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Generalized Jordan derivations on Frechet algebras

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Abstract. In this paper, we investigate generalized Jordan derivations on Frechet algebras. Moreover, we prove the generalized Hyers-Ulam-Rassias stability and superstability of generalized Jordan derivations on Frechet algebras. An important issue is so that we do not assume that the Frechet algebra is unital.

1. Introduction

Frechet algebras, named after Maurice Frechet, are special topological algebras. A topological algebra $\mathcal{A}$ is a Frechet algebra if it satisfies the following properties:

(1) it is complete as a uniform space;

(2) its topology may be induced by a countable family of submultiplicative semi-norms $\|\cdot\|_k, k = 0, 1, 2, \cdots$.

This means that a subset $\mathcal{U}$ of $\mathcal{A}$ is open if and only if for every u in $\mathcal{U}$ there exist a positive integer K and a nonnegative real number c such that $\{v : \|u - v\|_k < c, 0 \leq k \leq K\}$ is a subset of $\mathcal{U}$. Note that the topology on $\mathcal{A}$ can be induced by a translation invariant metric, i.e., a metric $\rho : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}^+$ such that $\rho(x, y) = \rho(x + a, y + a)$ for all $a, x, y \in \mathcal{A}$.

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation ζ must be close to an exact solution of ζ?* If the problem accepts a solution, we say that the equation ζ is stable. There are cases in which each 'approximate mapping' is actually a 'true mapping'. In such cases, we call the equation ζ superstable. The first stability problem concerning group homomorphisms was raised by Ulam [24] in 1940. We are given a group $\mathcal{G}$ and a metric group $\mathcal{G}'$ with metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : \mathcal{G} \longrightarrow \mathcal{G}'$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in \mathcal{G}$, then a homomorphism $h : \mathcal{G} \longrightarrow \mathcal{G}'$ exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in \mathcal{G}$? Ulam problem was partially solved by Hyers [17]. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

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for all $x, y \in E$, and for some $\varepsilon > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \varepsilon \tag{1}$$

for all $x \in E$. Also, if for each x the function $t \mapsto f(tx)$ from $\mathbb{R}$ to E' is continuous on $\mathbb{R}$, then T is linear. If f is continuous at a single point of E, then T is continuous everywhere in E. Moreover (1) is sharp.

In 1978, Th. M. Rassias [21] formulated and proved the following theorem, which implies Hyers Theorem as a special case. Suppose that E and E' are real normed spaces with E' complete, $f : E \longrightarrow E'$ is a mapping such that for each fixed $x \in E$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, and that there exist $\varepsilon > 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{2}$$

for all $x, y \in E$. Then there exists a unique linear mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{\varepsilon\|x\|^p}{1 - 2^{p-1}}$$

for all $x \in E$. In 1990, Th. M. Rassias [22] during the 27th International symposium on functional equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda following the same approach as in Th. M. Rassias [21], gave an affirmative solution to this question for $p > 1$. It was proved by Gajda [14], as well as by Th. M. Rassias and Semrl [23] that one can not prove a Th. M. Rassias type theorem when $p = 1$. In 1994, P. Gavruta [15] provided a further generalization of Th. M. Rassias Theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ in (2) by a general control function $\varphi(x, y)$ for the existence of a unique linear mapping. Badora [2] proved the generalized Hyers-Ulam stability of ring homomorphisms and generalized the result of Bourgin. Miura [20] proved the generalized Hyers-Ulam stability of Jordan homomorphisms. For more details about the results concerning stability of functional equation on Banach algebras, the reader refer to [3], [5], [8], and [10].

During the last decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [1], [4], [6], [16], [18], and [19].

