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Approximation numbers for relatively bounded operators

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Abstract. Let T be a densely defined closed operator between Banach spaces X and Y. A concept of approximation numbers, called *T–approximation numbers*, is considered for T–bounded operators between Banach spaces X and Z with their domains contained in X, and some properties of such T–approximation numbers are studied. The theorems proved in the paper include a result on approximation of T–approximation numbers of a T–bounded operator A using T–approximation numbers of A_n, where $\{A_n\}$ is a certain sequence of operators which converges to A in some sense. This result is analogous to a theorem proved recently by the authors in [3] for bounded linear operators.

1. Introduction

Let X, Y and Z be normed linear spaces. For a linear operator T between normed linear spaces, the domain of T is denoted by $D(T)$ and the range of T is denoted by $R(T)$. Let $L(X,Y)$ be the class of all linear operators with $D(T) \subseteq X$ and $R(T) \subseteq Y$, $CL(X,Y)$ be the class of all closed and densely defined operators in $L(X,Y)$ and let $BL(X,Y)$ be the class of all bounded linear operators from X to Y. The norm on a normed linear space is denoted by $\|\cdot\|$. For $k \in \mathbb{N}$, we denote by $\mathcal{F}_k(X,Y)$ the class of all operators in $BL(X,Y)$ of rank less than k. We use the abbreviations $L(X)$, $CL(X)$, $BL(X)$ and $\mathcal{F}_k(X)$ for $L(X,X)$, $CL(X,X)$, $BL(X,X)$ and $\mathcal{F}_k(X,X)$, respectively.

We recall that, for $T \in BL(X,Y)$ and $k \in \mathbb{N}$, the k^{th} **approximation number** of T is defined by

$$s_k(T) := \inf\{\|T - F\| : F \in \mathcal{F}_k(X,Y)\}.$$

The concept of approximation numbers of operators in $BL(X,Y)$ is a generalization of the concept of singular values of compact operators between Hilbert spaces. For a study on approximation numbers and their properties, one may refer to [12], where approximation numbers are used to study the geometry of Banach spaces. The convergence properties of approximation numbers are also found useful in estimating the error while solving operator equations [14].

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Attempts were made in literature to define concepts analogous to approximation numbers for operators which are not necessarily bounded linear. For example, in [2], the authors have extended the concept of approximation numbers to operators in certain subclass of $CL(X,Y)$; and in [5], similar concepts were defined for a larger subclass of $CL(X,Y)$, namely, *relatively bounded operators* between Hilbert spaces. In [14], a concept similar to approximation numbers was defined for bounded nonlinear operators, and some of its properties and applications to projection methods for solving some operator equations were given. A concept similar to approximation numbers was defined in [11] for elements in Banach algebras and in [13], certain numbers analogous to approximation numbers were defined for matrices over integral domains.

In Section 2, we introduce a concept of *T–approximation numbers* for T–bounded operators, where T is a densely defined closed operator between normed linear spaces, and study some properties. The concept of T–approximation numbers is akin to the concept introduced in [5] for operators between Hilbert spaces. In Section 3, we prove a modified form of a convergence result proved in [3] (Theorem 2.8) for approximation numbers of bounded linear operators. Then we prove a result analogous to Theorem 2.8 in [3] for the generalized approximation numbers defined for relatively bounded operators in Section 2.

2. Approximation Numbers for Relatively Bounded Operators

Let X and Y be Banach spaces and $T \in CL(X,Y)$. Recall that the *graph norm* on $D(T)$, denoted by $\|\cdot\|_T$, is defined by

$$\|x\|_T := \|x\| + \|Tx\|, \; x \in D(T).$$

It is known that $D(T)$ with $\|\cdot\|_T$ is a Banach space. We denote this Banach space by X_T. Also, if $A \in L(X,Z)$, then the space $D(A) \cap D(T)$ with the norm $\|\cdot\|_T$ is denoted by $D_T(A)$.

Now we introduce a concept of approximation numbers for *T-bounded operators* $A \in L(X,Z)$ using a specified closed operator $T \in CL(X,Y)$ and the corresponding space X_T, where Z is also a Banach space.

