Functional Analysis,
Approximation and
Computation
5:2 (2013), 11–16

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

On polynomially $*$-paranormal operators

Lingling Zhang[a], Atsushi Uchiyama[b], Kôtarô Tanahashi[c], Muneo Chō[d]

[a]*Department of Mathematics, North University of China, Taiyuan, P. R. China*
[b]*Department of Mathematical Science, Faculty of Science, Yamagata University, Yamagata, 990-8560, Japan*
[c]*Department of Mathematics, Tohoku Pharmaceutical University, Sendai 981-8558, Japan*
[d]*Department of Mathematics, Faculty of Science, Kanagawa University, Hiratsuka 259-1293, Japan*

Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. T is called a $*$-paranormal operator T if $\|T^*x\|^2 \leq \|T^2x\| \cdot \|x\|$ for all $x \in \mathcal{H}$. "$*$-paranormal" is a generalization of hyponormal ($TT^* \leq T^*T$), and it is known that a $*$-paranormal operator has several interesting properties. In this paper, we prove that if T is polynomially $*$-paranormal, i.e., there exists a nonconstant polynomial $q(z)$ such that $q(T)$ is $*$-paranormal, then T is isoloid and the spectral mapping theorem holds for the essential approximate point spectrum of T. Also, we prove related results.

1. Introduction

An operator T on a Hilbert space $\mathcal{H}$ is called paranormal and $*$-paranormal if $\|Tx\|^2 \leq \|T^2x\| \cdot \|x\|$ and $\|T^*x\|^2 \leq \|T^2x\| \cdot \|x\|$ for all $x \in \mathcal{H}$, respectively. There are interesting results concerning paranormal operators ([1], [2], [21]). It is well known that a paranormal operator T is normaloid, i.e., $\|T\| = r(T) = \sup\{|z| : z \in \sigma(T)\}$, moreover T is invertible then T^{-1} is also paranormal. In [3] Arora and Thukral showed that a $*$-paranormal operator is normaloid and $N(T - \lambda) \subset N((T - \lambda)^*)$ for all $\lambda \in \mathbb{C}$. Recently, in [22] Uchiyama and Tanahashi showed an example of invertible $*$-paranormal operator T such that T^{-1} is not normaloid, that is, this operator T^{-1} is not $*$-paranormal. Definitions of paranormal and $*$-paranormal are similar but those properties are different.

T is called polynomially $*$-paranormal if there exists a nonconstant polynomial $q(z)$ such that $q(T)$ is $*$-paranormal. A $*$-paranormal operator is an extension of a hyponormal operator. Several interesting properties were proved by many authors ([3], [9], [12], [16]).

Let $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, $R(T)$ and $N(T)$ denote the range and the null space of T, respectively. T is called left semi-Fredholm if $R(T)$ is closed and $\dim N(T) < \infty$ and T is called right semi-Fredholm if $R(T)$ is closed and $\dim N(T^*) = \dim R(T)^\perp < \infty$. T is called semi-Fredholm

2010 *Mathematics Subject Classification*. 47A10, 47B20.
Keywords. $*$-paranormal operator; polynomially $*$-paranormal operator; Weyl's theorem.
Received: 2 April 2013; Accepted: 22 April 2013
Communicated by Dragan S. Djordjević
This research is partially supported by Grant-in-Aid Scientific Research No. 24540195
Email addresses: zllww@126.com (Lingling Zhang), uchiyama@sci.kj.yamagata-u.ac.jp (Atsushi Uchiyama), tanahasi@tohoku-pharm.ac.jp (Kôtarô Tanahashi), chiyom01@kanagawa-u.ac.jp Corresponding author (Muneo Chō)

if it is either left or right semi-Fredholm and T is called Fredholm it is both left and right semi-Fredholm. The index of a semi-Fredholm operator T is defined by

$$\operatorname{ind} T = \dim N(T) - \dim R(T)^{\perp} = \dim N(T) - \dim N(T^*).$$

T is called Weyl if it is a Fredholm operator of index zero. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_w(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} \ : T - \lambda \ \text{ is not Fredholm }\},$$
$$\sigma_w(T) = \{\lambda \in \mathbb{C} \ : T - \lambda \ \text{ is not Weyl }\}.$$

It is known that $\sigma_e(T)$ and $\sigma_w(T)$ are non-empty compact sets and $\sigma_e(T) \subset \sigma_w(T) \subset \sigma(T)$ if $\dim \mathcal{H} = \infty$. $\pi_{00}(T)$ denotes the set of all isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T.