Let $\mathcal{A}$ be an algebra. A linear mapping $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is called a generalized derivation if there exists a derivation (in the usual sense) $d : \mathcal{A} \longrightarrow \mathcal{A}$ such that $\delta(ab) = \delta(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. Also, a linear mapping $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is called a generalized Jordan derivation if there exists a Jordan derivation (in the usual sense) $d : \mathcal{A} \longrightarrow \mathcal{A}$ such that $\delta(a^2) = \delta(a)a + ad(a)$ for all $a, b \in \mathcal{A}$. The stability of derivations was studied by Park in [13]. M. Moslehian [11] investigated the Hyers-Ulam-Rassias stability of generalized derivations from a unital normed algebra $\mathcal{A}$ to a unit linked Banach $\mathcal{A}$-bimodule. M. Eshaghi et al. [9] proved the Hyers-Ulam-Rassias stability and superstability of generalized Jordan derivations from a unital normed algebra $\mathcal{A}$ to a unit linked Banach $\mathcal{A}$-bimodule.

In this paper, our aim is to establish the generalized Hyers-Ulam-Rassias stability of generalized Jordan derivations on Frechet algebras associated with the following functional equation

$$f(\frac{a + b}{2}) + f(\frac{a - b}{2}) = f(a).$$

Note that for our methods there is no need to assume that the Frechet algebra is unital (see [9] and [11]).

2. Superstability

Throughout this paper, it is assumed that $\mathcal{A}$ is an arbitrary Frechet algebra equipped with a metric ρ such that $\rho(2^n x, 0) = 2^n \rho(x, 0)$ for all $x \in \mathcal{A}$ and all nonnegative integers n. It is clear that $\rho(2^n x, 0) = 2^n \rho(x, 0)$ holds true for all $x \in \mathcal{A}$ and all integers n.

In this section, we prove the superstability of generalized Jordan derivations on Frechet algebras. For given mappings $f, g : \mathcal{A} \longrightarrow \mathcal{A}$, we define the difference functions $D_\mu f, D_\mu g : \mathcal{A}^3 \longrightarrow [0, \infty)$ by

$$D_\mu f(a,b,c) := \rho(f(\frac{\mu a + \mu b}{2} + c^2) + f(\frac{\mu a - \mu b}{2}), \mu f(a) + f(c)c + cg(c)),$$

$$D_\mu g(a,b,c) := \rho(g(\mu a^2 + \mu b + \mu c), \mu g(a)a + \mu a g(a) + \mu g(b) + \mu g(c))$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $a, b, c \in \mathcal{A}$.

We need the following lemma in our main results.

Lemma 2.1. *[12] Let X and Y be linear spaces and let $f : X \longrightarrow Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and $\mu \in \mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

We now commence our work with the following superstability problem for generalized Jordan derivations in Frechet algebras.

Theorem 2.2. *Let $p, q < 1$ or $p, q > 1$ and ε be nonnegative real numbers. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exists a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ with $g(0) = f(0) = 0$ such that*

$$D_\mu f(a,b,c) \leq \varepsilon \rho(f(c), 0)^{2p}, \tag{3}$$

$$D_\mu g(a,b,c) \leq \varepsilon(\rho(a,0)^{2q} + \rho(b,0)^q + \rho(c,0)^q) \tag{4}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a generalized Jordan derivation.

Proof. Assume that $p, q < 1$. By putting $c = 0$ and $\mu = 1$ in (3) we get

$$f(\frac{a+b}{2}) + f(\frac{a-b}{2}) = f(a) \tag{5}$$

for all $a, b \in \mathcal{A}$. Letting $\frac{a+b}{2} = w_1$ and $\frac{a-b}{2} = w_2$ in (5) we conclude that f is additive. Setting $b = c = 0$ and $\mu = 1$ in (3) we obtain

$$f(\frac{a}{2}) = \frac{1}{2} f(a) \tag{6}$$

for all $a \in \mathcal{A}$. Let $b = c = 0$ in (3) and apply (6) to deduce that $f(\mu a) = \mu f(a)$ for all $a \in \mathcal{A}$ and $\mu \in \mathbb{T}^1$. Now Lemma 2.1 implies f is $\mathbb{C}$-linear. Putting $a = b = 0$ in (3) we get

$$\rho(f(c^2), f(c)c + cg(c)) \leq \varepsilon \rho(f(c), 0)^{2p} \tag{7}$$

for all $c \in \mathcal{A}$. Replace c by $2^n c$ and multiply both sides of (7) by $\frac{1}{2^{2n}}$ to get