Definition 2.1. *An operator $A \in L(X,Z)$ is said to be T-bounded or relatively bounded with respect to T if there exists a nonnegative real number α such that*

$$\|Ax\| \leq \alpha \|x\|_T \quad \forall\, x \in D(T) \cap D(A).$$

We may observe that the above definition is equivalent to the standard definition of T-boundedness (cf. Kato [8]), namely, $A \in L(X,Y)$ is T-bounded if and only if there exist nonnegative real numbers a and b such that

$$\|Ax\| \leq a\|x\| + b\|Tx\| \quad \forall\, x \in D(T) \cap D(A).$$

Now, suppose $A \in L(X,Z)$ is a T-bounded operator. Then it can be seen that the operator $\widehat{A} : D_T(A) \longrightarrow Z$, defined by

$$\widehat{A}x = Ax, \quad x \in D_T(A),$$

is a bounded operator. Using the operator $\widehat{A}$, we define the concept of approximation numbers for T-bounded operators. Note that T is always T-bounded and $\|\widehat{T}\| \leq 1$.

Definition 2.2. *Let $A \in L(X,Z)$ be a T-bounded operator and $k \in \mathbb{N}$. Then the k^{th} approximation number of A with respect to T, or the* T**-approximation number** *of A is defined by*

$$\widehat{s_k}(A) = s_k(\widehat{A}),$$

where $s_k(\widehat{A})$ denotes the k^{th} approximation number of the bounded operator $\widehat{A}$.

It follows from the definition and well known properties of approximation numbers (cf. [12]) that $\{\widehat{s_k}(A)\}$ is a nonincreasing sequence of nonnegative real numbers and $\widehat{s_1}(A) = \|\widehat{A}\|$. In order to prove some properties of the T-approximation numbers, we shall make use of the following lemma that gives some elementary properties of T-bounded operators. In all the proofs of the results that follow, we make use of known properties of approximation numbers given in [12].

Lemma 2.3. *Let $k \in \mathbb{N}$. The following properties hold for T-bounded operators.*

(a) Suppose A_1 and A_2 are T-bounded operators in $L(X, Z)$. Then $A_1 + A_2$ is a T-bounded operator in $L(X, Z)$ and

$$\widehat{A_1 + A_2} = \widehat{A_1} + \widehat{A_2} \quad \text{on } D_T(A_1 + A_2),$$

$$\|\widehat{A_1 + A_2}\| \leq \|\widehat{A_1}\| + \|\widehat{A_2}\|.$$

(b) If $A \in BL(X, Z)$, then A is T-bounded, $D_T(A) = X_T$ and $\|\widehat{A}\| \leq \|A\|$.

(c) If $A \in L(X, Z)$ is T-bounded and $B \in BL(Z, W)$, then $BA \in L(X, W)$ is T-bounded, $D_T(BA) = D_T(A)$, $\widehat{BA} = B\widehat{A}$ and $\|\widehat{BA}\| \leq \|B\| \|\widehat{A}\|$.

(d) Let $A \in L(X, Z)$ be T-bounded, invertible and $A^{-1} \in BL(Z, X)$. Then $\widehat{A}$ is invertible, $(\widehat{A})^{-1} = \widehat{A^{-1}}$ and

$$\|(\widehat{A})^{-1}\| \leq \|A^{-1}\| + \|TA^{-1}\|.$$

In particular, if T itself satisfies the above hypothesis, then

$$\|(\widehat{T})^{-1}\| \leq 1 + \|T^{-1}\|.$$

Proof. (a). The proof follows by observing that $D_T(A_1+A_2) = D_T(A_1) \cap D_T(A_2)$ and $\|(A_1+A_2)x\| \leq \|A_1x\| + \|A_2x\|$ for all $x \in D_T(A_1 + A_2)$.

(b). If $A \in BL(X, Z)$, then $D(A) = X$. Hence $D_T(A) = D(A) \cap D(T) = D(T)$. Also for $x \in D(T)$, $\|\widehat{A}x\| = \|Ax\| \leq \|A\| \|x\| \leq \|A\| \|x\|_T$. This proves (b).

(c). If $A \in L(X, Z)$ is T-bounded and $B \in BL(Z, W)$, then $D(BA) = D(A)$ and hence $D_T(BA) = D_T(A)$. Also for $x \in D_T(BA)$, we have $\widehat{BA}(x) = B(\widehat{A}x)$. Hence $\|\widehat{BA}\| \leq \|B\| \|\widehat{A}\|$.