H. Weyl [23] studied the spectrum of all compact perturbations of self-adjoint operators and proved that Weyl's theorem holds for self-adjoint operators. This result has been extended to hyponormal operators by [4], p-hyponormal operators ($(TT^*)^p \leq (T^*T)^p$ for $0 < p \leq 1$) by [5], [8], [20], log-hyponormal operators (T is invertible and $\log(TT^*) \leq \log(T^*T)$) by [6], for polynomially (algebraically) hyponormal operators by [11].

In this paper, we prove that polynomially $*$-paranormal operators are isoloid and the spectral mapping theorem holds for the essential approximate point spectrum of T for polynomially $*$-paranormal operators. This is a generalization of Han and Kim [12], in which they proved that if $T - \lambda$ is $*$-paranormal for all $\lambda \in \mathbb{C}$, then Weyl's theorem holds for T.

2. Results

Arora and Thukral [3] showed $N(T - \lambda) \subset N\left((T - \lambda)^*\right)$ for a $*$-paranormal operator T. In case of isolated points, the following result holds. It is due to [22].

Lemma 2.1. *Let $T \in B(\mathcal{H})$ be $*$-paranormal. Let $\lambda \in \sigma(T)$ be an isolated point and E_λ be the Riesz idempotent for λ. Then*

$$E_\lambda \mathcal{H} = N(T - \lambda) = N\left((T - \lambda)^*\right).$$

In particular, E_λ is self-adjoint, i.e., it is an orthogonal projection.

An operator $T \in B(\mathcal{H})$ is said to have finite ascent if $N(T^m) = N(T^{m+1})$ for some positive integer m, and finite descent if $R(T^n) = R(T^{n+1})$ for some positive integer n.

Lemma 2.2. *Let $T \in B(\mathcal{H})$ be $*$-paranormal. Then*

$$N(T - \lambda) = N((T - \lambda)^2)$$

for $\lambda \in \mathbb{C}$. Hence $T - \lambda$ has finite ascent for $\lambda \in \mathbb{C}$.

Proof. Let $x \in N((T - \lambda)^2)$. Since $N(T - \lambda) \subset N\left((T - \lambda)^*\right)$ for $\lambda \in \mathbb{C}$ by [3], we have $(T - \lambda)x \in N(T - \lambda) \subset N\left((T - \lambda)^*\right)$. Hence

$$\|(T - \lambda)x\|^2 = \langle (T - \lambda)^*(T - \lambda)x, x \rangle = 0.$$

Hence $N((T - \lambda)^2) \subset N(T - \lambda)$. The converse is clear. □

An operator $T \in B(\mathcal{H})$ is said to have the single valued extension property if there exists no nonzero analytic function f such that $(T - z)f(z) \equiv 0$. In this case, the local resolvent $\rho_T(x)$ of $x \in \mathcal{H}$ denotes the maximal open set on which there exists unique analytic function $f(z)$ satisfying $(T - z)f(z) \equiv x$. The local spectrum $\sigma_T(x)$ of $x \in \mathcal{H}$ is defined by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ and $X_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a given set $F \subset \mathbb{C}$. Larusen [14] proved that if $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$, then T has the single valued extension property.

Theorem 2.3. *If $T \in B(\mathcal{H})$ is polynomially $*$-paranormal, then $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$.*

Proof. Let $q(T)$ be $*$-paranormal for some nonconstant polynomial $q(z)$. Let

$$q(z) - q(\lambda) = a(z - \lambda)^m \Pi_{j=1}^n (z - \lambda_j)$$

where $a \neq 0, 1 \leq m$ and $\lambda_j \neq \lambda$. Then

$$q(T) - q(\lambda) = a(T - \lambda)^m \Pi_{j=1}^n (T - \lambda_j).$$

It suffices to show that

$$N((T - \lambda)^{m+1}) \subset N((T - \lambda)^m).$$

Let $x \in N((T - \lambda)^{m+1})$. Then

$$\begin{aligned}(q(T) - q(\lambda))x &= a(T - \lambda)^m \Pi_{j=1}^n (T - \lambda + \lambda - \lambda_j)x \\ &= a\Pi_{j=1}^n (\lambda - \lambda_j)(T - \lambda)^m x.\end{aligned}$$

Hence $(q(T) - q(\lambda))^2 x = a^2 \Pi_{j=1}^n (\lambda - \lambda_j)^2 (T - \lambda)^{2m} x = 0$ by assumption. Hence $x \in N((q(T) - q(\lambda))^2) = N(q(T) - q(\lambda))$ by Lemma 2. Thus

$$(q(T) - q(\lambda))x = a\Pi_{j=1}^n (\lambda - \lambda_j)(T - \lambda)^m x = 0$$

and $x \in N((T - \lambda)^m)$. □

Corollary 2.4. *If an operator $T \in B(\mathcal{H})$ is polynomially $*$- paranormal, then T has the single valued extension property. Hence, if $\lambda \in \sigma(T)$ is an isolated point of $\sigma(T)$, then*

$$\mathcal{H}_T(\{\lambda\}) = \{x \in \mathcal{H} : \|(T - \lambda)^n x\|^{\frac{1}{n}} \to 0\} = E_\lambda \mathcal{H}$$

where E_λ denotes the Riesz idempotent for λ.

