Functional Analysis,
Approximation and
Computation
5:2 (2013), 1–9

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Convergence theorems for a finite family of T-Zamfirescu operators

Priya Raphael[a], Shaini Pulickakunnel[b]

[a] *Department of Mathematics, Sam Higginbottom Institute of Agriculture, Technology and Sciences, Allahabad - 211007, India*
[b] *Department of Mathematics, Sam Higginbottom Institute of Agriculture, Technology and Sciences, Allahabad - 211007, India*

Abstract. In this paper, we introduce two iterative procedures and establish the strong convergence of these procedures using the common fixed points of a finite family of T-Zamfirescu operators in normed linear spaces. Our results improves and extends the corresponding result of Rafiq [19] and some other results in contemporary literature.

1. Introduction and Preliminaries

The classical Banach's contraction principle which was published in 1922 is one of the most useful results in fixed point theory. In 1968, Kannan [9] established a fixed point theorem, extending Banach's contraction principle to mappings that need not be continuous. Kannan's theorem was followed by a spate of papers, devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require continuity of the corresponding mappings. One of them, actually a sort of dual of Kannan's fixed point theorem is due to Chatterjea [5]. Another important result on fixed points for contractive type mapping in the setting of compact metric space is generally attributed to Edelstein [7].
The concepts of T-Banach contraction and T-contractive mappings were introduced by Beiranvand et al. [2] in 2009 and then they extended Banach's contraction principle and Edelstein's fixed point theorem. Followed by this, Moradi [11] introduced T-Kannan contractive mappings, extending in this way, the well-known Kannan fixed point theorem [9]. It was Morales and Rojas [12] who introduced the notion of T-Chatterjea mappings and obtained sufficient conditions for the existence of a unique fixed point of these mappings in complete cone metric spaces. Afterwards, Morales and Rojas [14] introduced T-Zamfirescu operators and obtained sufficient conditions for the existence of a unique fixed point of this type of mappings in the framework of complete cone metric spaces.
In the past few decades, the convergence problems of implicit or nonimplicit iterative procedures to a common fixed point for a finite family of nonexpansive mappings, asymptotically nonexpansive mappings, pseudocontractive mappings, quasinonexpansive operators or Zamfirescu operators in arbitrary Banach spaces, Hilbert spaces, uniformly convex Banach spaces or normed linear spaces have been considered by

2010 *Mathematics Subject Classification*. 47H09, 47H10, 47J25.
Keywords. Strong convergence; T-Zamfirescu operators; iterative procedures; common fixed points.
Received: 28 January 2013; Accepted: 14 March 2013
Communicated by Vladimir Müller
Email addresses: priyaraphael77@gmail.com Corresponding author (Priya Raphael), shainipv@gmail.com (Shaini Pulickakunnel)

several authors like Bauschke [1], Chang et al. [4], Chidume and Shahzad [6], Goebel and Kirk [8], Khan [10], Osilike [15], Plubtieng et al. [16], Pulickakunnel and Singh [17, 18], Rafiq [19], Saluja [20], Saluja and Nashine [21], Su and Li [22], Tan and Xu [23], Wangkeeree [24], Wittmann [25], Xu and Ori [26], Zhou and Chang [28] and many others.

In 2001, Xu and Ori [26] introduced the following implicit iteration process for a finite family of nonexpansive mappings:

Let K be a nonempty, closed, convex subset of a normed linear space E. Let $\{T_i : i \in I\}$ (I=1,2,...,N) be a finite family of nonexpansive mappings. For an initial point $x_0 \in K$, define the sequence $\{x_n\}$ as

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \quad n \geq 1,$$

where $T_n = T_{n(modN)}$ (here the mod N function takes values in I) and $\{\alpha_n\}_{n=1}^{\infty}$ a real sequence in (0,1) . They proved the weak convergence of this process to a common fixed point of a finite family of nonexpansive mappings defined in a Hilbert space.

