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Adjugates of commuting-block matrices

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Abstract. For commuting-block matrices, "the determinant of the determinant is the determinant"; here we find the corresponding result for the adjugate.

0. Introduction There are four familiar ways of looking at a 4×4 matrix $T = (T_{ij})$ with entries in the field K: as an array of sixteen numbers $T_{ij} \in K$; as a single entity $T \in G = K^{4\times4}$; as a row of four columns; as a column of four rows. For a fifth interpretation think of

0.1 $$T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in A^{2\times2} \text{ with } A = K^{2\times2}$$

as a 2×2 matrix of 2×2 matrices. If we write

0.2 $$|T| \equiv \det_A(T) = ab - mn$$

then we might think of $|T|$ as some kind of "A-valued determinant" for the matrix T. There is of course an ambiguity about the order in which to write the constituents of the products ab and mn: provided however

0.3 $$\{a, m, n, b\} \subseteq A \text{ is commutative}$$

then [4],[6] indeed $|T| \in A$ will function as a "determinant" for the invertibility of $T \in G$:

0.4 $$|T| \in A^{-1} \subseteq A \iff T \in G^{-1} \subseteq G\,.$$

The transparent way to see this is to introduce an analagous "adjugate" matrix:

0.5 $$T^{\sim} \equiv \mathrm{adj}_A(T) = \begin{pmatrix} b & -m \\ -n & a \end{pmatrix} \in G = A^{2\times2} \ :$$

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given the commutivity (0.3) it is clear that

0.6 $$T^{\sim}T = |T|I = TT^{\sim} \ ,$$

and given commutivity of eight matrices $\{a, m, n, b, a', m', n', b'\}$ the reverse product law

0.7 $$\mathrm{adj}_A(T'T) = \mathrm{adj}_A(T)\mathrm{adj}_A(T')$$

will be equally clear. From classical (numerical) determinant theory, the determination (0.4) says something about two different numerical determinants:

0.8 $$\det_K\det_A(T) = 0 \Longleftrightarrow \det_K(T) = 0 \ .$$

Thus it will come as no surprise that generally, given the commutivity (0.3),

0.9 $$\det_K\det_A(T) = \det_K(T) \ .$$

This is the result of Kovacs, Silver and Williams [6], established for $n \times n$ matrices of mutually commuting $m \times m$ matrices. In this note we set out to establish the corresponding result for adjugates:

0.10 $$\mathrm{adj}_K\det_A(T)\ \mathrm{adj}_A(T) = \mathrm{adj}_K(T) = \mathrm{adj}_A(T)\ \mathrm{adj}_K\det_A(T) \ .$$

Our leverage is a surprisingly simple formula for the adjugate of a "block triangle": '

1. Definition *Suppose G is a linear algebra, with identity I and invertible group G^{-1}, over the ring A: then an adjugate on G is a partially defined mapping*

1.1 $$T \mapsto T^{\sim} : D \to D \subseteq G \ ,$$

defined on a set containing the "scalars", and closed under the action of polynomials with central coefficients,

1.2 $$A \subseteq D \ ; \ p \in \mathrm{Centre}(A)[z] \Longrightarrow p(D) \subseteq D \ ,$$

which satisfies the following three conditions: if S, T and ST are in D then

1.3 $$I^{\sim} = I \in D \ ;$$

1.4 $$(ST)^{\sim} = T^{\sim}S^{\sim} \in D;$$

1.5 $$T^{\sim}T = TT^{\sim} = |T|I \in D \ .$$

The scalar-valued mapping $T \mapsto |T| \in A$ is the associated determinant.

For example if A is commutative and G is finite dimensional then there is a familiar, if a little complicated, adjugate defined on all of G. For semisimple complex Banach algebras we can define [1],[5] the determinant and adjugate on the coset $I + \mathrm{Socle}(G)$. On the other hand if we wish to treat $G = K^{4\times4}$ as an algebra over $A = K^{2\times2}$ then we will restrict ourselves to "internally commutative" $T \in G$, which have mutually commuting entries. For the product ST of (1.2) to satisfy this condition it will be sufficient that the pair (S, T) be "jointly internally commutative". We should remark [5] that the conditions of Definition 1 do not completely determine the adjugate $T^{\sim}$: for example if we multiply $T^{\sim}$ by a power $|T|^k$ of the determinant the conditions (1.3)-(1.5) will continue to hold.