Proof. Since T has the single valued extension property by Theorem 3 and [14], the first equality follows from [14] (Corollary 2.4) and the second equality follows from [18] (p.424). □

For Theorem 6, we need a following lemma ([22] Corollary 1). In [2], Aiena and Guillen proved Theorem 6 for polynomially paranormal operators.

Lemma 2.5. *Let $T \in B(\mathcal{H})$ be $*$-paranormal. If $\sigma(T) = \{\lambda\}$, then $T = \lambda \cdot I$.*

Theorem 2.6. *If $T \in B(\mathcal{H})$ is polynomially $*$-paranormal and $\sigma(T) = \{\lambda\}$, then $T - \lambda$ is nilpotent.*

Proof. Let $q(T)$ be $*$-paranormal for some non-constant polynomial $q(z)$. Let

$$q(z) - q(\lambda) = a(z - \lambda)^m \Pi_{j=1}^n (z - \lambda_j)$$

where $a \neq 0, 1 \leq m$ and $\lambda_j \neq \lambda$. Then

$$q(T) - q(\lambda) = a(T - \lambda)^m \Pi_{j=1}^n (T - \lambda_j).$$

Since $\sigma(q(T)) = q(\sigma(T)) = \{q(\lambda)\}$, $q(T) = q(\lambda)$ by Lemma 5 and

$$0 = q(T) - q(\lambda) = a(T - \lambda)^m \Pi_{j=1}^n (T - \lambda_j).$$

Since $a \neq 0$ and $\Pi_{j=1}^n (T - \lambda_j)$ is invertible, this implies $(T - \lambda)^m = 0$. □

For Theorems 8 and 14, we prepare the following lemma ([22] Lemma 2).

Lemma 2.7. *If T is $*$-paranormal and $\mathcal{M}$ is an invariant subspace for T, then $T|\mathcal{M}$ is also $*$-paranormal.*

Theorem 2.8. *Weyl's theorem holds for polynomially $*$-paranormal operators.*

Proof. Let $T \in B(\mathcal{H})$ be polynomially $*$-paranormal and $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $T - \lambda$ is Weyl and not invertible. If λ is an interior point of $\sigma(T)$, there exists an open set G such that $\lambda \in G \subset \sigma(T) \setminus \sigma_w(T)$. Hence $\dim N(T - \mu) > 0$ for all $\mu \in G$ and T does not have the single valued extension property by [10] Theorem 10. This is a contradiction. Hence λ is a boundary point of $\sigma(T)$, and hence an isolated point of $\sigma(T)$ by [7] Theorem XI 6.8. Thus $\lambda \in \pi_{00}(T)$.

Let $\lambda \in \pi_{00}(T)$ and E_λ be the Reisz idempotent for λ. Then $0 < \dim N(T - \lambda) < \infty$,

$$T = T|E_\lambda\mathcal{H} \oplus T|(I - E_\lambda)\mathcal{H}$$

and

$$\sigma(T|E_\lambda\mathcal{H}) = \{\lambda\},\ \sigma(T|(I - E_\lambda)\mathcal{H}) = \sigma(T) \setminus \{\lambda\}.$$

Let $q(z)$ be a nonconstant polynomial such that $q(T)$ is $*$-paranormal. Since $q(T) = q(T)|E_\lambda\mathcal{H} \oplus q(T)|(I - E_\lambda)\mathcal{H}$, $q(T)|E_\lambda\mathcal{H} = q(T|E_\lambda\mathcal{H})$ is $*$-paranormal by Lemma 7. Hence, $T|E_\lambda\mathcal{H}$ is polynomially $*$-paranormal and there exists a positive integer m such that $(T|E_\lambda\mathcal{H} - \lambda)^m = 0$ by Theorem 6. Hence

$$\begin{aligned}\dim E_\lambda\mathcal{H} &= \dim N((T|E_\lambda\mathcal{H} - \lambda)^m)\\ &\leq \dim N((T - \lambda)^m)\\ &\leq m \dim N(T - \lambda) < \infty.\end{aligned}$$

Thus E_λ is finite rank and $\lambda \in \sigma(T) \setminus \sigma_w(T)$ by [7] Proposition XI 6.9. □

The proof of next lemma is due to Y.M. Han and W. Y. Lee [11] (in the proof of Theorem 3).

