h(2x)) \leq 2C_{gh}\psi(2x,2x), \forall x \in Y \\ &\Rightarrow P(2g(2x) - 2h(2x)) \leq LC_{gh}\psi(x,x), \forall x \in Y \\ &\Rightarrow d(\sigma g, \sigma h) \leq LC_{gh}. \end{aligned}$$

Therefore, we see that

$$d(\sigma g, \sigma h) \leq Ld(g,h), \text{ for all } g, h \in S$$

that is, σ is strictly contractive mapping of S, with the Lipschitz constant L.

Now, replacing (x, y) by (x, x) in (53) and multiplying by 2, we get

$$P(f(x) - 2f(2x)) \leq 2\psi(x,x) \leq L\psi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in Y$. Hence (56) implies that $d(f, \sigma f) \leq 1$. Hence by applying the fixed point alternative Theorem 2.3, there exists a function $r : Y \to X$ satisfying the following:

(1) r is a fixed point of σ, that is

$$r(2x) = \frac{1}{2}r(x) \tag{58}$$

for all $x \in Y$. The mapping r is the unique fixed point of σ in the set

$$\mu = \{g \in S : d(f, g) < \infty\}.$$

This implies that r is the unique mapping satisfying (58) such that there exists $C \in (0, \infty)$ satisfying

$$P(f(x) - r(x)) \leq C\psi(x,x), \forall x \in Y.$$

(2) $d(\sigma^n f, r) \to 0$ as $n \to \infty$. Thus we have

$$\lim_{n\to\infty} 2^n f(2^n x) = r(x) \tag{59}$$

for all $x \in Y$.

(3) $d(r, f) \leq \frac{L}{1-L} d(r, \sigma f)$ which implies

$$d(r, f) \leq \frac{L}{1-L}.$$

Thus the inequality (55) holds. Hence from (53), (54) and (59), we have

$$\begin{aligned} P\left(R_j r(x,y)\right) &= \lim_{n\to\infty} 2^n P\left(R_j f(2^n x, 2^n y)\right) \\ &\leq \lim_{n\to\infty} 2^n \psi(2^n x, 2^n y) \\ &\leq \lim_{n\to\infty} 2^n \frac{L^n}{2^n}\psi(x,y) = 0 \end{aligned}$$

for all $x, y \in Y$, for $j = 1, 2$. So $R_j r(x, y) = 0$, for all $x, y \in Y$, for $j = 1, 2$. Hence r is a solution of functional equation (12), for $j = 1, 2$. By Theorem 2.1 [34], $r : Y \to X$ is a reciprocal mapping.

Next, we show that r is the unique reciprocal mapping satisfying (12), for $j = 1,2$ and (55). Suppose, let $R : Y \to X$ be another reciprocal function satisfying (12), for $j = 1,2$ and (55). Then from (12), for $j = 1,2$, we have that R is a fixed point of σ. Since $d(f,R) < \infty$, we have

$$R \in S^* = \{g \in S | d(f,g) < \infty\}.$$

From Theorem 2.3(3) and since both r and R are fixed points of σ, we have $r = R$. Therefore r is unique. Hence, there exists a unique reciprocal mapping $r : Y \to X$ satisfying (12), for $j = 1,2$ and (55), which completes the proof of Theorem 4.1. □

Theorem 4.2. *Suppose that the mapping $f : Y \to X$ satisfies the inequality (53), for all $x, y \in Y$, for $j = 1,2$, where $\psi : Y \times Y \to [0,\infty)$ is a given function. If there exists $L < 1$ such that*

$$\psi\left(\frac{x}{2}, \frac{y}{2}\right) \le 2L\psi(x,y) \tag{60}$$

for all $x, y \in Y$. Then there exists a unique reciprocal mapping $r : Y \to X$ such that

$$P(f(x) - r(x)) \le \frac{1}{1-L}\psi\left(\frac{x}{2}, \frac{x}{2}\right) \tag{61}$$

for all $x \in Y$.