2. Theorem *Suppose adjugate mappings*

2.1 $$a \mapsto a^{\sim} \ , \ b \mapsto b^{\sim}$$

are defined on domains D_A and D_B in linear algebras A and B over the ring K: then an adjugate mapping

2.2 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \mapsto \begin{pmatrix} a & m \\ n & b \end{pmatrix}^{\sim}$$

is partially defined on the block triangles $\begin{pmatrix} A & M \\ O & B \end{pmatrix} \cup \begin{pmatrix} A & O \\ N & B \end{pmatrix} \subseteq G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *by the formulae*

2.3 $$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}^{\sim} = \begin{pmatrix} |b|a^{\sim} & -a^{\sim}mb^{\sim} \\ 0 & |a|b^{\sim} \end{pmatrix} \text{ and } \begin{pmatrix} a & 0 \\ n & b \end{pmatrix}^{\sim} = \begin{pmatrix} |b|a^{\sim} & 0 \\ -b^{\sim}na^{\sim} & |a|b^{\sim} \end{pmatrix} ,$$

so that also

2.4 $$\begin{vmatrix} a & m \\ 0 & b \end{vmatrix} = \begin{vmatrix} a & 0 \\ n & b \end{vmatrix} = |a|\,|b| .$$

The domain of definition consists of those block triangles for which

2.5 $$a \in D_A \ ; \ b \in D_B \ ; \ \{|a|, |b|\} \subseteq \mathrm{comm}(a, b, m, n) .$$

Proof. We need to check conditions (1.3)-(1.5): for example

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} |b|a^{\sim} & -a^{\sim}mb^{\sim} \\ 0 & |a|b^{\sim} \end{pmatrix} = \begin{pmatrix} a|b|a^{\sim} & m|a|b^{\sim} - aa^{\sim}mb^{\sim} \\ 0 & b|a|b^{\sim} \end{pmatrix} , = \begin{pmatrix} |a||b| & 0 \\ 0 & |a||b| \end{pmatrix}$$

provided the determinants $|a|$ and $|b|$ commute with each of a, b and m •

When in particular we think of $A = K^{k\times k}$ and $B = K^{\ell\times\ell}$ as matrices over K, where $K = L^{m\times m}$ is itself a matrix algebra, then the determinant and the adjugate are given by the traditional formulae: if $T = (T_{ij}) \in L^{n\times n}$ then

2.6 $$\det_L(T) = \sum_{\pi\in\mathrm{Perm}(n)} \mathrm{sgn}(\pi) \prod_{j=1}^{n} T_{j\pi(j)} \ , \ \mathrm{adj}_L(T) = (T^{\sim}_{ij})$$

where $(-1)^{i+j}T^{\sim}_{ij}$ is the determinant of the matrix remaining when the row and column through the entry T_{ji} are deleted from T.

The block triangle formula respects the Kovacs/Silver/Williams formula:

3. Theorem *If there is equality*

3.1 $$\mathrm{adj}_L\det_K(T)\,\mathrm{adj}_K(T) = \mathrm{adj}_L(T) = \mathrm{adj}_K(T)\,\mathrm{adj}_L\det_K(T) ,$$

and hence also

3.2 $$\det_L\det_K(T) = \det_L(T) ,$$

with $T = a \in A$ and with $T = b \in B$ then this also holds for internally commutative

$$T \in \{\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} , \begin{pmatrix} a & 0 \\ n & b \end{pmatrix}\} \subseteq \begin{pmatrix} A & M \\ N & B \end{pmatrix} .$$

Proof. Writing $(\cdot)^{\sim} = \mathrm{adj}_K(\cdot)$ and $|\cdot| = \det_K(\cdot)$, so that (3.1) and (3.2) take the form

$$\mathrm{adj}_L|T|\,T^{\sim} = \mathrm{adj}_L(T) = T^{\sim}\mathrm{adj}_L|T| \ ; \ \det_L|T| = \det_L T ,$$

we have

$$\mathrm{adj}_L\begin{vmatrix} a & m \\ 0 & b \end{vmatrix}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}^{\sim} = \mathrm{adj}_L|a|\mathrm{adj}_L|b|\begin{pmatrix} |b|a^{\sim} & -a^{\sim}mb^{\sim} \\ 0 & |a|b^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} (\mathrm{adj}_L|b|)|b|(\mathrm{adj}_L|a|)a^\sim & -(\mathrm{adj}_L|a|)a^\sim m(\mathrm{adj}_L|b|)b^\sim \\ 0 & (\mathrm{adj}_L|a|)|a|(\mathrm{adj}_L|b|)b^\sim \end{pmatrix}$$

$$= \begin{pmatrix} (\det_L|b|)\mathrm{adj}_L(a) & \mathrm{adj}_L(a)m\ \mathrm{adj}_L(b) \\ 0 & (\det_L|a|)\mathrm{adj}_L(b) \end{pmatrix} = \mathrm{adj}_L \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} .$$

The argument for the lower triangle is the same •

Theorem 3 suggests an inductive proof of (3.1) for commuting block matrices. If $A = K$ then, following the argument of [6], write

