



Eigenvalues for a Schrödinger operator on a closed Riemannian manifold with holes

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Abstract. In this article we consider a closed Riemannian manifold (M, g) and A a subset of M . The purpose of this article is the comparison between the eigenvalues $(\lambda_k(M))_{k \geq 1}$ of a Schrödinger operator $P := -\Delta_g + V$ on the manifold (M, g) and the eigenvalues $(\lambda_k(M - A))_{k \geq 1}$ of P on the manifold $(M - A, g)$ with Dirichlet boundary conditions.

1. Introduction

The behaviour of the spectrum of a Riemannian manifold (M, g) under topological perturbation has been the subject of many research. The most famous exemple is the crushed ice problem [Kac], see also [Ann]. This problem consists to understand the behaviour of Laplacian eigenvalues with Dirichlet boundary on a domain with small holes. This subject was first studied by M. Kac [Kac] in 1974. Then, J. Rauch and M. Taylor [Ra-Ta] studied the case of Euclidian Laplacian in a compact set M of \mathbb{R}^n : they showed that the spectrum of $\Delta_{\mathbb{R}^n}$ is invariant by a topological excision of a M by a compact subset A with a Newtonian capacity zero. Later, S. Osawa, I. Chavel and E. Feldman [Ca-Fe1], [Ca-Fe2] treated the Riemannian manifold case. They used complex probalistic techniques based on Brownian motion. In [Ge-Zh], F. Gesztesy and Z. Zhao investigate the study the case of a Schrödinger operator with Dirichlet boundary conditions \mathbb{R}^n , they use probabilistic tools. In 1995, in a nice article [Cou] G. Courtois studied the case of Laplace Beltrami operator on closed Riemannian manifold. He used very simple techniques of analysis. In [Be-Co] J. Bertrand and B. Colbois explained also the case of Laplace Beltrami operator on compact Riemannian manifold. In this article we focus on the the Schrödinger operator $-\Delta_g + V$ case on a closed Riemannian manifold.

Assumption 1. *The manifold is closed (i.e. compact without boundary); the function V is bounded on the manifold M and $\min_M V > 0$.*

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In this work we show that under “little” topological excision of a part A from the manifold, the spectrum of $-\Delta_g + V$ on $M - A$ is close of the spectrum on M . More precisely, the “good” parameter for measuring the littleness of A is a type of electrostatic capacity defined by :

$$\text{cap}(A) := \inf \left\{ Q(u), u \in H^1(M), \int_M u \, d\mathcal{V}_g = 0, u - e_1 \in H_0^1(M - A) \right\}$$

where e_1 denotes the first eigenfunction of the operator $-\Delta_g + V$ on the manifold M , and Q is the following quadratic form :

$$Q(\varphi) := \int_M |d\varphi|^2 \, d\mathcal{V}_g + \int_M V |\varphi|^2 \, d\mathcal{V}_g$$

and $H_0^1(M - A)$ is the Sobolev space defined by :

$$H_0^1(M - A) := \overline{\{g \in H^1(M), g = 0 \text{ on a open neighborhood of } A\}}$$

the closure is for the norm $\|\cdot\|_{H^1(M)}$, $H^1(M)$ is the usual Sobolev space on M .

Indeed, more $\text{cap}(A)$ is small, more the spectrum $-\Delta_g + V$ on $M - A$ is close of the spectrum on M in the following sense :

Theorem 1.1. *Let (M, g) a closed Riemannian manifold. For all integer $k \geq 1$, there exists a constant C_k depending on the manifold (M, g) and on the potential V such that for all subset A of M we have :*

$$0 \leq \lambda_k(M - A) - \lambda_k(M) \leq C_k \sqrt{\text{cap}(A)}.$$

The organization of this paper is the following : in the part 2 we start by recall some classical results in spectral theory and about usual Sobolev spaces, next we define our specific Sobolev space $H_0^1(M - A)$ and the notion of Schrödinger capacity. In particular, we explain the link between the functional Hilbert space $H_0^1(M - A)$ and Schrödinger capacity $\text{cap}(A)$. The last part of this paper is a detailed proof of the main theorem.

2. Spectral problem background

2.1. Schrödinger operator on a Riemannian manifold

We recall here some generality on spectral geometry. In Riemannian geometry, the Laplace Beltrami operator is the generalisation of Laplacian $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ on \mathbb{R}^n . For a C^2 real valued function f on a Riemannian manifold and for a local chart $\phi : U \subset M \rightarrow \mathbb{R}^n$ of the manifold M , the Laplace Beltrami operator is given by the local expression :

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(\sqrt{g} g^{jk} \frac{\partial (f \circ \phi^{-1})}{\partial x_k} \right)$$

where $g = \det(g_{ij})$ and $g^{jk} = (g_{jk})^{-1}$.

The spectrum of this operator is a nice geometric invariant, see Berger, Gauduchon and Mazet [BGM] and [Be-Be]. The spectrum of Laplace Beltrami operator has many applications in geometry topology, physics ,etc ...

For every Riemannian manifold (M, g) with dimension $n \geq 1$ we have the “natural” Hilbert space $L^2(M) = L^2(M, d\mathcal{V}_g)$, \mathcal{V}_g is the Riemannian volume form associated to the metric g . For V a function from M to \mathbb{R} , we define the Schrödinger operator on the manifold (M, g) by the linear unbounded operator on the set of smooth compact supports real valued functions $C_c^\infty(M) \subset L^2(M)$ by : $-\Delta_g + V$.

2.2. Sobolev spaces

Let us denote by $C_c^\infty(M)$ the set of smooth functions with compact support in M . The set $C_c^\infty(M)$ is also called the set of *test functions* in the language of distributions. Recall first that the *Lebesgue space* $L^2(M)$ on the manifold (M, g) is defined by :

$$L^2(M) := \left\{ f : M \rightarrow \mathbb{R} \text{ measurable such that } \int_M |f|^2 d\mathcal{V}_g < +\infty \right\}.$$

This space is a Hilbert space for the scalar product :

$$\langle u, v \rangle_{L^2} := \int_M uv d\mathcal{V}_g.$$

Next the *Sobolev space* $H^1(M)$ is defined by :

$$H^1(M) := \overline{C_c^\infty(M)}$$

where the closure is for the norm $\|\cdot\|_{H^1} : \|u\|_{H^1} := \sqrt{\|u\|_{L^2}^2 + \|du\|_{L^2}^2}$.

Another point of view to define the space $H^1(M)$ is the following :

$$H^1(M) = \{u \in L^2(M); du \in L^2(M)\}$$

where the derivation is in the sense of distribution.

The space $H^1(M)$ is a Hilbert space for the scalar product :

$$\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \langle du, dv \rangle_{L^2}.$$

For finish, the Sobolev space $H_0^1(M, g)$ is defined by :

$$H_0^1(M) := \overline{C_c^\infty(M)}$$

the closure is for the norm $\|\cdot\|_{H^1(M)}$.