Proof. The proof is similar to the proof of Theorem 4.1. □

Corollary 4.3. *Let $f : Y \to X$ be a mapping and let there exist real numbers $\alpha \neq -1$ and $c_1 \ge 0$ such that (24) holds for all $x, y \in Y$, for $j = 1,2$. Then there exists a unique reciprocal mapping $r : Y \to X$ satisfying (12), for $j = 1,2$ and (25), for every $x \in Y$.*

Proof. The proof follows immediately by taking $\psi(x,y) = c_1 (\|x\|^\alpha + \|y\|^\alpha)$, for all $x, y \in Y$ and $L = 2^{\alpha+1}$, $L = 2^{-\alpha-1}$ in Theorems 4.1 and 4.2 respectively. □

Corollary 4.4. *Let $f : Y \to X$ be a mapping and let there exist real numbers a, b such that $\rho = a + b \neq -1$. Let there exist $c_2 \ge 0$ such that (38) holds for all $x, y \in Y$, for $j = 1,2$. Then there exists a unique reciprocal mapping $r : Y \to X$ satisfying (12), for $j = 1,2$ and (39), for every $x \in Y$.*

Proof. The required results in Corollary 4.4 can be easily derived by considering $\psi(x,y) = c_2\|x\|^a\|y\|^b$, for all $x, y \in Y$ and $L = 2^{\rho+1}$, $L = 2^{-\rho-1}$ in Theorems 4.1 and 4.2 respectively. □

Corollary 4.5. *Let $c_3 \ge 0$ and p, q be real numbers such that $\lambda = p + q \neq -1$, and $f : Y \to X$ be a mapping satisfying the functional inequality (40), for all $x, y \in Y$, for $j = 1,2$. Then there exists a unique reciprocal mapping $r : Y \to X$ satisfying (12), for $j = 1,2$ and (41), for every $x \in Y$.*

Proof. By choosing $\psi(x,y) = c_3 (\|x\|^p\|y\|^q + + (\|x\|^{p+q} + \|y\|^{p+q}))$, for all $x, y \in Y$ and $L = 2^{\lambda+1}$, $L = 2^{-\lambda-1}$ in Theorems 4.1 and 4.2 respectively, the proof of Corollary 4.5 is complete. □

Theorem 4.6. *Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\left\|R_j f(x,y)\right\| \le \psi(x,y) \tag{62}$$

for all $x, y \in X$, for $j = 1,2$, where $\psi : X \times X \to [0,\infty)$ is a given function. If there exists $L < 1$ such that (54) holds for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \to Y$ such that

$$\left\|f(x) - r(x)\right\| \le \frac{L}{1-L}\psi\left(\frac{x}{2}, \frac{x}{2}\right) \tag{63}$$

for all $x \in X$.

Proof. The proof is obtained by similar arguments as in Theorem 4.1. □

Theorem 4.7. *Suppose that the mapping $f : X \to Y$ satisfies the inequality (62), for all $x, y \in X$, for $j = 1, 2$, where $\psi : X \times X \to [0, \infty)$ is a given function. If there exists $L < 1$ such that (60) holds for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \to Y$ such that*

$$\left\| f(x) - r(x) \right\| \leq \frac{1}{1-L} \psi\left(\frac{x}{2}, \frac{x}{2}\right) \tag{64}$$

for all $x \in X$.

Proof. The proof is analogous to the proof of Theorem 4.2. □

Corollary 4.8. *Let $f : X \to Y$ be a mapping and let there exist real numbers $\alpha < -1$ and $c_1 \geq 0$ such that (47) holds for all $x, y \in X$, for $j = 1, 2$. Then there exists a unique reciprocal mapping $r : X \to Y$ satifying (12), for $j = 1, 2$ and (48), for every $x \in X$.*

Proof. The proof follows immediately by taking $\psi(x, y) = c_1 \left(P(x)^\alpha + P(y)^\alpha\right)$, for all $x, y \in X$ and $L = 2^{\alpha+1}$ in Theorem 4.6. □