(d). First note that since $A^{-1} \in BL(Z, X)$, TA^{-1} is a bounded operator. Also, $(\widehat{A})^{-1} = \widehat{A^{-1}}$, and for $z \in Z$,

$$\|(\widehat{A})^{-1}z\|_T = \|A^{-1}z\|_T = \|A^{-1}z\| + \|TA^{-1}z\| \leq (\|A^{-1}\| + \|TA^{-1}\|)\|z\|.$$

The particular case follows by taking T in place of A. □

Proposition 2.4. *For $k \in \mathbb{N}$, the following properties of T-approximation numbers hold.*

(a) $\widehat{s_k}(T) \leq 1$.

(b) $\widehat{s_k}(B) \leq s_k(B)$ if $B \in BL(X, Z)$.

(c) $\widehat{s}_{k_1+k_2-1}(A_1 + A_2) \leq \widehat{s}_{k_1}(A_1) + \widehat{s}_{k_2}(A_2)$ for T-bounded operators $A_1, A_2 \in L(X, Z)$.

(d) $\widehat{s_k}(BA) \leq \|B\| \widehat{s_k}(A)$ if $A \in L(X, Z)$ is T-bounded and $B \in BL(Z, W)$.

Proof. We have $\widehat{s_k}(T) = s_k(\widehat{T}) \leq \|\widehat{T}\| \leq 1$, which gives the conclusion in (a).

Part (b) follows from Lemma 2.3(b).

Now, let $A_1, A_2 \in L(X, Z)$ be T-bounded operators. Let $F_1 \in BL(D_T(A_1), Z)$ and $F_2 \in BL(D_T(A_2), Z)$ be such that $\text{rank}(F_1) < k_1$ and $\text{rank}(F_2) < k_2$. Then we see that $F_1 + F_2 \in \mathcal{F}_k(D_T(A_1 + A_2), Z)$ with $k = k_1 + k_2 - 1$ and

$$\|(\widehat{A_1 + A_2}) - (F_1 + F_2)\| \leq \|\widehat{A_1} - F_1\| + \|\widehat{A_2} - F_2\|.$$

Hence $s_k(\widehat{A_1 + A_2}) \leq \|\widehat{A_1} - F_1\| + \|\widehat{A_2} - F_2\|$ for every $F_1 \in BL(D_T(A_1), Z)$ and $F_2 \in BL(D_T(A_2), Z)$ with $\text{rank}(F_1) < k_1$ and $\text{rank}(F_2) < k_2$. Taking infimum over F_1 and F_2, we get

$$\widehat{s_k}(A_1 + A_2) \leq \widehat{s}_{k_1}(A_1) + \widehat{s}_{k_2}(A_2),$$

with $k = k_1 + k_2 - 1$, proving (c).

For proving (d), let $A \in L(X, Z)$ be a T-bounded operator, $B \in BL(Z, W)$, and let $F \in \mathcal{F}_k(D_T(A), Z)$. Then by Lemma 2.3 (c), BA is a T-bounded operator with $D_T(BA) = D_T(A)$ and $BF \in \mathcal{F}_k(D_T(A), W)$. Since

$$\|(\widehat{BA} - BF)x\| \leq \|B\| \, \|(\widehat{A} - F)x\| \quad \forall\, x \in D_T(A),$$

we have

$$\|\widehat{BA} - BF\| \leq \|B\| \, \|\widehat{A} - F\|.$$

Hence we obtain $\widehat{s_k}(BA) \leq \|B\| \widehat{s_k}(A)$. □

Proposition 2.5. *Let $A \in L(X, Z)$ be T-bounded, invertible, $D(A) \subseteq D(T)$ and $A^{-1} \in BL(Z, X)$. Then*

$$\widehat{s_k}(A) \geq \frac{1}{\|A^{-1}\| + \|TA^{-1}\|} \quad \forall\, k \in \mathbb{N}.$$

In particular, if T is invertible and $T^{-1} \in BL(Y, X)$, then

$$\widehat{s_k}(T) \geq \frac{1}{1 + \|T^{-1}\|} \quad \forall\, k \in \mathbb{N}.$$

Proof. We have, for all $k \in \mathbb{N}$, from Lemma 2.3,

$$1 = s_k(I_{X_T}) = s_k((\widehat{A})^{-1}\widehat{A}) \leq s_k(\widehat{A}) \, \|(\widehat{A})^{-1}\| \leq \widehat{s_k}(A)(\|A^{-1}\| + \|TA^{-1}\|).$$

From this, the required inequalities follow. □

We close this section by giving a result for the reference operator T.

