Functional Analysis,
Approximation and
Computation
5:2 (2013), 23–33

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Fixed point theorems for Rafiq T-contractive operator in CAT(0) spaces

G. S. Saluja[a]

[a] *Department of Mathematics, Govt. Nagarjuna P.G. College of Science, Raipur - 492010 (C.G.), India*

Abstract. In this paper, we study a three-step iterative scheme for Rafiq T-contractive operator in the framework of **CAT(0)** spaces. Also we establish strong convergence theorems for above said scheme and operator. Our results improve and extend some corresponding results from the existing literature (see, e.g., [29, 30] and some others).

1. Introduction and Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [4]), $\mathbb{R}$-trees (see [19]), Euclidean buildings (see [5]), the complex Hilbert ball with a hyperbolic metric (see [13]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [4].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [20, 21]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since, then the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [1], [8], [10]-[12], [14], [17]-[18], [22]-[23], [28], [31]-[32] and references therein). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \le 0$ since any CAT(k) space is a CAT(k') space for every $k' \ge k$ (see,e.g., [4]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and let $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denoted

2010 *Mathematics Subject Classification*. Primary 54H25; Secondary 54E40.
Keywords. Rafiq T-contractive operator; three-step iteration scheme; fixed point; strong convergence; CAT(0) space.
Received: 7 May 2013: Accepted: 1 September 2013
Communicated by Dragan S. Djordjević
Email address: saluja_1963@rediffmail.com, saluja1963@gmail.com (G. S. Saluja)

by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [4]).

CAT(0) space.

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d(\overline{x}, \overline{y}). \tag{1.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [16]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the mid point of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{1.2}$$

The inequality (1.2) is the (CN) inequality of Bruhat and Titz [6]. The above inequality has been extended in [11] as

$$d^2(z, \alpha x \oplus (1-\alpha) y) \leq \alpha d^2(z, x) + (1-\alpha) d^2(z, y) - \alpha(1-\alpha) d^2(x, y) \tag{1.3}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [4, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha) y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha) y) \leq \alpha d(z, x) + (1-\alpha) d(z, y), \tag{1.4}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha) y : \alpha \in [0, 1]\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

We recall the following definitions in a metric space (X, d).

A mapping $T \colon X \to X$ is called an a-contraction if

$$d(Tx, Ty) \leq a\, d(x, y) \tag{1.5}$$

where $a \in (0, 1)$ and for all $x, y \in X$.

The mapping T is called Kannan mapping [15] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)] \tag{1.6}$$

for all $x, y \in X$.

A similar definition is due to Chatterjea [9]: there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)] \tag{1.7}$$

for all $x, y \in X$.

Combining these three definitions, Zamfirescu [34] proved the following important result.

Theorem Z. Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping for which there exists the real number a, b and c satisfying $a \in (0,1)$, $b, c \in (0, \frac{1}{2})$ such that for any pair $x, y \in X$, at least one of the following conditions holds:

(Z_1) $d(Tx, Ty) \le a\, d(x, y)$,

(Z_2) $d(Tx, Ty) \le b\, [d(x, Tx) + d(y, Ty)]$,

(Z_3) $d(Tx, Ty) \le c\, [d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \ n = 0, 1, 2, \dots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

The conditions $(Z_1) - (Z_3)$ can be written in the following equivalent form

$$d(Tx, Ty) \le h \max\Big\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\Big\}, \qquad (QC)$$

for all $x, y \in X$ and $0 < h < 1$, has been obtained by Ciric [7] in 1974.

A mapping satisfying (QC) is called Ciric quasi-contraction. It is obvious that each of the conditions $(Z_1) - (Z_3)$ implies (QC).

An operator T satisfying the contractive conditions $(Z_1) - (Z_3)$ in the theorem Z is called Z-operator.

In 2000, Berinde [3] introduced a new class of operators on a normed space E satisfying

$$\left\|Tx - Ty\right\| \le \delta \left\|x - y\right\| + L\, \|Tx - x\|, \qquad (*)$$

for any $x, y \in E$, $0 \le \delta < 1$ and $L \ge 0$.