Followed by this, Rafiq [19] suggested the following implicit iteration process with errors for a finite family of Z-operators:

Let K be a nonempty, closed, convex subset of normed linear space E and $x_0 \in K$. Define the sequence $\{x_n\}$ as

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n + u_n, \quad n \geq 1, \tag{1}$$

where $T_n = T_{n(modN)}$ (here the mod N function takes values in I), $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in (0,1) and $\{u_n\}$ is a summable sequence in K. He established the strong convergence of this iteration process to a common fixed point for a finite family of Z-operators in normed linear spaces.

Here we recall the definitions of the following classes of generalized T-contraction type mappings as given by Morales and Rojas [13]:

Definition 1.1. *Let (M, d) be a metric space and $T, S : M \to M$ be two functions. A mapping S is said to be T-**Banach contraction** (TB contraction) if there exists $a \in [0, 1)$ such that*

$$d(TSx, TSy) \leq ad(Tx, Ty), \qquad \text{for all } x, y \in M.$$

When we substitute $T = I$, the identity map, in the above definition we obtain the definition of *Banach's contraction* [3].

Definition 1.2. *Let (M, d) be a metric space and $T, S : M \to M$ be two functions. A mapping S is said to be T-**Kannan contraction** (TK contraction) if there exists $b \in [0, \frac{1}{2})$ such that*

$$d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)], \qquad \text{for all } x, y \in M.$$

In Definition 1.2, if we take $T = I$, the identity map, then we get the definition of *Kannan operator* [9].

Definition 1.3. *Let (M, d) be a metric space and $T, S : M \to M$ be two functions. A mapping S is said to be T-**Chatterjea contraction** (TC contraction) if there exists $c \in [0, \frac{1}{2})$ such that*

$$d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)] \qquad \text{for all } x, y \in M.$$

If we substitute $T = I$, the identity map, in Definition 1.3 we obtain the definition of Chatterjea operator [5].

Definition 1.4. *Let (M, d) be a metric space and $T, S : M \to M$ be two functions. A mapping S is said to be T-**Zamfirescu operator** (TZ operator) if there are real numbers $0 \leq a < 1, 0 \leq b < \frac{1}{2}, 0 \leq c < \frac{1}{2}$ such that for all $x, y \in M$ at least one of the conditions is true:*
(TZ_1) : $d(TSx, TSy) \leq ad(Tx, Ty)$,
(TZ_2) : $d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)]$,
(TZ_3) : $d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)]$.

When we take $T = I$, the identity map, in the above definition we obtain the definition of Zamfirescu operator [27].

Motivated by Xu and Ori [26], Rafiq [19] and Morales and Rojas [13], we introduce and study the following:

Let E be a normed linear space, K a closed, convex subset of E and $x_0 \in K$. Let $T : K \to K$ and let $\{S_i\}_{i=1}^N : K \to K$ be N, T-Zamfirescu operators with $F = \cap_{i=1}^N F(S_i) \neq \phi$. Let $\{\alpha_n\}$ be a real sequence in (0,1). Define the sequence $\{Tx_n\}$ as follows:

$$Tx_n = \alpha_n Tx_{n-1} + (1-\alpha_n)TS_nx_n, \qquad \text{for all } n \geq 1, \tag{2}$$

where $S_n = S_{n(modN)}$ (the $modN$ function takes the values in $\{1, 2, ..., N\}$).

We also introduce the following two-step iterative process for a finite family of T-Zamfirescu operators $\{S_i\}_{i=1}^N : K \to K$ with $F = \cap_{i=1}^N F(S_i) \neq \phi$ where $T : K \to K$ and define the sequence $\{Tx_n\}$ as follows:

$$Tx_n = \alpha_n Tx_{n-1} + (1-\alpha_n)S_nTy_n, \tag{3}$$

$$Ty_n = \beta_n Tx_{n-1} + (1-\beta_n)TS_nx_n, \qquad \text{for all } n \geq 1,$$

where $S_n = S_{n(modN)}$ (the $modN$ function takes the values in $\{1, 2, ..., N\}$), $\{\alpha_n\}, \{\beta_n\}$ are real sequences in [0,1] and T and S_n, $(n = 1, 2, ..., N)$ are commuting mappings.