Corollary 4.9. *Let $f : X \to Y$ be a mapping and let there exist real exist real numbers a, b such that $\rho = a + b < -1$. Let there exist $c_2 \geq 0$ such that (49) holds for all $x, y \in X$, for $j = 1, 2$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying (12), for $j = 1, 2$ and (50), for every $x \in X$.*

Proof. The required results in Corollary 4.9 can be easily derived by considering $\psi(x, y) = c_2 P(x)^a P(y)^b$, for all $x, y \in X$ and $L = 2^{\rho+1}$ in Theorem 4.6. □

Corollary 4.10. *Let $c_3 \geq 0$ and p, q be real numbers such that $\lambda = p + q < -1$, and $f : X \to Y$ be a mapping satisfying the functional inequality (51) for all $x, y \in X$, for $j = 1, 2$. Then there exists a unique reciprocal mapping $r : X \to Y$ satisfying (12), for $j = 1, 2$ and (52), for every $x \in X$.*

Proof. By choosing $\psi(x, y) = c_3 \left(P(x)^p P(y)^q + (P(x)^{p+q} + P(y)^{p+q})\right)$, for all $x, y \in X$ and $L = 2^{\lambda+1}$ in Theorem 4.6, the proof of Corollary 4.10 is complete. □

5. Counter-examples

We present the following counter-examples modified by the well-known counter-example of Z. Gajda [5].
The following example illustrates the fact that functional equation (12), for $j = 1$ is not stable for $\alpha = -1$ in Corollaries 3.3 and 4.3.

Example 5.1. *Let $\varphi : \mathbb{R}\text{-}\{0\} \to \mathbb{R}$ be a mapping defined by*

$$\varphi(x) = \begin{cases} \frac{c_1}{x} & \text{for } x \in (1, \infty) \\ c_1 & \text{otherwise} \end{cases}$$

where c_1 is a constant, and define a mapping $g : \mathbb{R}\text{-}\{0\} \to \mathbb{R}$ by

$$g(x) = \sum_{n=0}^{\infty} \frac{\varphi\left(2^{-n}x\right)}{2^n}, \text{ for all } x \in \mathbb{R}\text{-}\{0\}.$$

Then the mapping g satisfies the inequality

$$\left| R_1 g(x, y) \right| \leq 5c_1 \left(|x|^{-1} + |y|^{-1}\right) \tag{65}$$

for all $x, y \in \mathbb{R}$-{0}. Therefore there do not exist a reciprocal mapping $r : \mathbb{R}\text{-}\{0\} \to \mathbb{R}$ with $r(x) = \frac{1}{x}$ and a constant $\delta > 0$ such that

$$|g(x) - r(x) \leq \delta |x|^{-1} \tag{66}$$

for all $x \in \mathbb{R}$-{0}.

Proof. $|g(x)| \leq \sum_{n=0}^{\infty} \frac{|\varphi(2^{-n})|}{|2^n|} \leq \sum_{n=0}^{\infty} \frac{c_1}{2^n} = 2c_1$. Hence g is bounded by $2c_1$. If $|x|^{-1} + |y|^{-1} \geq 1$, then the left hand side of (65) is less than $5c_1$. Now, suppose that $0 < |x|^{-1} + |y|^{-1} < 1$. Then there exists a positive integer m such that

$$\frac{1}{2^{m+1}} \leq |x|^{-1} + |y|^{-1} < \frac{1}{2^m}. \tag{67}$$

Hence $|x|^{-1} + |y|^{-1} < \frac{1}{2^m}$ implies

$$2^m |x|^{-1} + 2^m |y|^{-1} < 1$$

$$\text{or} \qquad \frac{x}{2^m} > 1, \ \frac{y}{2^m} > 1$$

$$\text{or} \qquad \frac{x}{2^{m-1}} > 2 > 1, \ \frac{y}{2^{m-1}} > 2 > 1$$

and consequently

$$\frac{1}{2^{m-1}}(x + y) > 1.$$

Therefore, for each value of $n = 0, 1, 2, \ldots, m-1$, we obtain

$$\frac{1}{2^n}(x), \frac{1}{2^n}(y), \frac{1}{2^n}(x + y) > 1$$

and $R_1\varphi\left(\frac{1}{2^n}x, \frac{1}{2^n}y\right) = 0$ for $n = 0, 1, 2, \ldots, m-1$. Using (67) and the definition of g, we obtain