He proved that this class is wider than the class of Zamfirescu operators and used the Mann iteration process to approximate fixed points of this class of operators in a normed space given in the form of following theorem:

Theorem B. Let C be a nonempty closed convex subset of a normed space E. Let $T\colon C \to C$ be an operator satisfying $(*)$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by: for $x_1 = x \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1 - b_n)x_n + b_n T^n x_n, \ n \ge 0,$$

where $\{b_n\}$ is a sequence in [0,1]. If $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

Noor iteration scheme.

In 2002, Xu and Noor [33] introduced a three-step iterative scheme as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = & (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = & (1-\beta_n)x_n + \beta_n T^n z_n, \\ z_n = & (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0 \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$.

Recently, Rafiq [29] proved the following result in generalized convex metric space by using (CR) contractive operator and Mann iteration process with errors as follows:

Theorem R. Let C be a nonempty closed convex subset of a generalized convex metric space X. Let $T\colon C \to C$ be an operator satisfying the condition

$$d(Tx, Ty) \leq h \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, d(x,Ty), d(y,Tx)\right\}, \qquad (CR)$$

for all $x, y \in X$ and $0 < h < 1$ (has been obtained by Ciric [7] in 1974). Let $\{x_n\}$ be the sequence defined by:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = & W(x_n, Tx_n, u_n; a_n, b_n, c_n) \end{cases}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences in [0,1] such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C. If $\sum_{n=1}^{\infty} b_n = \infty$ and $c_n = o(b_n)$, then $\{x_n\}$ converges to the unique fixed point of T.

The concept of T-Banach contraction and T-contractive mappings were introduced by Beiranvand et al. [2] in 2009 and they extended Banach's contraction principle and Edelstein fixed point theorem. Followed by this, Moradi [24] introduced T-Kannan contractive mappings, extending in the way, the well-known Kannan fixed point theorem [15].

Recently, Morales and Rojas [26], [27] have extended the concept of T-contraction mappings to cone metric space by proving fixed point theorems for T-Kannan, T-Zamfirescu and T-weakly contraction mappings. In [25], they studied the existence of fixed point for T-Zamfirescu operators in complete metric spaces and proved a convergence theorem of T-Picard iteration for the class of T-Zamfirescu operators. The result is as follows:

Theorem 1.1. *(See [25]) Let (M,d) be a complete metric space and $T, S\colon M \to M$ be two mappings such that T is continuous, one-to-one and subsequentially convergent. If S is a TZ operator, S has a unique fixed point. Moreover, if T is sequentially convergent, then for every $x_0 \in M$ the T-Picard iteration associated to S, $TS^n x_0$ converges to Tx^*, where x^* is the fixed point of S.*

Here we recall the definitions of the following classes of generalized T-contraction type mappings as given by Morales and Rojas [25].

Definition 1.2. *(See [25]) Let (X,d) be a metric space and $S, T\colon X \to X$ be two mappings. A mapping S is said be T-contraction, if there exists a real number $a \in [0,1)$ such that for all $x, y \in X$,*

$$d(TSx, TSy) \leq a\, d(Tx, Ty).$$

If we take $T = I$, the identity map, in the definition 1.2, then we obtain the definition of Banach's contraction.

The following example shows that a T-contraction mapping need not be a contraction mapping.

Example 1.3. *Let $X = [1, \infty)$ be with the usual metric. Define two mappings $T, S\colon X \to X$ as $Tx = \frac{1}{2x} + 2$ and $Sx = 3x$. Obviously, S is not contraction but S is T-contraction which is seen from the following:*

$$|TSx - TSy| = \left|\frac{1}{6x} - \frac{1}{6y}\right| = \frac{1}{3}|Tx - Ty|.$$

Definition 1.4. *(See [25]) Let (X, d) be a metric space and $S, T\colon X \to X$ be two mappings. A mapping S is said be T-Kannan contraction, if there exists a real number $b \in [0, \frac{1}{2})$ such that for all $x, y \in X$,*

$$d(TSx, TSy) \leq b\,[d(Tx, TSx) + d(Ty, TSy)].$$

If we take $T = I$, the identity map, in the definition 1.4, then we obtain the definition of Kannan operator [15].