$$\rho(\frac{f(2^{2n}c^2)}{2^{2n}}, \frac{f(2^n c)2^n c}{2^{2n}} + \frac{2^n c g(2^n c)}{2^{2n}}) \leq \frac{\varepsilon}{2^{2n}} \rho(f(2^n c), 0)^{2p}$$

for all $c \in \mathcal{A}$ and nonnegative integers n. Hence

$$\rho(f(c^2), f(c)c + c\frac{g(2^n c)}{2^n}) \leq \varepsilon(\frac{2^p}{2})^{2n} \rho(f(c), 0)^{2p} \tag{8}$$

for all $c \in \mathcal{A}$ and nonnegative integers n. Letting n tend to ∞ in (8) we conclude that

$$f(c^2) = f(c)c + c \lim_{n\to\infty} \frac{g(2^n c)}{2^n}$$

for all $c \in \mathcal{A}$. By Hyers' Theorem, the sequence $\{\frac{g(2^n c)}{2^n}\}$ is convergent. Set $d(c) := \lim_{n\to\infty} \frac{g(2^n c)}{2^n}$ for all $c \in \mathcal{A}$ and so

$$f(c^2) = f(c)c + cd(c) \tag{9}$$

for all $c \in \mathcal{A}$. We now claim that $d : \mathcal{A} \longrightarrow \mathcal{A}$ is a Jordan derivation. Putting $a = 0$ and replacing b, c by $2^n b, 2^n c$, respectively and multiplying both sides of (4) by $\frac{1}{2^n}$ we get

$$\rho(\frac{g(\mu 2^n b + \mu 2^n c)}{2^n}, \mu\frac{g(2^n b)}{2^n} + \mu\frac{g(2^n c)}{2^n}) \leq \frac{\varepsilon}{2^n}(\rho(2^n b, 0)^q + \rho(2^n c, 0)^q) \tag{10}$$

for all $b, c \in \mathcal{A}$ and $\mu \in \mathbb{T}^1$. Taking the limit as $n \to \infty$ and using Lemma 2.1 we find that d is $\mathbb{C}$-linear. Letting $b = c = 0$ and $\mu = 1$ in (4) we get

$$\rho(g(a^2), g(a)a + ag(a)) \leq \varepsilon\rho(a, 0)^{2q} \tag{11}$$

for all $a \in \mathcal{A}$. Replace a by $2^n a$ and multiply both sides of (11) by $\frac{1}{2^{2n}}$ to get

$$\rho(\frac{g(2^{2n}a^2)}{2^{2n}}, \frac{g(2^n a)}{2^n}a + a\frac{g(2^n a)}{2^n}) \leq \frac{\varepsilon}{2^{2n}}\rho(2^n a, 0)^{2q} \tag{12}$$

for all $a \in \mathcal{A}$ and all nonnegative integers n. Hence by letting $n \to \infty$ in (12) we conclude that d is a Jordan derivation. It then follows from (9) that f is a generalized Jordan derivation. Similarly, one can replace c in (7) by $\frac{c}{2^n}$ and multiply both sides by 2^{2n} to obtain the result for the case where $p, q > 1$. □

It is clear that a Banach algebra $\mathcal{A}$ is a Frechet algebra and its metric is induced by its norm and so by Theorem 2.2 we may solve the following superstability problem for generalized Jordan derivations on Banach algebras.

Corollary 2.3. *Let $\mathcal{A}$ be a Banach algebra and let $p, q < 1$ or $p, q > 1$ and ε be nonnegative real numbers. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exists a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ with $g(0) = f(0) = 0$ such that*

$$\|f(\frac{\mu a + \mu b}{2} + c^2) + f(\frac{\mu a - \mu b}{2}) - \mu f(a) - f(c)c - cg(c)\| \leq \varepsilon\|f(c)\|^{2p}, \tag{13}$$

$$\|g(\mu a^2 + \mu b + \mu c) - \mu g(a)a - \mu a g(a) - \mu g(b) - \mu g(c)\| \leq \varepsilon(\|a\|^{2q} + \|b\|^q + \|c\|^q) \tag{14}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a generalized Jordan derivation.

3. Stability

In this section we prove the generalized Hyers-Ulam stability of generalized Jordan derivations.