Substituting $T = I$, the identity map, in (3) we get the following iteration process for a finite family of Zamfirescu operators $\{S_i\}_{i=1}^N : K \to K$ with $F = \cap_{i=1}^N F(S_i) \neq \phi$:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)S_ny_n, \tag{4}$$

$$y_n = \beta_n x_{n-1} + (1-\beta_n)S_nx_n, \qquad \text{for all } n \geq 1,$$

where $S_n = S_{n(modN)}$ (the $modN$ function takes the values in $\{1, 2, ..., N\}$) and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in [0,1].

The aim of this paper is to establish strong convergence of the iterative processes defined by (2) and (3) using the common fixed points of a finite family of T-Zamfirescu operators in normed linear spaces. Our results generalizes the corresponding result of Rafiq [19].

In the sequel, we need the following lemma :

Lemma 1.1. Let $\{r_n\}, \{s_n\}$ and $\{t_n\}$ be sequences of nonnegative numbers satisfying

$$r_{n+1} \leq (1-s_n)r_n + s_nt_n, \qquad \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} r_n = 0$.

2. Main Results

Theorem 2.1. *Let E be a normed linear space, K a nonempty, closed, convex subset of E. Let $T : K \to K$ and let $\{S_i\}_{i=1}^N : K \to K$ be N, T-Zamfirescu operators with $F = \cap_{i=1}^N F(S_i) \neq \phi$. From an initial point $x_0 \in K$, define the sequence $\{Tx_n\}$ by the iterative process defined by (2) where $\{\alpha_n\}$ is a real sequence in $(0, 1)$ satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{Tx_n\}$ converges strongly to Tw where w is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in K.*

Proof. From the assumption $F = \cap_{i=1}^N F(S_i) \neq \phi$, it follows that the operators $\{S_i\}_{i=1}^N$ have a common fixed point in K, say w. Consider $x, y \in K$. Since each S_i, $i = 1, 2, ..., N$, is a T-Zamfirescu operator, each S_i satisfies at least one of the conditions (Tz_1), (Tz_2) and (Tz_3).

If (TZ_2) holds, then for any $x, y \in K$,

$$\begin{aligned}\left\|TS_ix - TS_iy\right\| &\leq b[\|Tx - TS_ix\| + \left\|Ty - TS_iy\right\|]\\ &\leq b[\|Tx - TS_ix\| + \left\|Ty - Tx\right\| + \|Tx - TS_ix\| + \|TS_ix - TS_iy\|],\end{aligned}$$

which implies

$$(1-b)\left\|TS_ix - TS_iy\right\| \leq b\left\|Tx - Ty\right\| + 2b\,\|Tx - TS_ix\|.$$

Since $0 \leq b < \frac{1}{2}$, we get

$$\left\|TS_ix - TS_iy\right\| \leq \frac{b}{1-b}\left\|Tx - Ty\right\| + \frac{2b}{1-b}\|Tx - TS_ix\|. \tag{5}$$

Similarly, if (TZ_3) holds, then we have for any $x, y \in K$,

$$\begin{aligned}\left\|TS_ix - TS_iy\right\| &\leq c[\left\|Tx - TS_iy\right\| + \left\|Ty - TS_ix\right\|]\\ &\leq c\left[\|Tx - TS_ix\| + \left\|TS_ix - TS_iy\right\| + \left\|Ty - Tx\right\| + \|Tx - TS_ix\|\right],\end{aligned}$$

which implies

$$(1-c)\left\|TS_ix - TS_iy\right\| \leq c\left\|Tx - Ty\right\| + 2c\,\|Tx - TS_ix\|.$$