Definition 1.5. *(See [25]) Let (X, d) be a metric space and $S, T\colon X \to X$ be two mappings. A mapping S is said be T-Chatterjea contraction, if there exists a real number $c \in [0, \frac{1}{2})$ such that for all $x, y \in X$,*

$$d(TSx, TSy) \leq c\,[d(Tx, TSy) + d(Ty, TSx)].$$

If we take $T = I$, the identity map, in the definition 1.5, then we obtain the definition of Chatterjea operator [9].

Definition 1.6. *(See [25]) Let (X, d) be a metric space and $S, T\colon X \to X$ be two mappings. A mapping S is said be T-Zamfirescu operator (TZ-operator), if there are real numbers $0 \leq a < 1$, $0 \leq b < \frac{1}{2}$, $0 \leq c < \frac{1}{2}$ such that for all $x, y \in X$ at least one of the following conditions holds:*

(TZ_1) $d(TSx, TSy) \leq a\, d(Tx, Ty)$,

(TZ_2) $d(TSx, TSy) \leq b\,[d(Tx, TSx) + d(Ty, TSy)]$,

(TZ_3) $d(TSx, TSy) \leq c\,[d(Tx, TSy) + d(Ty, TSx)]$.

If we take $T = I$, the identity map, in the definition 1.6, then we obtain the definition of Zamfirescu operator [34].

Definition 1.7. *(See [2]) Let T be a self mapping of a metric space (X, d). Then*

1. the mapping T is said to be sequentially convergent, if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is convergent.

2. the mapping T is said to be subsequentially convergent, if $\{y_n\}$ has a convergent subsequence whenever $\{Ty_n\}$ is convergent.

In this paper, inspired and motivated by [25, 29, 33, 34], we study a three-step iteration scheme and prove strong convergence theorem to approximate the fixed point for Rafiq T-contractive (RTC) operator in the framework of CAT(0) spaces.

Three-step iteration scheme in CAT(0) space.

Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ and let $S\colon C \to C$ be a T-contractive operator. Then for a given $x_1 = x_0 \in C$, compute the sequence $\{x_n\}$ by the iterative scheme as follows:

$$\begin{aligned} Tz_n &= \gamma_n TSx_n \oplus (1-\gamma_n)Tx_n, \\ Ty_n &= \beta_n TSz_n \oplus (1-\beta_n)Tx_n, \\ Tx_{n+1} &= \alpha_n STy_n \oplus (1-\alpha_n)Tx_n, \quad n \geq 0, \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are appropriate sequences in [0,1].

If we take $T = I$, the identity map, then (1.8) reduces to Noor [33] iteration scheme in CAT(0) space:

$$\begin{aligned} z_n &= \gamma_n Sx_n \oplus (1-\gamma_n)x_n, \\ y_n &= \beta_n Sz_n \oplus (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n Sy_n \oplus (1-\alpha_n)x_n, \quad n \geq 0, \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are appropriate sequences in [0,1].

If $\gamma_n = 0$ for all $n \geq 0$, then (1.9) reduces to Ishikawa iteration scheme in CAT(0) space:

$$\begin{aligned} y_n &= \beta_n Tx_n \oplus (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n Ty_n \oplus (1-\alpha_n)x_n, \quad n \geq 0, \end{aligned} \tag{1.10}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are appropriate sequences in [0,1].

We note that if $\beta_n = 0$ for all $n \geq 0$, then (1.10) reduces to Mann iteration scheme in CAT(0) space:

$$x_{n+1} = \alpha_n Tx_n \oplus (1-\alpha_n)x_n, \quad n \geq 0, \tag{1.11}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in (0,1).

We need the following useful lemmas to prove our main results in this paper.

Lemma 1.8. *(See [28]) Let (X, d) be a CAT(0) space.*

(i) For $x,\ y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,\ y]$ such that

$$d(x,\ z) = t\,d(x,\ y) \quad \text{and} \quad d(y,\ z) = (1-t)\,d(x,\ y). \qquad (A)$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x,\ y \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Lemma 1.9. *(See [29]) Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ and $\{s_n\}$ be sequences of nonnegative numbers satisfying the following conditions:*

$$p_{n+1} \leq (1-q_n)p_n + q_n r_n + s_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} q_n = \infty$, $\lim_{n\to\infty} r_n = 0$ and $\sum_{n=1}^{\infty} s_n < \infty$ hold, then $\lim_{n\to\infty} p_n = 0$.