Theorem 3.1. *Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exist a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ with $g(0) = f(0) = 0$ and a function $\varphi : \mathcal{A}^3 \longrightarrow [0, \infty)$ such that*

$$\tilde{\varphi}(a) := \sum_{i=1}^{\infty} \frac{1}{2^i}\varphi(2^i a, 0, 0) < \infty, \tag{15}$$

$$\lim_{i\to\infty} \frac{1}{2^i}\varphi(2^i a, 2^i b, 2^i c) = 0, \tag{16}$$

$$\max\{D_\mu f(a, b, c), D_\mu g(a, b, c)\} \leq \varphi(a, b, c) \tag{17}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique generalized Jordan derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\rho(\delta(a), f(a)) \leq \tilde{\varphi}(a) \tag{18}$$

for all $a \in \mathcal{A}$.

Proof. It follows from (17) that

$$D_\mu f(a,b,c) \leq \varphi(a,b,c), \tag{19}$$

$$D_\mu g(a,b,c) \leq \varphi(a,b,c). \tag{20}$$

By putting $b = c = 0$ and $\mu = 1$ in (19) we get

$$\rho(2f(\frac{a}{2}), f(a)) \leq \varphi(a,0,0) \tag{21}$$

for all $a \in \mathcal{A}$. If we replace a by $2a$ and multiply both sides of (21) by $\frac{1}{2}$ we get

$$\rho(\frac{f(2a)}{2}, f(a)) \leq \frac{1}{2}\varphi(2a,0,0) \tag{22}$$

for all $a \in \mathcal{A}$. Now we can use induction on n to show that

$$\rho(\frac{f(2^n a)}{2^n}, f(a)) \leq \sum_{i=1}^{n} \frac{1}{2^i}\varphi(2^i a,0,0) \tag{23}$$

for all $a \in \mathcal{A}$ and all nonnegative integers n. Hence

$$\begin{aligned}\rho(\frac{f(2^{n+m}a)}{2^{n+m}}, \frac{f(2^m a)}{2^m}) &\leq \sum_{i=1}^{n} \frac{1}{2^{i+m}}\varphi(2^{i+m}a,0,0)\\ &= \sum_{i=m+1}^{n+m} \frac{1}{2^i}\varphi(2^i a,0,0)\end{aligned}$$

for all $a \in \mathcal{A}$ and all nonnegative integers n, m with $n \geq m$. It follows from (15) that the sequence $\{\frac{f(2^n a)}{2^n}\}$ is Cauchy. Since $\mathcal{A}$ is complete this sequence converges. Set

$$\delta(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n}. \tag{24}$$

Putting $c = 0, \mu = 1$ and replacing a, b by $2^n a, 2^n b$, respectively and multiplying both sides of (19) by $\frac{1}{2^n}$ we get

$$\rho(\frac{f(2^n(\frac{a+b}{2}))}{2^n} + \frac{f(2^n(\frac{a-b}{2}))}{2^n}, \frac{f(2^n a)}{2^n}) \leq \frac{1}{2^n}\varphi(2^n a, 2^n b, 0) \tag{25}$$

for $a, b \in \mathcal{A}$ and all nonnegative integers n. Taking the limit as $n \to \infty$ we find that δ is additive. Letting $b = c = 0$, $\mu = 1$ and replacing a by $2^n a$ and multiplying both sides of (19) by $\frac{1}{2^n}$ we get

$$\rho(\frac{2f(\frac{2^n a}{2})}{2^n}, \frac{f(2^n a)}{2^n}) \leq \frac{1}{2^n}\varphi(2^n a,0,0) \tag{26}$$

for all $a \in \mathcal{A}$ and all nonnegative integers n. Taking the limit as $n \to \infty$ and using (16) we obtain

$$\delta(a) = 2\delta(\frac{a}{2}) \tag{27}$$

for all $a \in \mathcal{A}$. Letting $b = c = 0$ in (19) and using (27) we get $\delta(\mu a) = \mu\delta(a)$ and so Lemma 2.1 implies δ is $\mathbb{C}$-linear. Moreover, it follows from (23) and (24) as $n \to \infty$ that