Since $0 \leq c < \frac{1}{2}$, we get

$$\left\|TS_ix - TS_iy\right\| \leq \frac{c}{1-c}\left\|Tx - Ty\right\| + \frac{2c}{1-c}\|Tx - TS_ix\|. \tag{6}$$

Denote

$$\delta = max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}.$$

Then we have $0 \leq \delta < 1$ and from (TZ_1), (5) and (6) we get that the inequality

$$\left\|TS_ix - TS_iy\right\| \leq \delta\left\|Tx - Ty\right\| + 2\delta\,\|Tx - TS_ix\| \tag{7}$$

holds for all $x, y \in K$ and for every $i \in \{1, 2, ..., N\}$. Let $\{Tx_n\}$ be the iteration process defined by (2) and $x_0 \in K$ be an initial point.

Consider

$$\begin{aligned}\|Tx_n - Tw\| &= \|\alpha_n Tx_{n-1} + (1-\alpha_n)TS_nx_n - Tw\|\\ &= \|\alpha_n(Tx_{n-1} - Tw) + (1-\alpha_n)(TS_nx_n - Tw)\|\\ &\leq \alpha_n\|Tx_{n-1} - Tw\| + (1-\alpha_n)\|TS_nx_n - Tw\|.\end{aligned} \tag{8}$$

Since $S_iw = w, S_n = S_{n(modN)}$ and the $modN$ function takes values in $\{1, 2, ..., N\}$, taking $x = w, y = x_n$ in (7), we obtain

$$\|TS_nw - TS_nx_n\| \leq \delta\,\|Tw - Tx_n\| + 2\delta\,\|Tw - TS_nw\|,$$

which implies

$$\|TS_nx_n - Tw\| \leq \delta\,\|Tx_n - Tw\|. \tag{9}$$

Using (9) in (8), we have the inequality

$$\|Tx_n - Tw\| \leq \alpha_n\|Tx_{n-1} - Tw\| + (1-\alpha_n)\delta\,\|Tx_n - Tw\|,$$

which gives

$$[1 - (1-\alpha_n)\delta]\,\|Tx_n - Tw\| \leq \alpha_n\|Tx_{n-1} - Tw\|.$$

Thus we get

$$\|Tx_n - Tw\| \le \frac{\alpha_n}{1-(1-\alpha_n)\delta} \|Tx_{n-1} - Tw\|. \tag{10}$$

Let $A_n = \alpha_n, B_n = 1 - \delta(1-\alpha_n)$.
Now, consider

$$\begin{aligned} 1 - \frac{A_n}{B_n} &= 1 - \frac{\alpha_n}{1-\delta(1-\alpha_n)} \\ &= \frac{1-\delta(1-\alpha_n)-\alpha_n}{1-\delta(1-\alpha_n)} \\ &= \frac{(1-\delta)-\alpha_n(1-\delta)}{1-\delta(1-\alpha_n)} \\ &= \frac{(1-\delta)(1-\alpha_n)}{1-\delta(1-\alpha_n)} \\ &\ge (1-\delta)(1-\alpha_n). \end{aligned}$$

Thus we get

$$\frac{A_n}{B_n} \le 1 - (1-\delta)(1-\alpha_n). \tag{11}$$

From (10), we have the inequality

$$\|Tx_n - Tw\| \le \frac{A_n}{B_n} \|Tx_{n-1} - Tw\|.$$

Using (11) in the above inequality, we get

$$\|Tx_n - Tw\| \le [1-(1-\delta)(1-\alpha_n)] \|Tx_{n-1} - Tw\|.$$

Now, setting $r_n = \|Tx_{n-1} - Tw\|, s_n = (1-\delta)(1-\alpha_n)$ and using the facts that $0 \le \delta < 1, \alpha_n \in (0,1)$ and $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, from Lemma 1.1 we obtain

$$\lim_{n\to\infty} \|Tx_{n-1} - Tw\| = 0.$$

Hence, we conclude that $\{Tx_n\}$ converges strongly to Tw where w is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in K.