2. Strong convergence theorems in CAT(0) spaces

In this section, we establish some strong convergence results of a three-step iteration scheme to approximate a fixed point for Rafiq T-contractive operator (RTC) in the framework of CAT(0) spaces.

Theorem 2.1. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space. Let $S, T\colon C \to C$ be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and $S\colon C \to C$ is a T-contractive operator satisfying the condition*

$$d(TSx, TSy) \leq h \max\Big\{d(Tx, Ty), \frac{d(Tx, TSx) + d(Ty, TSy)}{2}, d(Tx, TSy), d(Ty, TSx)\Big\}, \quad (RTC)$$

for all $x, y \in X$ and $0 < h < 1$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.8). If $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the fixed point of the operator S in C.

Proof. From Theorem 1.1, we get that S has a unique fixed point in C, say u. Consider $x, y \in C$. Since S is a Rafiq T-contractive operator satisfying (RTC), then if

$$\begin{aligned} d(TSx, TSy) &\leq \frac{h}{2}[d(Tx, TSx) + d(Ty, TSy)] \\ &\leq \frac{h}{2}[d(Tx, TSx) + d(Ty, Tx) + d(Tx, TSx) + d(TSx, TSy)], \end{aligned}$$

implies

$$\Big(1 - \frac{h}{2}\Big)d(TSx, TSy) \leq \frac{h}{2}d(Tx, Ty) + hd(Tx, TSx),$$

which yields (using the fact that $0 < h < 1$)

$$d(TSx, TSy) \leq \Big(\frac{h/2}{1 - h/2}\Big)d(Tx, Ty) + \Big(\frac{h}{1 - h/2}\Big)d(Tx, TSx).$$

If

$$\begin{aligned} d(TSx, TSy) &\leq h\, d(Tx, TSy), \\ &\leq h[d(Tx, TSx) + d(TSx, TSy)], \end{aligned}$$

implies

$$(1 - h)d(TSx, TSy) \leq h\, d(Tx, TSx),$$

which also yields (using the fact that $0 < h < 1$)

$$d(TSx, TSy) \leq \Big(\frac{h}{1 - h}\Big) d(Tx, TSx), \tag{2.1}$$

and also for

$$\begin{aligned} d(TSx, TSy) &\leq h\, d(Ty, TSx), \\ &\leq h[d(Tx, Ty) + d(Tx, TSx)]. \end{aligned} \tag{2.2}$$

Denote

$$\delta = \max\Big\{h, \frac{h/2}{1 - h/2}\Big\} = h,$$

$$L = \max\Big\{h,\ \frac{h}{1-h/2},\ \frac{h}{1-h}\Big\} = \frac{h}{1-h}.$$

Thus, in all cases,

$$\begin{aligned} d(TSx, TSy) &\leq \delta\, d(Tx, Ty) + L\, d(Tx, TSx) \\ &= h\, d(Tx, Ty) + \Big(\frac{h}{1-h}\Big) d(Tx, TSx) \end{aligned} \tag{2.3}$$

holds for all $x,\ y \in C$.

Also from (RTC) with $y = u = Su$, we have

$$\begin{aligned} d(TSx, TSu) &\leq h \max\Big\{d(Tx, Tu),\ \frac{d(Tx, TSx)}{2},\ d(Tx, Tu),\ d(Tu, TSx)\Big\} \\ &\leq h \max\Big\{d(Tx, Tu),\ \frac{d(Tx, TSx)}{2},\ d(Tu, TSx)\Big\} \\ &= h \max\Big\{d(Tx, Tu),\ \frac{d(Tx, Tu) + d(Tu, TSx)}{2},\ d(Tu, TSx)\Big\} \\ &\leq h\ \max\{d(Tx, Tu),\ d(Tu, TSx)\}. \end{aligned} \tag{2.4}$$