So we have :

$$C_c^\infty(M) \subset H_0^1(M) \subset H^1(M) \subset L^2(M).$$

Recall that, for the norm $\|\cdot\|_{L^2(M)}$ we have :

$$\overline{C_c^\infty(M)} = L^2(M).$$

2.3. Spectral problem

The spectral problem is the following : find all pairs (λ, u) with $\lambda \in \mathbb{R}$ and $u \in L^2(M)$ such that :

$$-\Delta_g u + Vu = \lambda u$$

(with $u \in L^2(M)$ in the non-compact case).

In the case of manifold with boundary, we need boundary conditions on the functions u , for example the Dirichlet conditions : $u = 0$ on the boundary of M , or Neumann conditions : $\frac{\partial u}{\partial n} = 0$ on the boundary of M . In the case of closed manifolds (compact without boundary) we don't have conditions.

For our context (the closed case) the natural space to look here is the Sobolev space $H^1(M)$.

Recall here a classical theorem of spectral theory (see for example [Re-Si]) :

Theorem 2.1. *For the above problems, the operator $-\Delta_g + V$ is self-adjoint, the spectrum of the operator $-\Delta_g + V$ consists of a sequence of infinite increasing eigenvalues with finite multiplicity :*

$$\lambda_1(M) \leq \lambda_2(M) \leq \dots \leq \lambda_k(M) \leq \dots \rightarrow +\infty.$$

Moreover, the associate eigenfunctions $(e_k)_{k \geq 0}$ is a Hilbert basis of the space $L^2(M)$.

Definition 2.2. We define the quadratic form Q with domain $D(Q) := H^1(M)$ by :

$$Q(\varphi) := \int_M |d\varphi|^2 d\mathcal{V}_g + \int_M V |\varphi|^2 d\mathcal{V}_g.$$

Recall also (see for example [Co-Hi]) the minimax variational characterization for eigenvalues : for all $k \geq 1$

$$\lambda_k(M) = \min_{\substack{E \subset H^1(M) \\ \dim(E)=k}} \max_{\substack{\varphi \in E \\ \varphi \neq 0}} R(\varphi)$$

where $R(\varphi)$ is the Rayleigh quotient of the function φ :

$$R(\varphi) := \frac{Q(\varphi)}{\int_M \varphi^2 d\mathcal{V}_g}.$$

In our context, a consequence of the minimax principle is :

Proposition 2.3. *The first eigenvalue $\lambda_1(M)$ and e_1 the first eigenfunction of the operator $-\Delta_g + V$ on the manifold (M, g) satisfy $\lambda_1(M) \geq \min_M V > 0$ and $e_1 > 0$ or $e_1 < 0$ in M .*

Proof. It's clear that

$$\int_M |de_1|^2 d\mathcal{V}_g + \int_M V |e_1|^2 d\mathcal{V}_g \geq \min_M V \|e_1\|_{L^2(M)}^2$$

and on the other hand

$$\begin{aligned} \int_M |de_1|^2 d\mathcal{V}_g + \int_M V |e_1|^2 d\mathcal{V}_g &= - \int_M \Delta_g e_1 e_1 d\mathcal{V}_g + \int_M V |e_1|^2 d\mathcal{V}_g \\ &= \int_M (-\Delta_g + V) e_1 e_1 d\mathcal{V}_g = \lambda_1(M) \|e_1\|_{L^2(M)}^2 \end{aligned}$$

so $\lambda_1(M) \geq \min_M V$. Next, suppose the function e_1 changes sign into M , since $e_1 \in H^1(M)$, the function $f := |e_1|$ belongs to $H^1(M)$ and $|df| = |de_1|$ (see for example [Gi-Tr]), hence $R(f) = R(e_1)$. So, the function f is a first eigenfunction of $-\Delta_g + V$ on the manifold M which satisfies $f \geq 0$ on M , f vanish into M and $(-\Delta_g + V)f = \lambda_1(M)f \geq 0$ on M . Using the maximum principle [Pr-We], the function f can not achieved it minimum in an interior point of the manifold M , hence f does not vanish on M , so we obtain a contradiction. \square

3. Proof of the main theorem

3.1. Some other usefull spaces

We define on the space $H^1(M)$ the \star -norm by :

$$\|u\|_{\star}^2 := \int_M |du|^2 d\mathcal{V}_g + \int_M V |u|^2 d\mathcal{V}_g$$

so, without difficulty we have :

Proposition 3.1. *The application $\|\cdot\|_{\star}$ is a norm on the space $H^1(M)$; moreover this norm is equivalent to the Sobolev norm $\|\cdot\|_{H^1(M)}$. In particular $H^1(M)$, $\|\cdot\|_{\star}$ is a Banach space.*

Let us denotes by $C_c^{\infty}(M - A)$ the set of smooth functions with compact support on $M - A$. For a compact subset A of the manifold M the usual Sobolev space $H_0^1(M - A)$ is defined by the closure of $C_c^{\infty}(M - A)$ for the norm $\|\cdot\|_{H^1(M)}$:

$$H_0^1(M - A) := \overline{C_c^{\infty}(M - A)}.$$

What happens when the set A is not compact ? For example if A is a dense and countable subset of points of the manifold M , the space of test functions $C_c^{\infty}(M - A)$ is reduced to $\{0\}$. Therefore we cannot define the space $H_0^1(M - A)$. In this case, we propose a definition of $H_0^1(M - A)$ for any subset A of M .

Definition 3.2. *We define the Sobolev spaces $\mathcal{H}_0^1(M - A)$ and $H_0^1(M - A)$ by :*

$$\mathcal{H}_0^1(M - A) := \{g \in H^1(M), g = 0 \text{ on a open neighborhood of } A\};$$

$$H_0^1(M - A) := \overline{\mathcal{H}_0^1(M - A)}$$

where the closure is for the norm $\|\cdot\|_{H^1(M)}$.