$$\rho(\delta(a), f(a)) \leq \tilde{\varphi}(a) \tag{28}$$

for all $a \in \mathcal{A}$. It is known that the additive mapping δ satisfying (18) is unique. Putting $a = b = 0$ and replacing c by $2^n c$ and multiplying both sides of (19) by $\frac{1}{2^{2n}}$ we get

$$\rho(\frac{f(2^{2n}c^2)}{2^{2n}}, \frac{f(2^n c)}{2^n}c + c\frac{g(2^n c)}{2^n}) \leq \frac{1}{2^{2n}}\varphi(0, 0, 2^n c) \tag{29}$$

for all $c \in \mathcal{A}$ and all nonnegative integers n. By (24) and (16) the sequence $\{\frac{g(2^n c)}{2^n}\}$ is convergent. Set $d(c) := \lim_{n\to\infty} \frac{g(2^n c)}{2^n}$ for all $c \in \mathcal{A}$ and let n tend to ∞ in (29) to find that

$$\delta(c^2) = \delta(c)c + cd(c). \tag{30}$$

It remains to prove that d is a Jordan derivation. The rest of the proof is similar to the proof of Theorem 2.2 and we omit it. □

Corollary 3.2. *Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exist a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ with $g(0) = f(0) = 0$ and a function $\varphi : \mathcal{A}^3 \longrightarrow [0, \infty)$ such that satisfying* (17) *and*

$$\sum_{i=1}^{\infty} \frac{1}{2^i}\varphi(2^i a, 2^i b, 2^i c) < \infty \tag{31}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exist a unique generalized Jordan derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ and a function $\tilde{\varphi} : \mathcal{A} \longrightarrow [0, \infty)$ such that

$$\rho(\delta(a), f(a)) \leq \tilde{\varphi}(a) \tag{32}$$

for all $a \in \mathcal{A}$.

Proof. Put $\tilde{\varphi}(a) := \sum_{i=1}^{\infty} \frac{1}{2^i}\varphi(2^i a, 0, 0)$. The result follows from (31) and Theorem 3.1. □

Corollary 3.3. *Let $p < 1$ and ε be nonnegative real numbers. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exists a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ with $g(0) = f(0) = 0$ such that*

$$\max\{D_\mu f(a, b, c), D_\mu g(a, b, c)\} \leq \varepsilon(\rho(a, 0)^p + \rho(b, 0)^p + \rho(c, 0)^p) \tag{33}$$

for all $a, b, c \in \mathcal{A}$ and $\mu \in \mathbb{T}^1$. Then there exists a unique generalized Jordan derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\rho(\delta(a), f(a)) \leq \frac{2^p \varepsilon}{2 - 2^p}\rho(a, 0)^p \tag{34}$$

for all $a \in \mathcal{A}$.

Proof. Put $\varphi(a, b, c) := \varepsilon(\rho(a, 0)^p + \rho(b, 0)^p + \rho(c, 0)^p)$. Then the result follows from Corollary 3.2. □

By using Theorem 3.1 we may solve the following generalized Hyers-Ulam stability problem for generalized Jordan derivations in Banach algebras.

Corollary 3.4. *Let $\mathcal{A}$ be a Banach algebra. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exist a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ with $g(0) = f(0) = 0$ and a function $\varphi : \mathcal{A}^3 \longrightarrow [0, \infty)$ satisfying* (15) *and* (16) *such that*

$$\|f(\frac{\mu a + \mu b}{2} + c^2) + f(\frac{\mu a - \mu b}{2}) - \mu f(a) - f(c)c - cg(c)\| \leq \varphi(a, b, c), \tag{35}$$

$$\|g(\mu a^2 + \mu b + \mu c) - \mu g(a)a - \mu a g(a) - \mu g(b) - \mu g(c)\| \leq \varphi(a, b, c) \tag{36}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique generalized Jordan derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\rho(\delta(a), f(a)) \leq \tilde{\varphi}(a) \tag{37}$$

for all $a \in \mathcal{A}$.