If $d(TSx, TSu) = d(TSx, Tu) \leq h\, d(Tu, TSx)$, is impossible or implies $d(Tu, TSx) = 0$. Thus

$$d(TSx, TSu) = d(TSx, Tu) \leq h\, d(Tx, Tu). \tag{2.5}$$

Now (2.5) gives

$$d(TSx_n, Tu) \leq h\, d(Tx_n, Tu), \tag{2.6}$$

$$d(TSy_n, Tu) \leq h\, d(Ty_n, Tu), \tag{2.7}$$

and

$$d(TSz_n, Tu) \leq h\, d(Tz_n, Tu). \tag{2.8}$$

Using (1.8), (2.8) and Lemma 1.8(ii), we have

$$\begin{aligned} d(Tz_n, Tu) &= d(\gamma_n TSx_n \oplus (1-\gamma_n)Tx_n, Tu) \\ &\leq \gamma_n d(TSx_n, Tu) + (1-\gamma_n) d(Tx_n, Tu) \\ &\leq \gamma_n h d(Tx_n, Tu) + (1-\gamma_n) d(Tx_n, Tu) \\ &\leq [1 - \gamma_n + h\gamma_n] d(Tx_n, Tu). \end{aligned} \tag{2.9}$$

Again using (1.8), (2.7), (2.9) and Lemma 1.8(ii), we have

$$\begin{aligned} d(Ty_n, Tu) &= d(\beta_n TSz_n \oplus (1-\beta_n)Tx_n, Tu) \\ &\leq \beta_n d(TSz_n, Tu) + (1-\beta_n) d(Tx_n, Tu) \\ &\leq \beta_n h d(Tz_n, Tu) + (1-\beta_n) d(Tx_n, Tu) \\ &\leq \beta_n h[1 - \gamma_n + h\gamma_n] d(Tx_n, Tu) + (1-\beta_n) d(Tx_n, Tu) \\ &\leq [1 - \beta_n + h\beta_n(1 - \gamma_n + h\gamma_n)] d(Tx_n, Tu). \end{aligned} \tag{2.10}$$

Now,

$$1 - \beta_n + h\beta_n(1 - \gamma_n + h\gamma_n) = 1 - [\beta_n(1-h)(1 + \gamma_n h)].$$

Since $(1+\gamma_n h) \geq (1-h)$, we have

$$1-\beta_n + h\beta_n(1-\gamma_n + h\gamma_n) \quad \leq \quad 1-(1-h)^2\beta_n. \tag{2.11}$$

Using (2.11) in (2.10), we obtain

$$d(Ty_n, Tu) \quad \leq \quad [1-(1-h)^2\beta_n]\, d(Tx_n, Tu). \tag{2.12}$$

Now using (1.8), (2.6), (2.12), $ST = TS$ (by assumption of the theorem) and Lemma 1.8(ii), we have

$$\begin{aligned} d(Tx_{n+1}, Tu) &= d(\alpha_n STy_n \oplus (1-\alpha_n)Tx_n, Tu) \\ &\leq \alpha_n d(STy_n, Tu) + (1-\alpha_n)d(Tx_n, Tu) \\ &= \alpha_n d(TSy_n, Tu) + (1-\alpha_n)d(Tx_n, Tu) \\ &\leq \alpha_n h d(Ty_n, Tu) + (1-\alpha_n)d(Tx_n, Tu) \\ &\leq \alpha_n h[1-(1-h)^2\beta_n]d(Tx_n, Tu) + (1-\alpha_n)d(Tx_n, Tu) \\ &\leq [1-\alpha_n + h\alpha_n - h(1-h)^2\alpha_n\beta_n]d(Tx_n, Tu) \\ &= [1-(1-h)\alpha_n - h(1-h)^2\alpha_n\beta_n]d(Tx_n, Tu) \\ &= [1-(1-h)\alpha_n\{1+h(1-h)\beta_n\}]d(Tx_n, Tu). \end{aligned} \tag{2.13}$$

Since $(1+h(1-h)\beta_n) \geq (1-h)^2$, we have

$$1-(1-h)\alpha_n\{1+h(1-h)\beta_n\} \quad \leq \quad 1-(1-h)^3\alpha_n. \tag{2.14}$$