We have the :

Proposition 3.3. *If the set A is compact, the previous definition of the space $H_0^1(M - A)$ coincides with the usual ones.*

Proof. Let $f \in H_0^1(M - A) := \overline{\mathcal{H}_0^1(M - A)}$, then by definition : for all $\varepsilon \geq 0$ there exists $g \in \mathcal{H}_0^1(M - A)$ such that $\|f - g\|_{H^1(M)} \leq \varepsilon$. So, we will show that we can write g as a limit of sequence from the space $C_c^{\infty}(M - A)$ and conclude. Since $g \in \mathcal{H}_0^1(M - A)$ there exists an open set $U \supset A$ such that $g|_U = 0$. Consider two open sets U_1 and U_2 of the manifold M such that :

$$A \subset U_1, M - U \subset U_2, U_1 \cap U_2 = \emptyset;$$

and consider also a function $\varphi \in \mathcal{D}(M)$ such that :

$$\varphi|_{U_1} = 0, \varphi|_{U_2} = 1.$$

Of course, the function φ belongs to the space $C_c^{\infty}(M - A)$. Next, since $g \in \mathcal{H}_0^1(M - A) \subset H^1(M)$ and as the set of smooth functions $C^{\infty}(M)$ is dense in $H^1(M)$: there exists a sequence $(g_n)_n$ in $C^{\infty}(M)$ such that $\lim_{n \rightarrow +\infty} g_n = g$ for the norm $\|\cdot\|_{H^1(M)}$. Therefore we claim that : $\lim_{n \rightarrow +\infty} \varphi g_n = g$ for the norm $\|\cdot\|_{H^1(M)}$. Indeed, start by, for all integer n :

$$\begin{aligned} \|\varphi g_n - g\|_{H^1(M)}^2 &\leq \|g_n - g\|_{H^1(M-U)}^2 + \|\varphi g_n - g\|_{H^1(U)}^2 \\ &\leq \|g_n - g\|_{H^1(M)}^2 + \|\varphi g_n - g\|_{H^1(U)}^2. \end{aligned}$$

Next, we observe that, for all integer n :

$$\begin{aligned}
 \|\varphi g_n - g\|_{H^1(U)}^2 &= \|\varphi g_n\|_{H^1(U)}^2 \\
 &= \int_U |\varphi g_n|^2 d\mathcal{V}_g + \int_U |d\varphi g_n + \varphi dg_n|^2 d\mathcal{V}_g \\
 &\leq \int_U |\varphi g_n|^2 d\mathcal{V}_g + \int_U |d\varphi g_n|^2 d\mathcal{V}_g + \int_U |\varphi dg_n|^2 d\mathcal{V}_g + 2 \int_U |d\varphi g_n \varphi dg_n| d\mathcal{V}_g \\
 &\leq \|\varphi\|_\infty^2 \|g_n\|_{L^2(U)}^2 + \|d\varphi\|_{L^\infty(M)}^2 \|g_n\|_{L^2(U)}^2 \\
 &\quad + \|\varphi\|_\infty^2 \|dg_n\|_{L^2(U)}^2 + 2 \|d\varphi\|_\infty \|\varphi\|_\infty \int_U |g_n dg_n| d\mathcal{V}_g \\
 &\leq \|\varphi\|_\infty^2 \|g_n\|_{L^2(U)}^2 + \|d\varphi\|_\infty^2 \|g_n\|_{L^2(U)}^2 \\
 &\quad + \|\varphi\|_\infty^2 \|dg_n\|_{L^2(U)}^2 + 2 \|d\varphi\|_\infty \|\varphi\|_{L^\infty(M)} \|g_n\|_{L^2(U)} \|dg_n\|_{L^2(U)},
 \end{aligned}$$

by Cauchy-Schwarz inequality.

Finally we get for all integer n :

$$\|\varphi g_n - g\|_{H^1(U)}^2 \leq \|g_n\|_{H^1(U)}^2 \left(2 \|\varphi\|_\infty^2 + \|d\varphi\|_\infty^2 + 2 \|d\varphi\|_\infty \|\varphi\|_\infty \right).$$

As a consequence, we have for all integer n :

$$\begin{aligned}
 \|\varphi g_n - g\|_{H^1(M)}^2 &\leq \|g_n - g\|_{H^1(M-U)}^2 \\
 &\quad + \|g_n\|_{H^1(U)}^2 \left(2 \|\varphi\|_\infty^2 + \|d\varphi\|_\infty^2 + 2 \|d\varphi\|_\infty \|\varphi\|_\infty \right).
 \end{aligned}$$

Now, it suffices to note that $\|g_n\|_{H^1(U)}^2 = \|g_n - g\|_{H^1(U)}^2 \leq \|g_n - g\|_{H^1(M)}^2$ (since $g = 0$ on the open set U) and we have finally :

$$\begin{aligned}
 \|\varphi g_n - g\|_{H^1(M)}^2 &\leq \\
 \|g_n - g\|_{H^1(M)}^2 &\left(1 + 2 \|\varphi\|_\infty^2 + \|d\varphi\|_\infty^2 + 2 \|d\varphi\|_\infty \|\varphi\|_\infty \right).
 \end{aligned}$$

The sequence $(\varphi g_n)_n$ belong to $C_c^\infty(M-A)^\mathbb{N}$, and since $\lim_{n \rightarrow +\infty} g_n = g$ for the norm $\|\cdot\|_{H^1(M)}$ the previous inequality implies $\lim_{n \rightarrow +\infty} \varphi g_n = g$ for the norm $\|\cdot\|_{H^1(M)}$.

So we have shown that every function $f \in H_0^1(M-A) := \overline{\mathcal{H}_0^1(M-A)}$ is a limit (for the norm $\|\cdot\|_{H^1(M)}$) of a sequence of $C_c^\infty(M-A)$.

Conversely, since $C_c^\infty(M-A) \subset \mathcal{H}_0^1(M-A)$ we get :

$$H_0^1(M-A) := \overline{C_c^\infty(M-A)} \subset H_0^1(M-A) := \overline{\mathcal{H}_0^1(M-A)}.$$

□

Let us also denote the spaces $H^1_\star(M)$ and $S_A(M)$ by :

$$H^1_\star(M) := \left\{ f \in H^1(M), \int_M f d\mathcal{V}_g = 0 \right\};$$

and

$$S_A(M) := \left\{ u \in H^1_\star(M), u - e_1 \in H^1_0(M - A) \right\}.$$

In the definition of the space $H^1_\star(M)$ the condition $\int_M f d\mathcal{V}_g = 0$ is analog to a boundary condition. We observe that the space $H^1_\star(M)$ is a Hilbert space for the norm :

$$\|u\|_\star := \int_M |du|^2 d\mathcal{V}_g + \int_M V |u|^2 d\mathcal{V}_g;$$

and $S_A(M)$ is just an affine closed subset of $H^1(M)$.

3.2. Schrödinger capacity

Next, we introduce the Schrödinger capacity of the set A ;

Definition 3.4. Let us consider the Schrödinger capacity $cap(A)$ of the set A defined by

$$cap(A) := \inf \left\{ \int_M |du|^2 d\mathcal{V}_g + \int_M V |u|^2 d\mathcal{V}_g, u \in S_A(M) \right\}.$$

Let us remark that : there exists an unique function $u_A \in S_A(M)$ such that

$$cap(A) = \int_M |du_A|^2 d\mathcal{V}_g + \int_M V |u_A|^2 d\mathcal{V}_g.$$

Indeed : here the capacity $cap(A)$ is just the distance between the function 0 and the closed space $S_A(M)$. This distance is equal to $\|u_A\|_\star$ where u_A is the orthogonal projection of 0 on $S_A(M)$:

$$cap(A) = d_\star(0, S_A(M)) := \inf \{ \|u\|_\star, u \in S_A(M) \} = \|u_A\|_\star.$$

In the following lemma we give the relationships between the capacity $cap(A)$, the functions u_A, e_1 and the Sobolev spaces $H^1_0(M - A), H^1(M)$.