3.3 $$\begin{pmatrix} 1 & 0 \\ -n & a \end{pmatrix}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} a & m \\ 0 & ab-nm \end{pmatrix} ; \begin{pmatrix} a & m \\ n & b \end{pmatrix}\begin{pmatrix} 1 & -m \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ n & ab-nm \end{pmatrix} ,$$

remembering $a \in K \subseteq B$ in the bottom right hand corner. Thus we can write $ST = U$ and $TR = V$ with (3.1) holding for S, R, U and V, so that

3.4 $$\mathrm{adj}_L(T)\mathrm{adj}_L|S|S^\sim = \mathrm{adj}_L|U|U^\sim = \mathrm{adj}_L|T|\mathrm{adj}_L|S|S^\sim T^\sim .$$

This is hovering around what we are looking for:

4. Theorem *If $T = (T_{ij})$ is a commuting block matrix over $K = L^{m\times m}$, for a commutative ring L, then* (3.1) *and* (3.2) *hold. Proof.* The argument is by induction on $n \in \mathbf{N}$, where $T \in K^{n\times n}$. It is clear when $n = 1$, and to transmit the conclusion from $n = k$ to $n = k+1$ suppose T is a block triangle, with $A = K$ and $B = A^{k\times k}$. Both factorizations $ST = U$ and $TR = V$ from (3.3) are available; in the notation of Theorem 3

4.1 $$|T||S| = |U| \text{ and } \det_L(T)\det_L(S) = \det_L(U) ,$$

and hence

4.2 $$\det_L(S)\det_L(T) = \det_L(U) = \det_L|U| = \det_L(|T||S|) = \det_L|S|\det_L|T| = \det_L(S)\det_L|T| .$$

This therefore establishes (3.2): but now

4.3 $$T\,T^\sim \mathrm{adj}_L(T) = T\ \mathrm{adj}_L|T| ,$$

and hence if $T = (T_{ij})$ is not a left zero divisor in $K^{n\times n}$ the second half of (3.1) holds. Similarly if T is not a right zero divisor then the factorization $TR = V$ gives the first half of (3.1). But now, again as in [6], we may replace the ring L by the polynomial ring $L[t]$, and similarly K, A and B, and repeat the whole argument with $T - tI$ in place of T. Since $T - tI$ is never either a left or a right zero divisor in the appropriate polynomial ring with matrix coefficients, and

4.4 $$(T-tI)(T-tI)^\sim \mathrm{adj}_L(T-tI) = (T-tI)\mathrm{adj}_L|T-tI| ,$$

we obtain the analogue of (3.2) with $T - tI$ in place of T, and can now "set $t = 0$" •

This argument also shows that each of the formulae of Theorem 3 follows from the other. The extension to Banach algebras is straightforward.

The easiest way for $T = (T_{ij})$ to be "commuting block" is [6] for

4.5 $$T_{ij} = p_{ij}(S) \ :$$

each block T_{ij} is a polynomial in a common matrix S. When there are four blocks of the same size then we recover the formula (0.5). When either $B = A^{k\times k}$ or $A = B^{k\times k}$ as in Theorem 3 then we are in the situation of "Cholesky's algorithm" [2],[3] which can be used to test for positivity: if $A = K^{k\times k}$ and $B = K$ is the scalars

4.6 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix}^\sim = \begin{pmatrix} ba^\sim - d & -a^\sim m \\ -na^\sim & |a| \end{pmatrix}$$

and

4.7 $$\begin{vmatrix} a & m \\ n & b \end{vmatrix} = |a|b - na^{\sim}m \ ,$$

where the matrix $d = \Phi(m, a, n)$ is independent of b, linear in m and in n, and satisfies

4.8 $$md = 0 = dn \text{ and } |a|d = (na^{\sim}m)a^{\sim} - a^{\sim}mna^{\sim} \ .$$

In the case of Cholesky's algorithm $K = \mathbf{C}$, $a \geq 0$ is "positive", b is real and $n = m^*$, so that the whole matrix T is hermitian.

We conclude with a count of the multiplications required to calculate each of $|T|$ and $T^{\sim}$ in each of three different ways:

	(2.6)	(3.2)	(4.7)	(2.6)	(3.1)	(4.6)
4×4	40	18	25	144	48	90
5×5	206		56	1000		216
6×6	1236	63	183	7380	180	410
7×7	8659		233	60564		594
8×8	69260	146	377	554176	432	852

The first three columns count multiplications for the determinant $|T|$, first by the traditional method, second using the Kovacs/Silver/Williams formula, assuming commuting block structure, and third by means of the inductive procedure suggested by the Cholesky algorithm. The second three columns count multiplications for the adjugate $T^{\sim}$ in the same ways.

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