$$\begin{aligned}
\frac{|R_1 g(x, y)|}{(|x|^{-1} + |y|^{-1})} &\leq \sum_{n=m}^{\infty} \frac{\left|\varphi\left(2^{-n}\left(\frac{x+y}{2}\right)\right) - \varphi\left(2^{-n}(x + y)\right) - \frac{\varphi(2^{-n}x)\varphi(2^{-n}y)}{\varphi(2^{-n}x)+\varphi(2^{-n}y)}\right|}{2^n\left(|x|^{-1} + |y|^{-1}\right)} \\
&\leq \sum_{k=0}^{\infty} \frac{\frac{5}{2}c_1}{2^k 2^m\left(|x|^{-1} + |y|^{-1}\right)} \\
&\leq \sum_{k=0}^{\infty} \frac{\frac{5}{2}c_1}{2^k} = \frac{5}{2}c_1\left(1 - \frac{1}{2}\right)^{-1} = 5c_1, \text{ for all } x, y \in \mathbb{R}\text{-}\{0\}.
\end{aligned}$$

That is, the inequality (65) holds true. Now, assume that there exists a reciprocal mapping $r : \mathbb{R}\text{-}\{0\} \to \mathbb{R}$ satisfying (66). Therefore, we have

$$|g(x)| \leq (\delta + 1)|x|^{-1}. \tag{68}$$

However, we can choose a positive integer p with $pc_1 > \delta + 1$. If $x \in (1, 2^{p-1})$, then $2^{-n}x \in (1, \infty)$ for all $n = 0, 1, 2, \ldots, p-1$ and therefore

$$|g(x)| = \sum_{n=0}^{\infty} \frac{\varphi(2^{-n}x)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\frac{c_1}{2^{-n}x}}{2^n} = \frac{pc_1}{x} > (\delta + 1)x^{-1}$$

which contradicts (5.4). Therefore, the reciprocal type functional equation (12) for $j = 1$ is not stable for $\alpha = -1$ in Corollaries 3.3 and 4.3. □

The following example illustrates the fact that functional equation (12), for $j = 1$ is not stable for $\lambda = -1$ $\left(\text{when } p = -\frac{1}{2}, q = -\frac{1}{2}\right)$ in Corollaries 3.5 and 4.5.

Example 5.2. *Let $\varphi : \mathbb{R}\text{-}\{0\} \to \mathbb{R}$ be a mapping defined by*

$$\varphi(x) = \begin{cases} \frac{c_2}{x} & \text{for } x \in (1, \infty) \\ c_2 & \text{otherwise} \end{cases}$$

where c_2 is a constant, and define a mapping $g : \mathbb{R}\text{-}\{0\} \to \mathbb{R}$ by

$$g(x) = \sum_{n=0}^{\infty} \frac{\varphi\left(2^{-n}x\right)}{2^n}, \text{ for all } x \in \mathbb{R}\text{-}\{0\}.$$

Then the mapping g satisfies the inequality

$$\left|R_1 g(x, y)\right| \leq 5c_2 \left(|x|^{-\frac{1}{2}}|y|^{-\frac{1}{2}} + \left(|x|^{-1} + |y|^{-1}\right)\right) \tag{69}$$

for all $x, y \in \mathbb{R}\text{-}\{0\}$. Therefore there do not exist a reciprocal mapping $r : \mathbb{R}\text{-}\{0\} \to \mathbb{R}$ with $r(x) = \frac{1}{x}$ and a constant $\delta > 0$ such that

$$|g(x) - r(x) \leq \delta|x|^{-1} \tag{70}$$

for all $x \in \mathbb{R}\text{-}\{0\}$.