Lemma 3.5. For all subset A of the manifold M , the following properties are equivalent :

- (i) $cap(A) = 0$;
- (ii) $u_A = 0$;
- (iii) $e_1 \in H^1_0(M - A)$;
- (iv) $H^1_0(M - A) = H^1(M)$.

Proof. It is clear from the formula (3.1) that (i) \Leftrightarrow (ii) \Leftrightarrow (iii). Next, suppose the property (iii) holds : so there exists a sequence $(v_n)_n \in \mathcal{H}^1_0(M - A)^{\mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} v_n = e_1$ for the norm $\|\cdot\|_{H^1(M)}$. So, for all smooth function $\varphi \in C^\infty(M)$ we have $\lim_{n \rightarrow +\infty} (\varphi v_n)/e_1 = \varphi$ for the norm $\|\cdot\|_{H^1(M)}$, indeed for all integer n :

$$\left\| \frac{\varphi v_n}{e_1} - \varphi \right\|_{H^1(M)}^2 = \int_M \left| \frac{\varphi v_n}{e_1} - \varphi \right|^2 d\mathcal{V}_g + \int_M \left| d\left(\frac{\varphi v_n}{e_1}\right) - d\varphi \right|^2 d\mathcal{V}_g.$$

First, we have for all integer n :

$$\begin{aligned} \int_M \left| \frac{\varphi v_n}{e_1} - \varphi \right|^2 d\mathcal{V}_g &= \int_M \frac{1}{|e_1|^2} |\varphi (v_n - e_1)|^2 d\mathcal{V}_g \\ &\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \|\varphi\|_\infty^2 \|v_n - e_1\|_{L^2(M)}^2 \end{aligned}$$

so, since $\lim_{n \rightarrow +\infty} v_n = e_1$ for the norm $\|\cdot\|_{H^1(M)}$ we have

$$\lim_{n \rightarrow +\infty} \int_M \left| \frac{\varphi v_n}{e_1} - \varphi \right|^2 d\mathcal{V}_g = 0.$$

On the other hand, for all integer n :

$$\begin{aligned} \int_M \left| d\left(\frac{\varphi v_n}{e_1}\right) - d\varphi \right|^2 d\mathcal{V}_g &= \int_M \left| \frac{d(\varphi v_n)e_1 - \varphi v_n de_1}{e_1^2} - d\varphi \right|^2 d\mathcal{V}_g \\ &= \int_M \left(\frac{1}{e_1^2} \right) |d(\varphi) v_n e_1 + \varphi d(v_n) e_1 - \varphi v_n d(e_1) - d(\varphi) e_1^2|^2 d\mathcal{V}_g \\ &\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \|d\varphi v_n e_1 - d\varphi e_1^2 + \varphi dv_n e_1 - \varphi v_n de_1\|_{L^2(M)}^2 \\ &\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \left(\|d\varphi v_n e_1 - d\varphi e_1^2\|_{L^2(M)} + \|\varphi dv_n e_1 - \varphi v_n de_1\|_{L^2(M)} \right)^2 \\ &\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \left[\|d\varphi\|_\infty \|e_1\|_\infty \|v_n - e_1\|_{L^2(M)} + \right. \\ &\quad \left. \|\varphi\|_\infty \|e_1 (dv_n - de_1) + e_1 de_1 - v_n de_1\|_{L^2(M)} \right]^2 \\ &\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \left[\|d\varphi\|_\infty \|e_1\|_\infty \|v_n - e_1\|_{L^2(M)} + \right. \\ &\quad \left. \|\varphi\|_\infty \|e_1\|_\infty \|dv_n - de_1\|_{L^2(M)} + \|\varphi\|_\infty \|de_1\|_\infty \|e_1 - v_n\|_{L^2(M)} \right]^2; \end{aligned}$$

so, since $\lim_{n \rightarrow +\infty} v_n = e_1$ for the norm $\|\cdot\|_{H^1(M)}$ we have

$$\lim_{n \rightarrow +\infty} \int_M \left| d\left(\frac{\varphi v_n}{e_1}\right) - d\varphi \right|^2 d\mathcal{V}_g = 0.$$

Therefore, for all function $\varphi \in C^\infty(M)$ we have $\lim_{n \rightarrow +\infty} \frac{\varphi v_n}{e_1} = \varphi$ for the norm $\|\cdot\|_{H^1(M)}$.

Next, by density of $C^\infty(M)$ in $H^1(M)$: for all function $f \in H^1(M)$ we have $\lim_{n \rightarrow +\infty} \frac{f v_n}{e_1} = f$. Since the sequence $\left(\frac{f v_n}{e_1}\right)_n \in \mathcal{H}_0^1(M - A)^N$ we get finally that f belongs to space $H_0^1(M - A)$. Finally, it is easy to see that (iv) \Rightarrow (iii). \square

An obvious consequence of this lemma is the following result :

Proposition 3.6. *The spectrum of $-\Delta_g + V$ on the manifold (M, g) and on the manifold $(M - A, g)$ are equal if and only if $\text{cap}(A) = 0$.*

3.3. The Poincaré inequality

Now, let introduce the Poincaré inequality :

Theorem 3.7. *If $\lambda_1(M)$ denotes the first eigenvalue of the operator $-\Delta_g + V$ on the manifold (M, g) , the following inequality*

$$\|u_A\|_{L^2(M)}^2 \leq \frac{\text{cap}(A)}{\lambda_1(M)}$$

holds for all subset A of M .