Using (2.14) in (2.13), we obtain

$$\begin{aligned} d(Tx_{n+1}, Tu) &\leq [1-(1-h)^3\alpha_n]\, d(Tx_n, Tu) \\ &= (1-B_n)\, d(Tx_n, Tu), \end{aligned} \tag{2.15}$$

where $B_n = (1-h)^3\alpha_n$, since $0 < h < 1$ and by assumption of the theorem $\sum_{n=1}^{\infty}\alpha_n = \infty$, it follows that $\sum_{n=1}^{\infty} B_n = \infty$, therefore by Lemma 1.9, we get that $\lim_{n\to\infty} d(Tx_n, Tu) = 0$. Therefore $\{Tx_n\}$ converges strongly to Tu, where u is the fixed point of the operator S in C. This completes the proof. □

Since T-Kannan contractive condition included in the Rafiq T-contractive operator, by Theorem 2.1, we obtain the corresponding convergence result of the iteration process defined by (1.8) for the above said class of operators as corollary:

Corollary 2.2. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space. Let S, $T\colon C \to C$ be two commuting mappings such that T is continuous, one-to-one, subsequentially convergent and $S\colon C \to C$ is a T-Kannan contractive operator satisfying the condition*

$$d(TSx, TSy) \leq b\left[d(Tx, TSx) + d(Ty, TSy)\right],$$

for all x, $y \in X$ and $b \in (0, \frac{1}{2})$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.8). If $\sum_{n=1}^{\infty}\alpha_n = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the fixed point of the operator S in C.

If we take $T = I$, the identity map, in equation (1.8) and (2.3), then we obtain the following results as corollary:

Corollary 2.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space and let $S\colon C \to C$ be an operator satisfying (2.3). Let $\{x_n\}$ be defined by the iteration scheme (1.9). If $\sum_{n=1}^{\infty}\alpha_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof of this Corollary follows by taking $T = I$, the identity map in Theorem 2.1. This completes the proof. □

Corollary 2.4. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space and let $S\colon C \to C$ be an operator satisfying (2.3). Let $\{x_n\}$ be defined by the iteration scheme (1.10). If $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof of this Corollary follows by taking $T = I$, the identity map, and $\gamma_n = 0$ for all $n \geq 0$ in Theorem 2.1. This completes the proof. □

Remark 2.5. *Theorem 2.1 extends the corresponding result of [29, 30] to the case of three-step iteration process and from convex metric space or uniformly convex Banach space to the setting of CAT(0) spaces by using Rafiq T-contractive operator.*

Remark 2.6. *Theorem 2.1 also extends Theorem B to the case of three-step iteration process and from normed space to the setting of CAT(0) spaces by using Rafiq T-contractive operator.*