Proof. The proof is analogous to the proof of Example 5.1. □

The following example illustrates the fact that functional equation (12), for $j = 2$ is not stable for $\alpha = -1$ in Corollaries 3.3 and 4.3.

Example 5.3. *Let φ and g be mappings defined as in Example 5.1. Then the mapping g satisfies the inequality*

$$\left|R_2 g(x, y)\right| \leq 7c_1 \left(|x|^{-1} + |y|^{-1}\right) \tag{71}$$

for all $x, y \in \mathbb{R}\text{-}\{0\}$. Therefore there do not exist a reciprocal mapping $r : \mathbb{R}\text{-}\{0\} \to \mathbb{R}$ with $r(x) = \frac{1}{x}$ and a constant $\delta > 0$ such that

$$|g(x) - r(x) \leq \delta|x|^{-1} \tag{72}$$

for all $x \in \mathbb{R}\text{-}\{0\}$.

Proof. The proof is similar to the proof of Example 5.1. □

The following example illustrates the fact that functional equation (12), for $j = 2$ is not stable for $\lambda = -1$ $\left(\text{when } p = -\frac{1}{2}, q = -\frac{1}{2}\right)$ in Corollaries 3.5 and 4.5.

Example 5.4. *Let φ and g be mappings defined as in Example 5.1. Then the mapping g satisfies the inequality*

$$\left|R_2 g(x, y)\right| \leq 7c_2 \left(|x|^{-\frac{1}{2}}|y|^{-\frac{1}{2}} + \left(|x|^{-1} + |y|^{-1}\right)\right) \tag{73}$$

for all $x, y \in \mathbb{R}\text{-}\{0\}$. Therefore there do not exist a reciprocal mapping $r : \mathbb{R}\text{-}\{0\} \to \mathbb{R}$ with $r(x) = \frac{1}{x}$ and a constant $\delta > 0$ such that

$$|g(x) - r(x) \leq \delta|x|^{-1} \tag{74}$$

for all $x \in \mathbb{R}\text{-}\{0\}$.