Proof. The case $\text{cap}(A) = 0$ is an obvious consequence of the lemma in section 3.2. Suppose here that $\text{cap}(A) > 0$, then $\|u_A\|_{L^2(M)} > 0$. The first eigenvalue $\lambda_1(M)$ of the operator $-\Delta_g + V$ on the manifold (M, g) is given by :

$$\begin{aligned} \lambda_1(M) &= \min_{\substack{E \subset H^1(M) \\ \dim(E)=1}} \max_{\substack{\varphi \in E \\ \varphi \neq 0}} \frac{\int_M |d\varphi|^2 + V|\varphi|^2 d\mathcal{V}_g}{\int_M |\varphi|^2 d\mathcal{V}_g} \\ &= \min_{\substack{\varphi \in H^1(M) \\ \varphi \neq 0}} \frac{\int_M |d\varphi|^2 + V|\varphi|^2 d\mathcal{V}_g}{\int_M |\varphi|^2 d\mathcal{V}_g} \end{aligned}$$

Since u_A belongs to the space $H^1(M)$ we get $\lambda_1(M) \leq \frac{\text{cap}(A)}{\|u_A\|_{L^2(M)}^2}$. \square

3.4. The main theorem

Recall our main result :

Theorem 3.8. *Let (M, g) a compact Riemannian manifold. For all integer $k \geq 1$, there exists a constant C_k depending on the manifold of (M, g) and the potential V such that for all subset A of M we have :*

$$0 \leq \lambda_k(M - A) - \lambda_k(M) \leq C_k \sqrt{\text{cap}(A)}.$$

Remark 3.9. *We can easily adapt the proof for a compact Riemannian manifold with boundary.*

Proof. Let us denote by $(e_k)_{k \geq 1}$ an orthonormal basis of the space $L^2(M)$ with eigenfunctions of the operator $-\Delta_g + V$ on the manifold (M, g) . For all integer $k \geq 1$, we consider the sets

$$F_k := \text{span} \{e_1, e_2, \dots, e_k\}$$

and

$$E_k := \left\{ f \left(1 - \frac{u_A}{e_1} \right), f \in F_k \right\}.$$

First, observe that $E_k \subset H_0^1(M - A)$. For all $j \in \{1, \dots, k\}$ we introduce also the functions $\phi_j := e_j \left(1 - \frac{u_A}{e_1} \right) \in E_k$.

• **Step 1 :** we compute the L^2 -inner product $\langle \phi_i, \phi_j \rangle_{L^2(M)}$ for all pairs $(i, j) \in \{1, \dots, k\}^2$:

$$\begin{aligned} \langle \phi_i, \phi_j \rangle_{L^2(M)} &= \int_M e_i e_j \left(1 - \frac{u_A}{e_1} \right)^2 d\mathcal{V}_g \\ &= \delta_{i,j} - 2 \int_M \frac{e_i e_j}{e_1} u_A d\mathcal{V}_g + \int_M e_i e_j \frac{u_A^2}{e_1^2} d\mathcal{V}_g. \end{aligned}$$

Thus, for all pair $(i, j) \in \{1, \dots, k\}^2$ we get :

$$\left| \langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j} \right| \leq 2 \int_M \left| \frac{e_i e_j}{e_1} u_A \right| d\mathcal{V}_g + \int_M \left| e_i e_j \frac{u_A^2}{e_1^2} \right| d\mathcal{V}_g,$$

hence, by Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \left| \langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j} \right| &\leq 2 \max_{1 \leq i, j \leq k} \left\| \frac{e_i e_j}{e_1^2} \right\|_{\infty} \|u_A\|_{L^2(M)} + \max_{1 \leq i, j \leq k} \left\| \frac{e_i e_j}{e_1^2} \right\|_{\infty} \|u_A\|_{L^2(M)}^2 \\ &\leq 2 \max_{1 \leq i, j \leq k} \left\| \frac{e_i e_j}{e_1} \right\|_{\infty} \sqrt{\text{vol}(M)} \|u_A\|_{L^2(M)} + \max_{1 \leq i, j \leq k} \left\| \frac{e_i e_j}{e_1^2} \right\|_{\infty} \|u_A\|_{L^2(M)}^2 \end{aligned}$$

hence by Poincaré inequality we have

$$\left| \langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j} \right| \leq B_{k,M} \left(\sqrt{\text{cap}(A)} + \text{cap}(A) \right)$$

where $B_k = B_k(e_1, e_2, \dots, e_k, \lambda_1(M), M) \geq 0$, and since the eigenfunctions e_1, e_2, \dots, e_k and the eigenvalue $\lambda_1(M)$ depends only on (M, g) and V , for all integer k the constant B_k depends only on (M, g) and V , ie : $B_k = B_k(M, V)$.

Therefore, there exists $\varepsilon_k \in]0, 1[$ (depends on the constant B_k) such that for all $A \subset M$ we have :

$$\text{cap}(A) \leq \varepsilon_k \Rightarrow \dim(E_k) = k \text{ and } \forall j \in \{1, \dots, k\}, \left| \|\phi_j\|_{L^2(M)}^2 - 1 \right| \leq D_k \sqrt{\text{cap}(A)}$$

where (and for the same reasons as in the study of B_k) for all integer k , the constant D_k depends only on M and V , ie $D_k = D_k(M, V)$.

• **Step 2** : Let a function $\phi = f \left(1 - \frac{u_A}{e_1} \right) \in E_k$, with $f \in F_k$. Without loss generality we can assume that $\|f\|_{L^2(M)} = 1$, indeed : we have $R(\phi) = R\left(\frac{\phi}{\|f\|_{L^2(M)}}\right)$ and in our context we interest in the Rayleigh quotient of ϕ (see the end of the final step of the proof).

Set $v_A := \frac{u_A}{e_1}$, we have :

$$\begin{aligned} \int_M |d\phi|^2 d\mathcal{V}_g &= \int_M |df - d(fv_A)|^2 d\mathcal{V}_g \\ &= \int_M |df|^2 d\mathcal{V}_g + \int_M |dfv_A + fdv_A|^2 d\mathcal{V}_g - 2 \int_M dfd(fv_A) d\mathcal{V}_g \\ &= \int_M |df|^2 d\mathcal{V}_g + \int_M |dfv_A|^2 d\mathcal{V}_g + \int_M |fdv_A|^2 d\mathcal{V}_g \\ &\quad + 2 \int_M dfdv_A f v_A d\mathcal{V}_g - 2 \int_M |df|^2 v_A d\mathcal{V}_g - 2 \int_M dfdv_A f d\mathcal{V}_g \\ &= \int_M |df|^2 d\mathcal{V}_g + \int_M |dfv_A|^2 d\mathcal{V}_g + \int_M |fdv_A|^2 d\mathcal{V}_g \\ &\quad - 2 \int_M |df|^2 v_A d\mathcal{V}_g - 2 \int_M dfdv_A f (1 - v_A) d\mathcal{V}_g. \end{aligned}$$

Recall we have $dv_A = \frac{du_A e_1 - u_A de_1}{e_1^2}$, and :

$$\int_M V |\phi|^2 d\mathcal{V}_g = \int_M V |f|^2 d\mathcal{V}_g - 2 \int_M V |f|^2 v_A d\mathcal{V}_g + \int_M V |v_A f|^2 d\mathcal{V}_g$$

hence

$$\begin{aligned} \int_M |d\phi|^2 d\mathcal{V}_g + \int_M V|\phi|^2 d\mathcal{V}_g &= \underbrace{\int_M |df|^2 d\mathcal{V}_g + \int_M V|f|^2 d\mathcal{V}_g}_{:=A(f)} + \underbrace{\int_M |dfv_A|^2 d\mathcal{V}_g}_{:=B(f)} \\ &+ \underbrace{\int_M |fdv_A|^2 d\mathcal{V}_g + \int_M V|v_A f|^2 d\mathcal{V}_g}_{:=C(f)} - 2 \left(\underbrace{\int_M |df|^2 v_A d\mathcal{V}_g + \int_M V|f|^2 v_A d\mathcal{V}_g}_{:=D(f)} \right) \\ &- 2 \underbrace{\int_M dfdv_A f(1-v_A) d\mathcal{V}_g}_{:=E(f)}. \end{aligned}$$