Lemma 2.9. *Let $T \in B(\mathcal{H})$ and $\lambda \in \mathbb{C}$. If $T - \lambda$ is semi-Fredholm and it has finite ascent, then* ind $(T - \lambda) \leq 0$.

Proof. If $T - \lambda$ has finite descent, then ind $(T - \lambda) = 0$ by [19] Theorem V 6.2. If $T - \lambda$ does not have finite descent, then

$$n \cdot \text{ind } (T - \lambda) = \dim N(T - \lambda)^n - \dim R((T - \lambda)^n)^\perp \to -\infty.$$

Hence ind $(T - \lambda) < 0$. □

Corollary 2.10. *If $T \in B(\mathcal{H})$ is polynomially $*$-paranormal and $T - \lambda$ is semi-Fredholm for some $\lambda \in \mathbb{C}$, then* ind $(T - \lambda) \leq 0$.

The following lemma is proved by [1] Corollary 3.72.

Lemma 2.11. *Let $T \in B(\mathcal{H})$ and $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$. Then*

$$\sigma_w(f(T)) = f(\sigma_w(T))$$

for all functions $f(z)$ which are analytic on some open neighborhood G of $\sigma(T)$.

Hence we have the following corollary by Theorem 3.

Corollary 2.12. *Let $T \in B(\mathcal{H})$ be polynomially $*$-paranormal. Then*

$$\sigma_w(f(T)) = f(\sigma_w(T))$$

for all functions $f(z)$ which are analytic on some open neighborhood G of $\sigma(T)$.

Theorem 2.13. *Let $T \in B(\mathcal{H})$ be isoloid and satisfy Weyl's theorem. If $T - \lambda$ has finite ascent for every $\lambda \in \mathbb{C}$, then Weyl's theorem holds for $f(T)$, where $f(z)$ is an analytic function on some open neighborhood of $\sigma(T)$.*

Proof. Since T is isoloid,

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$$

by [15]. Since T satisfies Weyl's theorem, by Lemma 11 it holds

$$f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T)).$$

Thus Weyl's theorem holds for $f(T)$. □

Theorem 2.14. *Polynomially $*$-paranormal operators are isoloid.*

Proof. Let $T \in B(\mathcal{H})$ be polynomially $*$-paranormal. Let λ be an isolated point of $\sigma(T)$ and E_λ be the Riesz idempotent for λ. Then

$$T = T|E_\lambda\mathcal{H} \oplus T|(I - E_\lambda)\mathcal{H}$$

and

$$\sigma(T|E_\lambda\mathcal{H}) = \{\lambda\},\ \sigma(T|(I - E_\lambda)\mathcal{H}) = \sigma(T) \setminus \{\lambda\}.$$

Since $T|E_\lambda\mathcal{H}$ is polynomially $*$-paranormal by Lemma 7 and there exists a positive integer m such that $(T|E_\lambda\mathcal{H} - \lambda)^m = 0$ by Theorem 6, hence

$$N((T|E_\lambda\mathcal{H} - \lambda)^m) = E_\lambda\mathcal{H}.$$

Since, for every $x \in E_\lambda\mathcal{H}$, $x \oplus 0 \in N((T - \lambda)^m)$, this implies $N((T - \lambda)^m) \neq \{0\}$ and $N(T - \lambda) \neq \{0\}$. Thus λ is an eigenvalue of T. □

Corollary 2.15. *If $T \in B(\mathcal{H})$ is polynomially $*$-paranormal, then Weyl's theorem holds for $f(T)$, where $f(z)$ is an analytic function on some open neighborhood of $\sigma(T)$.*

Proof. Since T is isoloid by Theorem 14,

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$$

by [15]. Theorem 8 and Corollary 12 imply that

$$f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T)).$$

Thus Weyl's theorem holds for $f(T)$. □

The essential approximate point spectrum $\sigma_{ea}(T)$ is defined by

$$\sigma_{ea}(T) = \cap\{\sigma_a(T + K) : K \text{ is a compact operator }\}$$

where $\sigma_a(T)$ is the approximate point spectrum of T. We consider the set

$$\Phi_+^-(H) = \{T \in B(H) : T \text{ is left semi-Fredholm and } \operatorname{ind} T \leq 0\}.$$

V. Rakočević [17] proved that

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(H)\}$$

and the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every function $f(z)$ which is analytic on some open neighborhood of $\sigma(T)$.