Since T-Kannan contraction and T-Chatterjea contraction are both included in the class of T-Zamfirescu operators, by Theorem 2.1, we obtain the corresponding convergence results of the iteration process defined by (2) for finite families of these classes of operators as corollaries.

Corollary 2.2. *Let E be a normed linear space, K a nonempty, closed, convex subset of E. Let $T : K \to K$ and let $\{S_i\}_{i=1}^N : K \to K$ be N, T-Kannan contractions with $F = \cap_{i=1}^N F(S_i) \neq \phi$. From an initial point $x_0 \in K$, define the sequence $\{Tx_n\}$ by the iterative process defined by (2) where $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{Tx_n\}$ converges strongly to Tw where w is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in K.*

Corollary 2.3. *Let E be a normed linear space, K a nonempty, closed, convex subset of E. Let $T : K \to K$ and let $\{S_i\}_{i=1}^N : K \to K$ be N, T-Chatterjea contractions with $F = \cap_{i=1}^N F(S_i) \neq \phi$. From an initial point $x_0 \in K$, define the sequence $\{Tx_n\}$ by the iterative process defined by (2) where $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{Tx_n\}$ converges strongly to Tw where w is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in K.*

When we substitute $T = I$, the identity map, in Theorem 2.1 we get the result proved by Rafiq [19], as a corollary to our result where the error term is not taken into consideration.

Corollary 2.4. [19, *Theorem 2.1*] *Let E be a normed linear space, K a nonempty, closed, convex subset of E. Let $\{T_i\}_{i=1}^N : K \to K$ be N, Zamfirescu operators with $F = \cap_{i=1}^N F(T_i) \neq \phi$. From an initial point $x_0 \in K$, define the sequence $\{x_n\}$ by the iterative process defined by (1) where $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and $\|u_n\| = O(1-\alpha_n)$. Then $\{x_n\}$ converges strongly to a common fixed point of the operators $\{T_i\}_{i=1}^N$ in K.*

Theorem 2.5. *Let E be a normed linear space, K a nonempty, closed, convex subset of E. Let $T : K \to K$ and let $\{S_i\}_{i=1}^N : K \to K$ be N, T-Zamfirescu operators with $F = \cap_{i=1}^N F(S_i) \neq \phi$ where T and $S_n, (n = 1, 2, ..., N)$ are commuting mappings. From an initial point $x_0 \in K$, define the sequence $\{Tx_n\}$ by the iterative process defined by (3) where $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0,1]$ satisfying $\sum_{n=1}^{\infty}\beta_n(1-\alpha_n) = \infty$. Then $\{Tx_n\}$ converges strongly to Tw where w is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in K.*

Proof. From the assumption $F = \cap_{i=1}^N F(S_i) \neq \phi$, it follows that the operators $\{S_i\}_{i=1}^N$ have a common fixed point in K, say w. Consider $x, y \in K$. Since each $S_i, i = 1, 2, ..., N$, is a T-Zamfirescu operator, following as in the proof of Theorem 2.1, we get that the inequality,

$$\left\|TS_ix - TS_iy\right\| \leq \delta \left\|Tx - Ty\right\| + 2\delta \left\|Tx - TS_ix\right\| \tag{12}$$

holds for all $x, y \in K$ and for every $i \in \{1, 2, ..., N\}$. Let $\{Tx_n\}$ be the iteration process defined by (3) and $x_0 \in K$ be an initial point.
Consider

$$\begin{aligned} \|Tx_n - Tw\| &= \left\|\alpha_n Tx_{n-1} + (1-\alpha_n)S_nTy_n - Tw\right\| \\ &= \left\|\alpha_n(Tx_{n-1} - Tw) + (1-\alpha_n)(S_nTy_n - Tw)\right\| \\ &\leq \alpha_n \|Tx_{n-1} - Tw\| + (1-\alpha_n)\left\|S_nTy_n - Tw\right\|. \end{aligned} \tag{13}$$