◆ Study of $A(f) := \int_M |df|^2 d\mathcal{V}_g + \int_M V|f|^2 d\mathcal{V}_g \geq 0$: since $f \in F_k$ we can write $f = \sum_{i=1}^k \alpha_i e_i$ where $(\alpha_i)_{1 \leq i \leq k} \in \mathbb{R}^k$

and with $\sum_{i=1}^k \alpha_i^2 = 1$ (since $\|f\|_{L^2(M)} = 1$), thus we get

$$\begin{aligned} A(f) &= \left\langle \sum_{j=1}^k \alpha_j de_j, \sum_{i=1}^k \alpha_i de_i \right\rangle_{L^2(M)} + \left\langle \sqrt{V} \sum_{j=1}^k \alpha_j e_j, \sqrt{V} \sum_{i=1}^k \alpha_i e_i \right\rangle_{L^2(M)} \\ &= \sum_{i,j} \alpha_i \alpha_j \left(\langle de_j, de_i \rangle_{L^2(M)} + \int_M V e_j e_i d\mathcal{V}_g \right) \\ &= \sum_{i,j} \alpha_i \alpha_j \left(-\langle e_j, \Delta_g e_i \rangle_{L^2(M)} + \int_M V e_j e_i d\mathcal{V}_g \right) \\ &= \sum_{i,j} \alpha_i \alpha_j \langle e_j, (-\Delta_g + V) e_i \rangle_{L^2(M)} \\ &= \sum_{i,j} \alpha_i \alpha_j \lambda_i(M) \langle e_j, e_i \rangle_{L^2(M)} = \sum_{i=1}^k \alpha_i^2 \lambda_i(M) \leq \lambda_k(M). \end{aligned}$$

Hence, for all integer k , and for all function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$ we have

$$0 \leq A(f) \leq \lambda_k(M).$$

◆ Study of $B(f) := \int_M |d(f)v_A|^2 d\mathcal{V}_g$: here $v_A = \frac{u_A}{e_1}$ and $dv_A = \frac{du_A e_1 - u_A de_1}{e_1^2}$, so we get $B \leq \|df\|_\infty^2 \|v_A\|_{L^2(M)}^2$ and, with the Poincar inequality :

$$\|v_A\|_{L^2(M)}^2 \leq \left\| \frac{1}{e_1} \right\|_\infty^2 \|u_A\|_{L^2(M)}^2 \leq \left\| \frac{1}{e_1} \right\|_\infty^2 \frac{\text{cap}(A)}{\lambda_1(M)}$$

hence, for all integer k , and for all function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$ we have

$$0 \leq B(f) \leq E_k \text{cap}(A)$$

where $E_k = E_k(e_1, \lambda_1(M)) > 0$, moreover since the eigenfunction e_1 and the eigenvalue $\lambda_1(M)$ depends only on (M, g) and V , for all integer k the constant E_k depends only on (M, g) and V , ie : $E_k = E_k(M, V)$.

◆ Study of $C(f)$: here $C(f)$ is equal to $\underbrace{\int_M |fdv_A|^2 d\mathcal{V}_g}_{:=C_1(f)} + \underbrace{\int_M V |v_A f|^2 d\mathcal{V}_g}_{:=C_2(f)}$. Let us observe first $C_1(f)$:

$$C_1(f) \leq \|f\|_\infty^2 \|dv_A\|_{L^2(M)}^2$$

and

$$\begin{aligned} \|dv_A\|_{L^2(M)}^2 &= \int_M \left| \frac{du_A e_1 - u_A de_1}{e_1^2} \right|^2 d\mathcal{V}_g \\ &\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \int_M |du_A e_1 - u_A de_1|^2 d\mathcal{V}_g \\ &\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \left(\int_M |du_A e_1|^2 d\mathcal{V}_g + 2 \int_M |du_A de_1 e_1 u_A| d\mathcal{V}_g + \int_M |de_1 u_A|^2 d\mathcal{V}_g \right) \\ &\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \left(\|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2 + 2 \|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2 \right). \end{aligned}$$

Next we have also :

$$\begin{aligned} C_2(f) &= \int_M V |v_A f|^2 d\mathcal{V}_g \leq \|f\|_\infty^2 \int_M V |v_A|^2 d\mathcal{V}_g \\ &\leq \|f\|_\infty^2 \left\| \frac{1}{e_1} \right\|_\infty^2 \int_M V |u_A|^2 d\mathcal{V}_g. \end{aligned}$$

Hence we get :

$$\begin{aligned} C(f) &\leq \|f\|_\infty^2 \left\| \frac{1}{e_1} \right\|_\infty^2 \left[\|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2 \right. \\ &\quad \left. + 2 \|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2 \right] \\ &\quad + \|f\|_\infty^2 \left\| \frac{1}{e_1} \right\|_\infty^2 \int_M V |u_A|^2 d\mathcal{V}_g \\ &\leq \|f\|_\infty^2 \left\| \frac{1}{e_1} \right\|_\infty^2 \left[\|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2 + 2 \|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2 \right. \\ &\quad \left. + \int_M |du_A|^2 d\mathcal{V}_g + \int_M V |u_A|^2 d\mathcal{V}_g \right] \\ &\leq \|f\|_\infty^2 \left\| \frac{1}{e_1} \right\|_\infty^2 \left[\|du_A\|_{L^2(M)}^2 + \|V\|_\infty \|u_A\|_{L^2(M)}^2 \right. \\ &\quad \left. + 2 \|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2 \right]; \end{aligned}$$

so, since $\|du_A\|_{L^2(M)}^2 \leq \text{cap}(A)$ and $\|u_A\|_{L^2(M)}^2 \leq \frac{\text{cap}(A)}{\lambda_1(M)}$ we get for all integer k , and for all function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$:

$$0 \leq C(f) \leq F_k \text{cap}(A)$$

where $F_k = F_k(f, e_1, \lambda_1(M)) > 0$. Here, for k fixed, the constant F_k depends also on f , and f depends on the functions f_1, f_2, \dots, f_k (which are depends only on M and V) and on the scalars $\alpha_1, \alpha_2, \dots, \alpha_k$; since $\sum_{i=1}^k \alpha_i^2 = 1$, all the $(\alpha_i)_{1 \leq i \leq k}$ are bounded in \mathbb{R} , so finally, for all integer k the constant F_k can be bounded by a constant (we denotes also by $F_k = F_k(M, V)$) which depends only on M and V .