Theorem 2.6. *Suppose X and Y are Hilbert spaces and T is invertible with $T^{-1} \in BL(Y, X)$. Then*

$$\widehat{s_k}(T) := s_k(\widehat{T}) \geq \frac{1}{1 + s_k(T^{-1})} \quad \forall\, k \in \mathbb{N}.$$

If, in addition T^{-1} is compact, then

$$\widehat{s_k}(T) = 1 \quad \forall\, k \in \mathbb{N}.$$

Proof. By Lemma 2.3(d), $(\widehat{T})^{-1}$ is a bounded operator and $\|(\widehat{T})^{-1}\| \leq 1 + \|T^{-1}\|$. Now let $P \in \mathcal{F}_k(Y)$ be an orthogonal projection. Then for $y \in Y$,

$$\begin{aligned} \|(\widehat{T})^{-1}y - (\widehat{T})^{-1}Py\|_T &= \|(\widehat{T})^{-1}y - (\widehat{T})^{-1}Py\| + \|T(\widehat{T})^{-1}y - T(\widehat{T})^{-1}Py\| \\ &= \|T^{-1}y - T^{-1}Py\| + \|y - Py\| \\ &\leq (1 + \|T^{-1} - T^{-1}P\|)\|y\|. \end{aligned}$$

Hence $\|(\widehat{T})^{-1} - (\widehat{T})^{-1}P\| \leq 1 + \|T^{-1} - T^{-1}P\|$, so that, by Proposition 2.4.2 in [1],

$$s_k((\widehat{T})^{-1}) \leq \inf\{1 + \|T^{-1} - T^{-1}P\| : P \in \mathcal{F}_k(Y)\} = 1 + s_k(T^{-1}).$$

Therefore, using the relation $s_{k_1+k_2-1}(A_1A_2) \leq s_{k_1}(A_1)s_{k_2}(A_2)$ for any two bounded operators A_1 and A_2 between normed linear spaces (cf. [12]),

$$1 = s_{2k-1}((\widehat{T})^{-1}\widehat{T}) \leq s_k((\widehat{T})^{-1})s_k(\widehat{T}) \leq (1 + s_k(T^{-1}))s_k(\widehat{T}).$$

Thus, we obtain the required inequality.

Next, assume that T^{-1} is a compact operator. Then, by Theorem XI.10.1 in [7], we know that $\lim\limits_{k \to \infty} s_k(T^{-1}) = 0$. Hence, we have

$$1 \leq \lim_{k \to \infty} (1 + s_k(T^{-1}))s_k(\widehat{T}) = \lim_{k \to \infty} s_k(\widehat{T}) \leq \|\widehat{T}\| \leq 1.$$

This shows that $\lim\limits_{k \to \infty} s_k(\widehat{T}) = 1$. Since $\{s_k(\widehat{T})\}$ is a nonincreasing sequence, we obtain $s_k(\widehat{T}) = 1$ for all $k \in \mathbb{N}$. □

3. On the convergence of T–approximation numbers

We recall the following theorem proved in [3].

Theorem 3.1. (cf. [3], Theorem 2.8) *Let X be separable, and Z be a reflexive Banach space. Let $A \in BL(X, Z)$, and $\{P_n\}$ and $\{Q_n\}$ be sequences of operators in $BL(X)$ and $BL(Z)$ respectively such that $\|P_n\| \|Q_n\| \leq 1$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, let $A_n := Q_n A P_n$. If $A_n x \longrightarrow Ax$ for $x \in X$ in the weak sense of convergence, then for each $k \in \mathbb{N}$,*

$$\lim_{n \longrightarrow \infty} s_k(A_n) = s_k(A).$$

In this section, we explore the possibility of obtaining an analogous result for a T–bounded operator $A \in L(X, Z)$. For this purpose, first we prove a modified version of the above theorem in a specific case.