Since $S_iw = w, S_n = S_{n(modN)}$ and the *modN* function takes values in $\{1, 2, ..., N\}$, taking $x = w, y = y_n$ in (12), we obtain

$$\left\|TS_nw - TS_ny_n\right\| \leq \delta \left\|Tw - Ty_n\right\| + 2\delta \left\|Tw - TS_nw\right\|,$$

which implies

$$\left\|TS_ny_n - Tw\right\| \leq \delta \left\|Ty_n - Tw\right\|.$$

Since T and S_n are commuting mappings, the above inequality becomes

$$\left\|S_nTy_n - Tw\right\| \leq \delta \left\|Ty_n - Tw\right\|. \tag{14}$$

Using (14) in (13), we obtain

$$\|Tx_n - Tw\| \leq \alpha_n \|Tx_{n-1} - Tw\| + (1-\alpha_n)\delta \left\|Ty_n - Tw\right\|. \tag{15}$$

From (3), it follows that

$$\begin{aligned} \left\|Ty_n - Tw\right\| &= \left\|\beta_n Tx_{n-1} + (1-\beta_n)TS_nx_n - Tw\right\| \\ &= \left\|\beta_n(Tx_{n-1} - Tw) + (1-\beta_n)(TS_nx_n - Tw)\right\| \\ &\leq \beta_n \|Tx_{n-1} - Tw\| + (1-\beta_n)\|TS_nx_n - Tw\|. \end{aligned} \tag{16}$$

Taking $x = w$ and $y = x_n$ in (12), we get

$$\|TS_nw - TS_nx_n\| \leq \delta \|Tw - Tx_n\| + 2\delta \|Tw - TS_nw\|,$$

which gives

$$\|TS_n x_n - Tw\| \leq \delta \|Tx_n - Tw\| . \tag{17}$$

Using (17) in (16), we get

$$\left\|Ty_n - Tw\right\| \leq \beta_n \|Tx_{n-1} - Tw\| + (1 - \beta_n)\delta \|Tx_n - Tw\| . \tag{18}$$

Using (18) in (15), we obtain that

$$\begin{aligned} \|Tx_n - Tw\| &\leq \alpha_n \|Tx_{n-1} - Tw\| + (1 - \alpha_n)\delta\beta_n \|Tx_{n-1} - Tw\| \\ &\quad + (1 - \alpha_n)(1 - \beta_n)\delta^2 \|Tx_n - Tw\| \\ &= [\alpha_n + (1 - \alpha_n)\delta\beta_n] \|Tx_{n-1} - Tw\| + (1 - \alpha_n)(1 - \beta_n)\delta^2 \|Tx_n - Tw\| . \end{aligned}$$

Thus we get the inequality,

$$[1 - (1 - \alpha_n)(1 - \beta_n)\delta^2] \|Tx_n - Tw\| \leq [\alpha_n + (1 - \alpha_n)\delta\beta_n] \|Tx_{n-1} - Tw\| ,$$

which implies that

$$\|Tx_n - Tw\| \leq \frac{\alpha_n + (1 - \alpha_n)\delta\beta_n}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2} \|Tx_{n-1} - Tw\| . \tag{19}$$

Now, let $A_n = \alpha_n + (1 - \alpha_n)\delta\beta_n$ and $B_n = 1 - (1 - \alpha_n)(1 - \beta_n)\delta^2$. Then,

$$\begin{aligned} 1 - \frac{A_n}{B_n} &= 1 - \frac{\alpha_n + (1 - \alpha_n)\delta\beta_n}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2} \\ &= \frac{1 - [(1 - \alpha_n)(1 - \beta_n)\delta^2 + \alpha_n + (1 - \alpha_n)\delta\beta_n]}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2} . \end{aligned}$$