◆ Study of $|D(f)|$: we have

$$\begin{aligned} |D| &= \left| \int_M |df|^2 v_A d\mathcal{V}_g + \int_M V |f|^2 v_A d\mathcal{V}_g \right| \\ &\leq \|df\|_\infty^2 \left\| \frac{1}{e_1} \right\|_\infty \int_M \left| \frac{u_A}{e_1} \right| d\mathcal{V}_g + \left\| \frac{V |f|^2}{e_1} \right\|_\infty \int_M |u_A| d\mathcal{V}_g \\ &\leq \max \left(\|df\|_\infty^2 \left\| \frac{1}{e_1} \right\|_\infty, \left\| \frac{V |f|^2}{e_1} \right\|_\infty \right) \int_M |u_A| d\mathcal{V}_g \\ &\leq \max \left(\|df\|_\infty^2 \left\| \frac{1}{e_1} \right\|_\infty, \left\| \frac{V |f|^2}{e_1} \right\|_\infty \right) \sqrt{\text{Vol}(M)} \|u_A\|_{L^2(M)} \\ &\leq \max \left(\|df\|_\infty^2 \left\| \frac{1}{e_1} \right\|_\infty, \left\| \frac{V |f|^2}{e_1} \right\|_\infty \right) \sqrt{\text{Vol}(M)} \sqrt{\frac{\text{cap}(A)}{\lambda_1(M)}}. \end{aligned}$$

Hence, for all integer k , and for all function $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$:

$$|D(f)| \leq G_k \sqrt{\text{cap}(A)}$$

where (and for the same reasons as in the study of F , see the constant F_k) for all integer k , the constant G_k depends only on M and V , ie $G_k = G_k(M, V)$.

◆ Study of $|E(f)|$: recall that $E(f) = \int_M df dv_A f (1 - v_A) d\mathcal{V}_g$, hence

$$|E(f)| \leq \int_M |df dv_A| |f| d\mathcal{V}_g + \int_M |df dv_A| |f v_A| d\mathcal{V}_g.$$

For the first term $\int_M |df dv_A| |f| d\mathcal{V}_g$ we have :

$$\int_M |df dv_A| |f| d\mathcal{V}_g \leq \|f\|_\infty \|df\|_\infty \sqrt{\text{Vol}(M)} \|dv_A\|_{L^2(M)};$$

we have see in the study of $C(f)$ that

$$\begin{aligned} &\|dv_A\|_{L^2}^2 \\ &\leq \left\| \frac{1}{e_1} \right\|_\infty^2 \left(\|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2 + 2 \|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2 \right) \end{aligned}$$

so with $K := \|f\|_\infty \|df\|_\infty \sqrt{\text{Vol}(M)} \left\| \frac{1}{e_1} \right\|_\infty$ we get

$$\int_M |df dv_A| |f| d\mathcal{V}_g$$

$$\begin{aligned} &\leq K \sqrt{\|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2 + 2 \|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty^2 \|u_A\|_{L^2(M)}^2} \\ &\leq K \sqrt{\text{cap}(A) \|e_1\|_\infty^2 + 2 \|de_1\|_\infty \|e_1\|_\infty \sqrt{\text{cap}(A)} \sqrt{\frac{\text{cap}(A)}{\lambda_1(M)}} + \|de_1\|_\infty^2 \frac{\text{cap}(A)}{\lambda_1(M)}} \\ &\leq H_k \sqrt{\text{cap}(A)} \end{aligned}$$

where (same reasons as above), for all integer k , the constant H_k depends only on M and V , ie $H_k = H_k(M, V)$.
Next, for the second term : $\int_M |df dv_A| |fv_A| d\mathcal{V}_g$ we have :

$$\begin{aligned} \int_M |df dv_A| |fv_A| d\mathcal{V}_g &\leq \|df\|_\infty \|f\|_\infty \|dv_A\|_{L^2(M)} \|v_A\|_{L^2(M)} \\ &\leq \|df\|_\infty \|f\|_\infty \|dv_A\|_{L^2(M)} \left\| \frac{1}{e_1} \right\|_\infty \|u_A\|_{L^2(M)} \\ &\leq \|df\|_\infty \|f\|_\infty \left\| \frac{1}{e_1} \right\|_\infty \sqrt{\frac{\text{cap}(A)}{\lambda_1(M)}} H_k \sqrt{\text{cap}(A)} \\ &\leq H'_{k,M} \text{cap}(A). \end{aligned}$$

where (same reasons as above), for all integer k , the constant H_k depends only on M and V , ie $H'_k = H_k(M, V)$.
So, for all integer k :

$$|E(f)| \leq H''_{k,M} \left(\sqrt{\text{cap}(A)} + \text{cap}(A) \right),$$

where $H''_k := H''_k(M, V)$.

Finally, with the study of $A(f), B(f), C(f), |D(f)|$ and $|E(f)|$, for all integer k , for any function $\phi = f \left(1 - \frac{u_A}{e_1}\right) \in E_k$, with $f \in F_k$ such that $\|f\|_{L^2(M)} = 1$ we get :

$$\int_M |d\phi|^2 d\mathcal{V}_g + \int_M V |\phi|^2 d\mathcal{V}_g \leq \lambda_k(M) + I_k \left(\sqrt{\text{cap}(A)} + \text{cap}(A) \right)$$

where, for all integer k , the constant I_k depends only on M and V , ie : $I_k = I_k(M, V)$.

• **Step 3** : Now we claim that : for all $A \subset M$ such that $\text{cap}(A) \leq \varepsilon_k$ and for any function $\phi \in E_k$ we have :

$$\|\phi\|_{L^2(M)}^2 \geq 1 - J'_{k,M} \sqrt{\text{cap}(A)}$$

where, for all integer k , the constant $J'_{k,M}$ depend only on M and V , ie : $J'_{k,M} = J'_{k,M}(M, V)$.