Let $A \in BL(X, Z)$ and for $n \in \mathbb{N}$, let $P_n \in BL(X)$ and $Q_n \in BL(Z)$ be operators satisfying $\|P_n\| \|Q_n\| \leq 1$. Let $A_n := Q_n A P_n$ and $\tilde{A}_n := A_n \mid_{R(P_n)}: R(P_n) \longrightarrow R(Q_n)$, $n \in \mathbb{N}$. We have shown in [4] that if, in addition, the operators P_n are projections, then $s_k(A_n) = s_k(\tilde{A}_n)$ for all $n \in \mathbb{N}$. In applications, often P_n and Q_n are projections of finite rank, and in that case, one may be more interested in the restrictions $\tilde{A}_n$ rather than A_n. Since it is not easy to obtain a sequence of finite rank projections of norm 1 on general Banach spaces, we prove an analogue of Theorem 3.1 by assuming that P_n's are projections and Q_n satisfies $\|Q_n\| \leq 1$ for all $n \in \mathbb{N}$. The proof of this result is similar to the proof of Theorem 2.8 given in [3]. However we give it here for the sake of completeness. This result (Theorem 3.3) also helps in proving Theorem 3.4. For its proof, we shall make use of the following lemma.

Lemma 3.2. *Let Z be a normed linear space, $A \in BL(X, Z)$ and $\{P_n\}$ be a sequence of projections in $BL(X)$ such that $P_n x \longrightarrow x$ for every $x \in X$. Suppose there exists $M > 0$ such that $\|A_{|_{R(P_n)}}\| \leq M$ for all $n \in \mathbb{N}$. Then $\|A\| \leq M$.*

Proof. Since $\|A_{|_{R(P_n)}} P_n x\| \leq \|A_{|_{R(P_n)}}\| \|P_n x\| \leq M \|P_n x\|$ for all $x \in X$ and $n \in \mathbb{N}$, we have, for every $x \in X$,

$$\|Ax\| = \lim_{n \longrightarrow \infty} \|AP_n x\| = \lim_{n \longrightarrow \infty} \|A_{|_{R(P_n)}} P_n x\| \leq M(\lim_{n \longrightarrow \infty} \|P_n x\|) = M\|x\|$$

so that $\|A\| \leq M$. □

Theorem 3.3. *Let Z be Banach space and $A \in BL(X, Z)$. For $n \in \mathbb{N}$, let $\{P_n\}$ be a sequence of projections in $BL(X)$ such that $R(P_n) \subseteq R(P_{n+1})$ and $P_n x \longrightarrow x$ for each $x \in X$, and let $\{Q_n\}$ be a sequence of operators in $BL(Z)$ such that $\|Q_n\| \leq 1$ and $Q_n z \longrightarrow z$ for each $z \in Z$. For $n \in \mathbb{N}$, let $A_n := Q_n A P_n$ and $\tilde{A}_n := A_n \mid_{R(P_n)}: R(P_n) \longrightarrow R(Q_n)$. Then for each $k \in \mathbb{N}$,*

$$s_k(\tilde{A}_n) \leq s_k(A) \quad \forall\, n \in \mathbb{N}.$$

If, in addition, X is separable and Z is reflexive, then for each $k \in \mathbb{N}$,

$$\lim_{n \longrightarrow \infty} s_k(\tilde{A}_n) = s_k(A).$$

Proof. For $F \in \mathcal{F}_k(X, Z)$ and $n \in \mathbb{N}$, we have $Q_n F P_n \in \mathcal{F}_k(X, Z)$ and

$$\begin{aligned} \|\tilde{A}_n - Q_n F P_n \mid_{R(P_n)} \| &= \|Q_n A P_n \mid_{R(P_n)} - Q_n F P_n \mid_{R(P_n)} \| \\ &= \|Q_n A \mid_{R(P_n)} - Q_n F \mid_{R(P_n)} \| \\ &\leq \|Q_n (A - F)\| \\ &\leq \|A - F\|. \end{aligned}$$

Hence $s_k(\tilde{A}_n) \leq \|\tilde{A}_n - Q_n F P_n \mid_{R(P_n)} \| \leq \|A - F\|$. This is true for all $F \in \mathcal{F}_k(X, Z)$ so that $s_k(\tilde{A}_n) \leq s_k(A)$.