Proof. The proof is similar to the proof of Example 5.2. □

References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math.Soc. Japan, 2(1950), 64-66.
[2] C. Alsina, On the stability of a functional equation, General Inequalities, Vol.5, Oberwolfach, Birkhauser, Basel, (1987), 263-271.
[3] P.W. Cholewa, Remarks on the stability of functional equations, Aequationes Math. 27 (1984), 76-86.
[4] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific Publishing Co., Singapore, New Jersey, London, 2002.
[5] Z. Gajda, On stability of additive mappings, Int. J. Math. Math. Sci. 14 (1991), 431-434.
[6] A. Grabiec, The generalized Hyers-Ulam stability of a class of functional equations, Publ. Math. Debrecen, 48 (1996), 217-235.
[7] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951), 241-244.
[8] J.A. Fridy, On statistical convergence, Analysis 5 (1985), 301-313.
[9] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), 431-436.
[10] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222-224.
[11] D.H. Hyers, G. Isac and Th.M. Rassias, Stability of Functional Equations in Several variables, Birkahuser, Basel, 1998.
[12] G. Isac and Th.M. Rassias, Stability of ψ- additive mappings: applications to nonlinear analysis, Int. J. Math. Math. Sci. 19(2) (1996), 219-228.
[13] S.M. Jung, Hyers-Ulam-Rassias stability of Jensen's equation and its application, Proc. Amer. Math. Soc., 126 (1998), 3137-3143.
[14] S.M. Jung, Hyers-Ulam-Rassias stability of Functional Equations in Mathematical Analysis, Hardonic press, Palm Harbor, 2001.
[15] S. Karakus, Statistical convergence on probabilistic normed spaces, Math. Commun. 12 (2007), 11-23.
[16] E. Kolk, The statistical convergence in Banach spaces, Tartu Ul. Toime. 928, (1991), 41-52.
[17] B. Margolis and J. Diaz, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 74 (1968), 305-309.
[18] D. Mihet and V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, J. Math. Anal. Appl. 343 (2008), 567-572.
[19] S. A. Mohiuddine, Stability of Jensen functional equation in intuitionist fuzzy normed space, Chaos, Solitons & Fractals, 42(5) (2009), 2989-2996.
[20] S. A. Mohiuddine, M. Cancan, and H. Sevli, Intuitionistic fuzzy stability of a Jensen functional equation via fixed point technique, Mathematical and Computer Modelling, 54(9-10) (2011), 2403-2409.
[21] S. A. Mohiuddine and H. Sevli, Stability of Pexiderized quadratic functional equation in intuitionistic fuzzy normed space, Journal of Computational and Applied Mathematics, 235 (8) (2011), 2137-2146.
[22] S. A. Mohiuddine and A. Alotaibi, Fuzzy stability of a cubic functional equation via fixed point technique, Advances in Difference Equations, 2012, 2012:48.
[23] M. Mursaleen and S.A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, J. Computat. Anal. Math. 233 (2009), 142-149.
[24] M. Mursaleen and S. A. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos, Solitons & Fractals, 41 (2009), 2414-2421.
[25] M. Mursaleen and S. A. Mohiuddine, On stability of a cubic functional equation in intuitionistic fuzzy normed spaces, Chaos, Solitons & Fractals, 42(5) (2009), 2997-3005.
[26] Ch. Park and D.Y. Shin, Functional equations in paranormed spaces, Adv. differences equations, 2012, 2012:123 (23 pages).
[27] Ch. Park, Stability of an AQCQ-functional equation in paranormed space, Adv. differences equations, 2012, 2012:148 (19 pages).
[28] J.M. Rassias, On approximation of approximately linear mappings by linear mappings, J. Funct. Anal. 46 (1982), 126-130.
[29] K. Ravi, M. Arunkumar and J.M. Rassias, Ulam stability for the orthogonally general Euler-Lagrange type functional equation, Int. J. Math. Stat. 3(A08)(2008), 36-46.
[30] K. Ravi and B.V. Senthil Kumar, Ulam-Gavruta-Rassias stability of Rassias Reciprocal functional equation, Global Journal of App. Math. and Math. Sci. 3(1-2) (Jan-Dec 2010), 57-79.
[31] K. Ravi, J.M. Rassias and B.V. Senthil Kumar, Ulam stability of Generalized Reciprocal Funtional Equation in several variables, Int. J. App. Math. and Stat. 19 (D10) (2010), 1-19.
[32] K. Ravi, J.M. Rassias and B.V. Senthil Kumar, A fixed point approach to the generalized Hyers-Ulam stability of reciprocal difference and adjoint Funtional Equations, Thai Journal of Math. 8(3) (2010), 469-481.
[33] K. Ravi, J.M. Rassias and B.V. Senthil Kumar, Ulam stability of Reciprocal Difference and Adjoint Funtional Equations, The Australian J. Math. Anal.Appl. 8(1), Art.13 (2011), 1-18.
[34] K. Ravi, E. Thandapani and B.V. Senthil Kumar, Stability of reciprocal type functional equations, PanAmerican Math. Journal, 21(1) (2011), 59-70.
[35] T. Salat, On the statistically convergent sequences of real numbers, Math. Slovaca 30 (1980), 139-150.
[36] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951), 73-34.
[37] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc.Amer.Math. Soc. 72 (1978), 297-300.
[38] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, Acta. Appl. Math. 62 (2000), 23-130.
[39] S.M. Ulam, A collection of mathematical problems, Interscience Publishers, Inc.New York, 1960.
[40] A. Wilansky, Modern Methods in Topological vector space, McGraw-Hill International Book Co., New York (1978).