Since $1 - (1 - \alpha_n)(1 - \beta_n)\delta^2 \leq 1$, from the above inequality, we get

$$1 - \frac{A_n}{B_n} \geq 1 - [(1 - \alpha_n)(1 - \beta_n)\delta^2 + \alpha_n + (1 - \alpha_n)\delta\beta_n],$$

which implies

$$\frac{A_n}{B_n} \leq (1 - \alpha_n)(1 - \beta_n)\delta^2 + \alpha_n + (1 - \alpha_n)\delta\beta_n.$$

Since $0 \leq \delta < 1$ and $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$, this gives

$$\begin{aligned} \frac{A_n}{B_n} &\leq (1 - \alpha_n)(1 - \beta_n) + \alpha_n + (1 - \alpha_n)\delta\beta_n \\ &= 1 - \beta_n(1 - \alpha_n) + (1 - \alpha_n)\delta\beta_n \\ &= 1 - \beta_n(1 - \alpha_n)(1 - \delta). \end{aligned} \tag{20}$$

From(19), we have the inequality

$$\|Tx_n - Tw\| \leq \frac{A_n}{B_n} \|Tx_{n-1} - Tw\| .$$

Using (20) in the above inequality, we obtain

$$\|Tx_n - Tw\| \leq [1 - \beta_n(1 - \alpha_n)(1 - \delta)] \|Tx_{n-1} - Tw\| .$$

Now, setting $r_n = \|Tx_{n-1} - Tw\|, s_n = \beta_n(1-\alpha_n)(1-\delta)$ and using the facts that $0 \le \delta < 1, \{\alpha_n\}, \{\beta_n\} \in [0,1]$ and $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) = \infty$, it follows from Lemma 1.1 that

$$\lim_{n\to\infty} \|Tx_n - Tw\| = 0.$$

Hence, $\{Tx_n\}$ converges strongly to Tw where w is the common fixed point of the operators $\{S_i\}_{i=1}^N$.

By Theorem 2.5, we get the corresponding convergence results of the iterative process defined by (3) for finite families of T-Kannan contraction and T-Chatterjea contraction as corollaries to our result as both these classes of operators are included in the class of T-Zamfirescu operators.

Corollary 2.6. *Let E be a normed linear space, K a nonempty, closed, convex subset of E. Let $T : K \to K$ and let $\{S_i\}_{i=1}^N : K \to K$ be N, T-Kannan contractions with $F = \cap_{i=1}^N F(S_i) \neq \phi$ where T and $S_n, (n = 1, 2, ..., N)$ are commuting mappings. From an initial point $x_0 \in K$, define the sequence $\{Tx_n\}$ by the iterative process defined by (3) where $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) = \infty$. Then $\{Tx_n\}$ converges strongly to Tw where w is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in K.*

Corollary 2.7. *Let E be a normed linear space, K a nonempty, closed, convex subset of E. Let $T : K \to K$ and let $\{S_i\}_{i=1}^N : K \to K$ be N, T-Chatterjea contractions with $F = \cap_{i=1}^N F(S_i) \neq \phi$ where T and $S_n, (n = 1, 2, ..., N)$ are commuting mappings. From an initial point $x_0 \in K$, define the sequence $\{Tx_n\}$ by the iterative process defined by (3) where $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) = \infty$. Then $\{Tx_n\}$ converges strongly to Tw where w is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in K.*

Corollary 2.8. *Let E be a normed linear space, K a nonempty, closed, convex subset of E. Let $\{T_i\}_{i=1}^N : K \to K$ be N, Zamfirescu operators with $F = \cap_{i=1}^N F(T_i) \neq \phi$. From an initial point $x_0 \in K$, define the sequence $\{x_n\}$ by the iterative process defined by (4) where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of the operators $\{T_i\}_{i=1}^N$ in K.*

References

[1] H. H. Bauschke, *The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space*, J. Math. Anal. Appl. **202** (1996), 150-159.