Next theorem shows that the spectral mapping theorem holds for the essential approximate point spectrum of polynomially $*$-paranormal operators.

Theorem 2.16. *Let $T \in B(\mathcal{H})$ be polynomially $*$-paranormal. Then*

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$$

for all functions $f(z)$ which are analytic on some open neighborhood G of $\sigma(T)$.

Proof. It suffices to show that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$.
We may assume that f is nonconstant. Let $\lambda \notin \sigma_{ea}(f(T))$ and

$$f(z) - \lambda = g(z)\Pi_{j=1}^{n}(z - \lambda_j)$$

where $\lambda_j \in G$ and $g(z) \neq 0$ for all $z \in G$. Then

$$f(T) - \lambda = g(T)\Pi_{j=1}^{n}(T - \lambda_j).$$

Since $\lambda \notin \sigma_{ea}(f(T))$ and all operators on the right side of above equality commute, each $(T - \lambda_j)$ is left semi-Fredholm and ind $(T - \lambda_j) \leq 0$ by Corollary 10. Thus $\lambda_j \notin \sigma_{ea}(T)$ and $\lambda \notin f(\sigma_{ea}(T))$. □

References

[1] P. Aiena, *Fredholm and local spectral theory, with application to multipliers*, Kluwer Acad. Publishers, 2004.
[2] P. Aiena and J. R. Guillen, *Weyl's theorem for perturbations of paranormal operators*, Proc. Amer. Math. Soc. **135**(2004), 2443-2451.
[3] S. C. Arora and J. K. Thukral, *On a class of operators*, Glasnik Mathematicki, **21**(41)(1986), 381–386.
[4] S. K. Berberian, *An extension of Weyl's theorem to a class of not necessary normal operators*, Michigan Math. J., **16**(1969), 273–279.
[5] M. Chō, M. Ito and S. Oshiro, *Weyl's theorem holds for p-hyponormal operators,* Glasgow Math. J., **39**(1997), 217–220.
[6] M. Chō and K. Tanahashi, *Spectral properties of log-hyponormal operators,* Scientiae Mathematicae, **2**(1999), 1–8.
[7] J. B. Conway, A Course in Functional Analysis 2nd ed., Springer-Verlag, New York, 1990.
[8] B. P. Duggal, *The Weyl spectrum of p-hyponormal operators,* Integr. Equat. Oper. Theory, **29**(1997), 197–201.
[9] B. P. Duggal, I. H. Jeon and I. H. Kim, *On $*$-paranormal contractions and properties for $*$-class A operators*, Linear Algebra and its Applications, **436**(2012), 954–962.
[10] J. K. Finch, *The single valued extension property on a Banach space,* Pacific J. Math., **58**(1975), 61–69.
[11] Y. M. Han and W. Y. Lee, *Weyl's theorem holds for algebraically hyponormal operators,* Proc. Amer. Math. Soc., **128**(2000), 2291–2296.
[12] Y. M. Han and A. H. Kim, *A note on $*$-paranormal operators*, Integr. Equat. Oper. Theory, **49**(2004), 435–444.
[13] V. Istratescu, T. Saitō and T. Yoshino, *On a class of operators,* Tôhoku Math. J., **18** (1966), 410–413.
[14] K. B. Laursen, *Operators with finite ascent,* Pacific J. Math., **152**(1992), 323–336.
[15] W. Y. Lee and S. H. Lee, *A spectral mapping theorem for the Weyl spectrum,* Glasgow Math. J., **38**(1996), 61–64.
[16] S. M. Patel, *Contributions to the study of spectraloid operators*, Ph. D. Thesis, Delhi University, 1974.
[17] V. Rakočević, *Approximate point spectrum and commuting compact perturbations,* Glasgow Math. J., **28**(1986), 193–198.
[18] F. Riesz and B. Sconfigurationz. Nagy, Functional Analysis, Frederick Ungar, New York, 1955.
[19] A. E. Taylor and D. C. Lay, Introduction to Functional Analysis 2ed, Krieger, Florida, 1980.
[20] A. Uchiyama and T. Yoshino, *Weyl's theorem for p-hyponormal or M-hyponormal operators,* Glasgow Math. J., **43**(2001), 375–381.
[21] A. Uchiyama, *On the isolated point of the spectrum of paranormal operators*, Integr. Equat. Oper. Theory, **55**(2006), 145–151.
[22] A. Uchiyama and K. Tanahashi, *A note on $*$-paranormal operators and related classes of operators,* preprint.
[23] H. Weyl, *Über beschränkte quadratische Formen, deren Differenz vollsteig ist,* Rend. Circ. Mat. Palermo, **27**(1909), 373–392.