Now, let $d_n = s_k(\tilde{A}_n)$ and $d = s_k(A)$. Clearly, if $d = 0$, then $d_n = 0$ for all $n \in \mathbb{N}$. So, let $d > 0$, and let X be separable and Z be reflexive Banach spaces. Assume that (d_n) does not converge to d. Then there exists an $\varepsilon > 0$ and an infinite set $N_1 \subseteq \mathbb{N}$ such that $d_n < d - \varepsilon$ for all $n \in N_1$. This implies that $\|\tilde{A}_n - \tilde{F}_n\| < d - \varepsilon$

for some $\tilde{F}_n \in \mathcal{F}_k(R(P_n), Z)$ for all $n \in N_1$. Hence, using the boundedness of $\{\|P_n\|\}$ and $\{\|Q_n\|\}$, it follows that $\{\|\tilde{F}_n P_n\|\}$ is also bounded.

Since X is separable, Z is reflexive and $\{\tilde{F}_n P_n\}$ is in $\mathcal{F}_k(X, Z)$, by Lemma 2.4 in [3], there exists a subsequence $\{\tilde{F}_{n_j} P_{n_j}\}$ of $\{\tilde{F}_n P_n\}$ and an operator $F \in \mathcal{F}_k(X, Z)$ such that $\tilde{F}_{n_j} P_{n_j}$ converges to F in the weak operator topology.

Now let $x \in R(P_{n_{j_0}})$ for some $j_0 \in \mathbb{N}$ with $\|x\| = \|P_{n_{j_0}} x\| = 1$, and $f \in Z'$ be such that $\|f\| = 1$. Since $\tilde{A}_{n_j} P_{n_j} x \longrightarrow Ax$ as $j \to \infty$, there is a $j_1 \in \mathbb{N}$ such that

$$|f(\tilde{A}_{n_j} P_{n_j} x) - f(Ax)| \leq \|\tilde{A}_{n_j} P_{n_j} x - Ax\| < \frac{\varepsilon}{3} \quad \forall\, j \geq j_1.$$

Similarly since $\tilde{F}_{n_j} P_{n_j} x \longrightarrow Fx$ in the weak sense, there exists a $j_2 \in \mathbb{N}$ such that

$$|f(\tilde{F}_{n_j} P_{n_j} x) - f(Fx)| < \frac{\varepsilon}{3} \quad \forall\, j \geq j_2.$$

Now let $j_3 = max\{j_0, j_1, j_2\}$. Then, using the fact that $\|P_{n_{j_3}} x\| = \|x\| = 1$, we have

$$\begin{aligned}
|f(Ax - Fx)| &\leq |f(Ax) - f(\tilde{A}_{n_{j_3}} P_{n_{j_3}} x)| \\
&\quad + |f(\tilde{A}_{n_{j_3}} P_{n_{j_3}} x) - f(\tilde{F}_{n_{j_3}} P_{n_{j_3}} x)| + |f(\tilde{F}_{n_{j_3}} P_{n_{j_3}} x) - f(Fx)| \\
&< \frac{\varepsilon}{3} + \|\tilde{A}_{n_{j_3}} P_{n_{j_3}} x - \tilde{F}_{n_{j_3}} P_{n_{j_3}} x\| + \frac{\varepsilon}{3} \\
&\leq \frac{\varepsilon}{3} + \|\tilde{A}_{n_{j_3}} - \tilde{F}_{n_{j_3}}\| + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + d - \varepsilon + \frac{\varepsilon}{3} \\
&= d - \frac{\varepsilon}{3}.
\end{aligned}$$

Since the above is true for all $x \in R(P_{n_{j_0}})$ with $\|x\| = 1$, and $f \in Z'$ with $\|f\| = 1$, we have $\|(A-F)|_{R(P_{n_{j_0}})}\| \leq d - \frac{\varepsilon}{3}$. Since $j_0 \in \mathbb{N}$ was arbitrary, $\|(A - F)|_{R(P_{n_{j_0}})}\| \leq d - \frac{\varepsilon}{3}$ for all $j_0 \in \mathbb{N}$. Hence, by Lemma 3.2, $\|A - F\| \leq d - \frac{\varepsilon}{3}$. This leads to $d \leq d - \frac{\varepsilon}{3}$, which is a contradiction. Hence $d_n \longrightarrow d$ as $n \longrightarrow \infty$. □

Now, the promised result on T-approximation numbers:

Theorem 3.4. *Let X be a separable Banach space, Z be a reflexive Banach space which is separable and $A \in CL(X, Z)$ be a T-bounded operator with $D(A)=D(T)$. For $n \in \mathbb{N}$, let $P_n \in BL(X)$ with $R(P_n) \subseteq D(T)$ be finite rank projections satisfying*

1. *$P_n x \longrightarrow x$ for each $x \in X$*
2. *$R(P_n) \subseteq R(P_{n+1}) \subseteq D(T)$, $n \in \mathbb{N}$*
3. *$TP_n x \longrightarrow Tx$ for each $x \in D(T)$*

and $Q_n \in BL(Z)$ be such that $\|Q_n\| \leq 1$ and $Q_n z \longrightarrow z$ for all $z \in Z$. Let $A_n := Q_n A P_n$ and $\tilde{A}_n := A_n|_{R(P_n)}$: $R(P_n) \longrightarrow R(Q_n)$. Then for $n \in \mathbb{N}$, $\tilde{A}_n$ is T–bounded and for each $k \in \mathbb{N}$,

$$\lim_{n \longrightarrow \infty} \widehat{s_k}(\tilde{A}_n) = \widehat{s_k}(A).$$

Proof. Let $n \in \mathbb{N}$. Since A is a closed operator and $P_n \in BL(X)$ such that $R(P_n) \subseteq D(A)$, we have $AP_n \in BL(X, Z)$ for all $n \in \mathbb{N}$. Therefore, $\tilde{A}_n := A_n|_{R(P_n)}$: $R(P_n) \longrightarrow R(Q_n)$ is also a bounded operator. In particular, $\tilde{A}_n$ is T–bounded. We may recall that $\widehat{A}$ is a bounded operator from X_T to Z. Now, define $\widetilde{P}_n : X_T \longrightarrow X_T$ by

$$\widetilde{P}_n x = P_n x \quad \forall\, x \in X_T.$$

Clearly $\widetilde{P}_n^2 = \widetilde{P}_n$ for all $n \in \mathbb{N}$.

Now, since T is a closed operator and $P_n \in BL(X)$ such that $R(P_n) \subseteq D(T)$, we have $TP_n \in BL(X, Y)$ for all $n \in \mathbb{N}$. Also, by conditions (1) and (3) on P_n, it follows that $(\|P_n\|)$ and $(\|TP_n\|)$ are bounded. Therefore, from the relations

$$\|\widetilde{P}_n x\|_T = \|P_n x\|_T = \|P_n x\| + \|TP_n x\|,$$

$$\|\widetilde{P}_n x - x\|_T = \|P_n x - x\|_T = \|P_n x - x\| + \|TP_n x - Tx\|,$$

we see that $\widetilde{P}_n \in BL(X_T)$ for all $n \in \mathbb{N}$ and $(\|\widetilde{P}_n\|)$ is bounded.

Using the separability of X and Z, it can be seen that the space X_T is also separable. Hence, by Theorem 3.3, we have $\lim\limits_{n\longrightarrow\infty} s_k(Q_n\widetilde{\widetilde{AP}_n}|_{R(\widetilde{P}_n)}) = s_k(\widehat{A})$. Hence the proof is complete if we show that $s_k(Q_n\widetilde{\widetilde{AP}_n}|_{R(\widetilde{P}_n)}) = s_k(\widehat{\widetilde{A}_n})$ for all $n \in \mathbb{N}$.

To see this, note that the operator $\widehat{\widetilde{A}_n}$ is from the space $R(P_n)$ with $\|\cdot\|_T$ to $R(Q_n)$ and $Q_n\widetilde{\widetilde{AP}_n}|_{R(\widetilde{P}_n)}$ is from $R(\widetilde{P}_n)$ with $\|\cdot\|_T$ to Z, and for $x \in R(P_n)$,

$$\widehat{\widetilde{A}}_n x = Q_n A x = Q_n\widetilde{\widetilde{AP}_n}|_{R(\widetilde{P}_n)} x.$$

Hence $\|\widehat{\widetilde{A}}_n\| = \|Q_n\widetilde{\widetilde{AP}_n}|_{R(\widetilde{P}_n)}\|$ for all $n \in \mathbb{N}$. Now for each finite rank operator of rank less than k from $R(P_n)$ with norm $\|\cdot\|_T$ to Z, we have

$$\|\widehat{\widetilde{A}}_n - F\| = \|Q_n\widetilde{\widetilde{AP}_n}|_{R(\widetilde{P}_n)} - F\|.$$