References

[1] A. Akbar, M. Eslamian, *Common fixed point results in CAT(0) spaces*, Nonlinear Anal.: TMA, 74 (2011), no.5 1835–1840.
[2] A. Beiranvand, S. Moradi, M. Omid, H. Pazandeh, *Two fixed point theorems for spcial mappings*, arXiv:0903.1504v1 [math.FA].
[3] V. Berinde, *Iterative approximation of fixed points*, Baia Mare: Efemeride, 2000.
[4] M.R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Vol. 319 of Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany, 1999.
[5] K.S. Brown,*Buildings*, Springer, New York, NY, USA, 1989.
[6] F. Bruhat, J. Tits, *Groups reductifs sur un corps local*, Institut des Hautes Etudes Scientifiques. Publications Mathematiques, 41 (1972), 5–251.
[7] L.B. Ciric, *A generalization of Banach principle*, Proc. Amer. Math. Soc. 45 (1974), 727–730.
[8] P. Chaoha, A. Phon-on, *A note on fixed point sets in CAT(0) spaces*, J. Math. Anal. Appl. 320 (2006), no.2, 983–987.
[9] S.K. Chatterjee, *Fixed point theorems compactes*, Rend. Acad. Bulgare Sci. 25 (1972), 727–730.
[10] S. Dhompongsa, A. Kaewkho, B. Panyanak, *Lim's theorems for multivalued mappings in CAT(0) spaces*, J. Math. Anal. Appl. 312 (2005), no.2, 478–487.
[11] S. Dhompongsa, B. Panyanak, *On △-convergence theorem in CAT(0) spaces*, Comput. Math. Appl. 56 (2008), no.10, 2572–2579.
[12] R. Espinola, A. Fernandez-Leon, *CAT(k)-spaces, weak convergence and fixed point*, J. Math. Anal. Appl. 353 (2009), no.1, 410–427.
[13] K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Vol. 83 of Monograph and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, NY, USA, 1984.
[14] N. Hussain, M.A. Khamsi, *On asymptotic pointwise contractions in metric spaces*, Nonlinear Anal.: TMA, 71 (2009), no.10, 4423–4429.
[15] R. Kannan, *Some results on fixed point theorems*, Bull. Calcutta Math. Soc. 10 (1968), 71–76.
[16] M.A. Khamsi, W.A. Kirk, *An introduction to metric spaces and fixed point theory*, Pure Appl. Math, Wiley-Interscience, New York, NY, USA, 2001.
[17] S.H. Khan, M. Abbas, *Strong and △-convergence of some iterative schemes in CAT(0) spaces*, Comput. Math. Appl. 61 (2011), no.1, 109–116.
[18] A.R. Khan, M.A. Khamsi, H. Fukhar-ud-din, *Strong convergence of a general iteration scheme in CAT(0) spaces*, Nonlinear Anal.: TMA, 74 (2011), no.3, 783–791.
[19] W.A. Kirk, *Fixed point theory in CAT(0) spaces and $\mathbb{R}$-trees*, Fixed Point and Applications, (2004), no.4, 309–316.
[20] W.A. Kirk, *Geodesic geometry and fixed point theory*, in Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), Vol. 64 of Coleccion Abierta, 195-225, University of Seville Secretary of Publications, Seville, Spain, 2003.
[21] W.A. Kirk, *Geodesic geometry and fixed point theory II*, in International Conference on Fixed point Theory and Applications, 113-142, Yokohama Publishers, Yokohama, Japan, 2004.
[22] W. Laowang, B. Panyanak, *Strong and △ convergence theorems for multivalued mappings in CAT(0) spaces*, J. Inequal. Appl. (2009), Article ID 730132, 16 pages.
[23] L. Leustean, *A quadratic rate of asymptotic regularity for CAT(0)-spaces*, J. Math. Anal. Appl. 325 (2007), no. 1, 386–399.
[24] S. Moradi, *Kannan fixed point theorem on complete metric spaces and on generalized metric spaces depended on another function*, arXiv:0903.1577v1 [math.FA].
[25] J. Morales, E. Rojas, *Some results on T-Zamfirescu operators*, Revista Notas de Matematics, 5 (2009), no.1, 64–71.
[26] J. Morales, E. Rojas, *Cone metric spaces and fixed point theorems of T-Kannan contractive mappings*, Int. J. Math. Anal. 4 (2010), no.4, 175–184.
[27] J. Morales, E. Rojas, *T-Zamfirescu and T-weak contraction mappings on cone metric spaces*, arXiv:0909.1255v1 [math.FA].
[28] Y. Niwongsa, B. Panyanak, *Noor iterations for asymptotically nonexpansive mappings in CAT(0) spaces*, Int. J. Math. Anal. 4 (2010), no.13, 645–656.
[29] A. Rafiq, *Fixed points of Ciric quasi-contractive operators in generalized convex metric spaces*, General Math. 14 (2006), no.3, 79–90.
[30] B.E. Rhoades, *Fixed point iteration using infinite matrices*, Trans. Amer. Math. Soc. 196 (1974), 161–176.
[31] S. Saejung, *Halpern's iteration in CAT(0) spaces*, Fixed Point Theory and Applications (2010), Article ID 471781, 13 pages.
[32] N. Shahzad, *Fixed point results for multimaps in CAT(0) spaces*, Topology and its Applications, 156 (2009), no.5, 997–1001.

[33] B.L. Xu, M.A. Noor, *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 267 (2002), no.2, 444–453.
[34] T. Zamfirescu, *Fixed point theorems in metric space*, Arch. Math. (Basel), 23 (1972), 292–298.