Indeed : let $\phi \in E_k$, we have seen below in step 1 that :

$$\text{cap}(A) \leq \varepsilon_k \Rightarrow \dim(E_k) = k \text{ and } \forall j \in \{1, \dots, k\}, \left| \|\phi_j\|_{L^2(M)}^2 - 1 \right| \leq D_k \sqrt{\text{cap}(A)}$$

therefore, since $\phi \in E_k$, we can write $\phi = (1 - v_A)f$ with $f = \sum_{i=1}^k \alpha_i e_i$ where $(\alpha_i)_{1 \leq i \leq k} \in \mathbb{R}^k$. As in the step two

we can assume that $\|f\|_{L^2(M)} = 1$, hence we have $\sum_{i=1}^k \alpha_i^2 = 1$. Next, compute $\|\phi\|_{L^2(M)}^2$:

$$\|\phi\|_{L^2(M)}^2 = \left\| \sum_{i=1}^k (1 - v_A) \alpha_i e_i \right\|_{L^2(M)}^2 = \left\| \sum_{i=1}^k \alpha_i \phi_i \right\|_{L^2(M)}^2$$

$$= \sum_{i=1}^k \alpha_i^2 \|\phi_i\|_{L^2(M)}^2 + \sum_{i,j \text{ } i \neq j} \alpha_i \alpha_j \langle \phi_i, \phi_j \rangle_{L^2(M)}.$$

And since

$$\begin{aligned} \sum_{i=1}^k \alpha_i^2 \|\phi_i\|_{L^2(M)}^2 &= \sum_{i=1}^k \alpha_i^2 \left[1 - 2 \int_M e_i^2 v_A d\mathcal{V}_g + \int_M e_i^2 v_A^2 d\mathcal{V}_g \right] \\ &= 1 - \sum_{i=1}^k \alpha_i^2 \left[2 \int_M e_i^2 v_A d\mathcal{V}_g - \int_M e_i^2 v_A^2 d\mathcal{V}_g \right] \\ &= 1 - \sum_{i=1}^k \alpha_i^2 \int_M e_i^2 (2v_A - v_A^2) d\mathcal{V}_g; \end{aligned}$$

hence

$$\|\phi\|_{L^2(M)}^2 = 1 - \sum_{i=1}^k \alpha_i^2 \int_M e_i^2 (2v_A - v_A^2) d\mathcal{V}_g + \sum_{i,j \text{ } i \neq j} \alpha_i \alpha_j \langle \phi_i, \phi_j \rangle_{L^2(M)}$$

we have seen in step 1 that, for $\text{cap}(A)$ small enough :

$$\left| \langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j} \right| \leq B_k \left(\sqrt{\text{cap}(A)} + \text{cap}(A) \right)$$

hence, since all the $(\alpha_i)_{1 \leq i \leq k}$ are bounded in \mathbb{R} , and for $\text{cap}(A)$ small enough, we can find a constant $B'_{k,M}$ which depends only on M and V , ie $B'_k = B'_k(M, V)$ such that, for $\text{cap}(A)$ small enough :

$$\left| \sum_{i,j \text{ } i \neq j} \alpha_i \alpha_j \langle \phi_i, \phi_j \rangle_{L^2(M)} \right| \leq B'_k \sqrt{\text{cap}(A)}$$

and finally, in the same spirit as in the estimations in section 2, there exists a constant $B''_{k,M}$ which depends only on M and V , ie $B''_k = B''_k(M, V)$ such that, for $\text{cap}(A)$ small enough :

$$\left| \sum_{i=1}^k \alpha_i^2 \int_M e_i^2 (2v_A - v_A^2) d\mathcal{V}_g \right| \leq B''_k \sqrt{\text{cap}(A)}$$

so finally we obtain :

$$\|\phi\|_{L^2(M)}^2 \geq 1 - B'''_k \sqrt{\text{cap}(A)}$$

where the constant B'''_k depend only on M and V , ie : $B'''_k := B'''_k(M, V)$.

• **Final step** : As a consequence from step 2 and 3, for all function $\phi \in E_k$ we get :

$$\frac{\int_M |d\phi|^2 d\mathcal{V}_g + \int_M V |\phi|^2 d\mathcal{V}_g}{\int_M \phi^2 d\mathcal{V}_g} \leq \frac{\lambda_k(M) + I_k \left(\text{cap}(A) + \sqrt{\text{cap}(A)} \right)}{1 - B'''_k \sqrt{\text{cap}(A)}}$$

hence for $\text{cap}(A)$ small enough (ie : $\text{cap}(A) \leq \varepsilon_k$) we have

$$\frac{\int_M |d\phi|^2 d\mathcal{V}_g + \int_M V |\phi|^2 d\mathcal{V}_g}{\int_M \phi^2 d\mathcal{V}_g} \leq \lambda_k(M) + L_k \sqrt{\text{cap}(A)}$$

where $L_k := L_k(M, V)$. Next, since for all $k \geq 1$

$$\lambda_k(M - A) = \min_{\substack{E \subset H_0^1(M-A) \\ \dim(E)=k}} \max_{\substack{\varphi \in E \\ \varphi \neq 0}} \frac{\int_M |d\varphi|^2 d\mathcal{V}_g + \int_M V |\varphi|^2 d\mathcal{V}_g}{\int_M \varphi^2 d\mathcal{V}_g}$$

and since $\phi \in H_0^1(M - A)$, we get for all $k \geq 1$

$$\lambda_k(M - A) \leq \frac{\int_M |d\phi|^2 d\mathcal{V}_g + \int_M V |\phi|^2 d\mathcal{V}_g}{\int_M \phi^2 d\mathcal{V}_g} \leq \lambda_k(M) + C_k \sqrt{\text{cap}(A)}.$$

And the statement of the theorem is established. \square

References

- [1] C. Anné, *Problème de la glace pilée*, Séminaire de théorie spectrale et géométrie de Grenoble 3: 1-13, 1984-1985.
- [2] M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une varit Riemannienne*, Lectures Notes in Mathematics 194, Springer-Verlag 1971.
- [3] P. Bérard and M. Berger, *Le spectre d'une variété riemannienne en 1982*, Kaigai (Tokyo), 1983.
- [4] J. Bertrand and B. Colbois, *Capacités et inégalités de Faber-Krahn dans l'espace euclidien*, J. Funct. Anal., 232-1, 1-28, 2006.
- [5] I. Chavel and E. Feldman, *Spectra of domains in compact manifolds*, J. Funct. Anal., 30, 198-222, 1978.
- [6] I. Chavel and E. Feldman, *Spectra of manifolds less a small domain*, Duke Math J. , 56, 399-414, 1988.
- [7] R. Courant and D. Hilbert, *Methods of mathematical physics*, Intersciences Publishers, New York, 1953.
- [8] G. Courtois, *Spectrum of manifolds with holes*, J. Funct. Anal., 134, 194-221, 1995.
- [9] M. Kac, *Probabilistic methods in some problems of scattering theory*, Rocky Mountain J. Math 4, 511-538, 1974.
- [10] F. Gesztesy and Z. Zhao, *Domain perturbations, Browian motion, capacities, and ground states of Dirichlet Schrödinger operator*, Math. Z., 215, pp. 143-250, 1994.
- [11] S. Gilbard and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag Berlin, 1983.
- [12] M. Protter and H. Weinberger, *Maximum principles in differential equations*, Springer-Verlag, 1984.
- [13] J. Rauch and M. Taylor, *Potential and scattering on widly perturbed domains*, J. Funct. Anal., 18, 27-59, 1975.
- [14] M. Reed and B. Simon, *Methods of modern mathematical physics*, Academic Press, 1975.