[2] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, *Two fixed-point theorems for special mappings*, arXiv:0903.1504v1 [math.FA].

[3] V. Berinde, *Iterative approximation of fixed points*, Springer-Verlag, Berlin Heidelberg, 2007.

[4] S. S. Chang, K. K. Tan, H. W. J. Lee and Chi Kin Chan, *On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **313** (2006), 273-283.

[5] S. K. Chatterjea, *Fixed point theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727-730.

[6] C. E. Chidume and N. Shahzad, *Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings*, Nonlinear Analysis, **62** (2005), 1149-1156.

[7] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. **12** (1961), 7-10.

[8] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171-174.

[9] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **10** (1968), 71-76.

[10] S. H. Khan, *A modified iterative process for common fixed points of two finite families of nonexpansive mappings*, An. St. Univ. Ovidius Constanta, **19 (1)** (2011), 161-174.

[11] S. Moradi, *Kannan fixed point theorem on complete metric spaces and on generalized metric spaces depended on another function*, arXiv:0903.1577v1 [math.FA].

[12] J. Morales and E. Rojas, *Cone metric spaces and fixed point theorems of T-Kannan contractive mappings*, Int. J. Math. Anal. **4 (4)** (2010), 175-184.

[13] J. Morales and E. Rojas, *Some results on T-Zamfirescu operators*, Revista Notas de Matematica, **5 (1)** (2009), 64-71.

[14] J. Morales and E. Rojas, *T-Zamfirescu and T-weak contraction mappings on cone metric spaces*, arXiv:0909.1255v1 [math.FA].

[15] M. O. Osilike, *Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps*, J. Math. Anal. Appl. **294** (2004), 73-81.

[16] S. Plubtieng, R. Wangkeeree and R. Punpaeng, *On the convergence of modified Noor iterations with errors for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **322 (2)** (2006), 1018-1029.

[17] S. Pulickakunnel and N. Singh, *Approximation of common fixed points for a finite family of Zamfirescu operators*, General Mathematics, **18 (4)** (2010), 175-184.

[18] S. Pulickakunnel and N. Singh, *A new three-step iterative process for common fixed points of a finite family of Z-operators*, Journal of Contemporary Applied Mathematics, **1 (1)** (2011), 13-19.

[19] A. Rafiq, *Strong convergence of a modified implicit iteration process for a finite family of Z-operators*, Int. J. Math. Math. Sci. **2006** (2006), Article ID 10328, 1-6.

[20] G. S. Saluja, *Strong convergence for common fixed points of a pair of quasi-nonexpansive and asymptotically quasi-nonexpansive mappings*, Functional Analysis, Approximation and Computation, **2:1** (2010), 33-51.

[21] G. S. Saluja and H. K. Nashine, *Strong convergence of an implicit iteration process for a finite family of strictly asymptotically pseudocontractive mappings*, CUBO A Mathematical Journal, **13 (1)** (2011), 137-147.

[22] Y. Su and S. Li, *Composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps*, J. Math. Anal. Appl. **320** (2006), 882-891.

[23] K. K. Tan and H. K. Xu, *The nonlinear ergodic theorem for asymptotically nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc. **114** (1992), 399-404.

[24] R. Wangkeeree, *Implicit iteration process for a finite family of nonexpansive nonself mappings in Banach spaces*, Int. J. Math. Sci. Engg. Appls. **1 (1)** (2007), 1-12.

[25] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486-491.

[26] H. K. Xu and R. G. Ori, *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim. **22** (2001), 767-773.

[27] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. **23** (1972), 292-298.

[28] Y. Zhou and S. S. Chang, *Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces*, Numer. Funct. Anal. Optim. **23** (2002), 911-921.