Taking infimum over F, we get $s_k(\widehat{\widetilde{A}}_n) = s_k(Q_n\widetilde{\widetilde{AP}_n}\,|_{R(\widetilde{P}_n)})$ and this completes the proof. □

Definition 3.5. *Let $T \in CL(X, Y)$ and for $n \in \mathbb{N}$, let P_n be projection operators satisfying the assumptions (1)-(3) in Theorem 3.4. Then we say that the sequence $\{P_n\}$ is admissible for the operator T.*

The concept of admissible sequence of projections was defined and used in [9]. In Example 3.6 below, we illustrate Theorem 3.4 and the last part of Theorem 2.5 by considering an admissible sequence of projections. For this purpose, we shall make use of a result from [4] which states that if T is a bounded linear operator from a normed linear space X to a normed linear space Y and if there exists $\alpha \geq 0$ such that $\|T(x)\| \geq \alpha\|x\|$ for every x in a subspace M of X, then $s_k(T) \geq \alpha$ for all k not exceeding the dimension of M. This is Theorem 2.8 of [4].

Example 3.6. Let $X = \ell^p$, $1 < p < \infty$ and $D(T) := \{(x_1, x_2, \ldots) \in \ell^p : (x_1, 2x_2, 3x_3, \ldots) \in \ell^p\}$. Let $T : D(T) \longrightarrow \ell^p$ be the operator defined by

$$Tx = (x_1, 2x_2, 3x_3 \ldots), \quad x = (x_1, x_2, x_3, \ldots) \in D(T).$$

Then it is seen that T is a closed and densely defined operator with compact inverse. Now consider the projections P_n defined by

$$P_n x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \quad x = (x_1, x_2, x_3 \ldots) \in \ell^p.$$

Then $P_n \in BL(\ell^p)$ satisfy all the conditions given in Theorem 3.4. Now, let $m < n$ and define

$$X_{m,n} = \{x \in D(T) : x_j = 0 \text{ for } j \leq m \text{ and } j > n\}.$$

Let $x \in X_{m,n}$. Then

$$(m+1)^p\|x\|^p \leq \|Tx\|^p = \sum_{j=m+1}^{n} j^p|x(j)|^p \leq n^p\|x\|^p$$

and hence

$$(m+1)\|x\| \leq \|Tx\| \leq n\|x\|.$$

Also $\|x\|_T = \|x\| + \|Tx\| \leq (n+1)\|x\|$. Hence

$$\|\widehat{T}x\| \geq (m+1)\|x\| \geq \frac{m+1}{n+1}\|x\|_T.$$

Let $n = m + k$. Then $\dim(X_{m,n}) = k$ and hence (by [4], Theorem 2.8)

$$s_k(\widehat{T}) \geq \frac{m+1}{m+k+1} \geq \frac{1}{1+\frac{k}{m+1}} \quad \forall\, m \in \mathbb{N}.$$

This gives $s_k(\widehat{T}) = \widehat{s_k}(T) = 1$ for all $k \in \mathbb{N}$. By a similar argument for $T_n := P_n T P_n$, we get

$$1 \geq \widehat{s_k}(T_n) \geq \frac{1}{1+\frac{k}{n-k+1}} \quad \forall\, n \in \mathbb{N},\ n \geq k.$$

Hence $\widehat{s_k}(T_n)$ converges to 1 as n tends to ∞.

Remark 3.7. In Theorem 3.3 and Theorem 3.4 one can remove the assumption of reflexivity on the codomain Z by assuming it to be the dual space of some separable Banach space.

Remark 3.8. In Definition 2.2, instead of taking $\|\cdot\|_T$ on $D(T)$, one can take $\|\cdot\|_{T,p}$, $1 \leq p < \infty$ defined by $\|x\|_{T,p} = (\|x\|^p + \|Tx\|^p)^{\frac{1}{p}}$ and define generalized approximation numbers analogous to T-approximation numbers. Similar results can be proved for these numbers also. Note that, for $p = 1$ this definition coincides with the Definition 2.2, and for $p = 2$ it coincides with the τ^**-numbers* defined in [5] for relatively bounded operators between Hilbert spaces.

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