Corollary 3.5. *Let $\mathcal{A}$ be a Banach algebra. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exist a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ with $g(0) = f(0) = 0$ and a function $\varphi : \mathcal{A}^3 \longrightarrow [0, \infty)$ such that*

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \varphi(2^i a, 2^i b, 2^i c) < \infty, \tag{38}$$

$$\|f(\frac{\mu a + \mu b}{2} + c^2) + f(\frac{\mu a - \mu b}{2}) - \mu f(a) - f(c)c - cg(c)\| \leq \varphi(a, b, c), \tag{39}$$

$$\|g(\mu a^2 + \mu b + \mu c) - \mu g(a)a - \mu a g(a) - \mu g(b) - \mu g(c)\| \leq \varphi(a, b, c) \tag{40}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exist a unique generalized Jordan derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ and a function $\tilde{\varphi} : \mathcal{A} \longrightarrow [0, \infty)$ such that

$$\rho(\delta(a), f(a)) \leq \tilde{\varphi}(a) \tag{41}$$

for all $a \in \mathcal{A}$.

Theorem 3.6. *Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exist a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ and a function $\varphi : \mathcal{A}^3 \longrightarrow [0, \infty)$ satisfying* (17) *such that*

$$\tilde{\varphi}(a) := \sum_{i=1}^{\infty} 2^i \varphi(\frac{a}{2^i}, 0, 0) < \infty, \tag{42}$$

$$\lim_{i \to \infty} 2^i \varphi(\frac{a}{2^i}, \frac{b}{2^i}, \frac{c}{2^i}) = 0 \tag{43}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique generalized Jordan derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\rho(\delta(a), f(a)) \leq \tilde{\varphi}(a) \tag{44}$$

for all $a \in \mathcal{A}$.

Proof. Letting $a = b = c = 0$ in (43) we get $\lim_{i \to \infty} 2^i \varphi(0, 0, 0) = 0$ and so $\varphi(0, 0, 0) = 0$. Now put $a = b = c = 0$ in (20) to find that $D_\mu g(0, 0, 0) = 0$. Thus, $(2\mu - 1)g(0) = 0$. Since $\mu \in \mathbb{T}^1$, $g(0) = 0$. Put $a = b = c = 0$ and $\mu = 1$ in (19) to get $D_\mu f(0, 0, 0) = 0$. Since $g(0) = 0$, we conclude that $f(0) = 0$. By suitable replacements in (19) it is clear that the sequence $\{2^m f(\frac{a}{2^m})\}$ converges for all $a \in \mathcal{A}$. Define $\delta(a) := \lim_{m \to \infty} 2^m f(\frac{a}{2^m})$. The rest of the proof is similar to the proof of Theorem 3.1 and we omit it. □

Corollary 3.7. *Let $p > 1$ and $\varepsilon > 0$ be real numbers. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exists a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying* (33)*. Then there exists a unique generalized Jordan derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ such that*

$$\rho(\delta(a), f(a)) \leq \frac{2\varepsilon}{2^p - 2} \rho(a, 0)^p \tag{45}$$

for all $a \in \mathcal{A}$.

Proof. Putting $\varphi(a, b, c) := \varepsilon(\rho(a, 0)^p + \rho(b, 0)^p + \rho(c, 0)^p)$ in Theorem 3.6 we get the desired result. □

Corollary 3.8. *Let $\mathcal{A}$ be a Banach algebra. Suppose that $p \neq 1$ and $\varepsilon > 0$ are nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exists a map $g : \mathcal{A} \longrightarrow \mathcal{A}$ with $g(0) = f(0) = 0$ satisfying* (33) *such that*

$$\|f(\frac{\mu a + \mu b}{2} + c^2) + f(\frac{\mu a - \mu b}{2}) - \mu f(a) - f(c)c - cg(c)\| \leq \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p), \tag{46}$$

$$\|g(\mu a^2 + \mu b + \mu c) - \mu g(a)a - \mu a g(a) - \mu g(b) - \mu g(c)\| \leq \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p) \tag{47}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique generalized Jordan derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\|\delta(a) - f(a)\| \leq \frac{2\varepsilon}{|2 - 2^p|} \|a\|^p \tag{48}$$

for all $a \in \mathcal{A}$.

Proof. It follows from Corollary 3.3 and Corollary 3.7 by putting $\rho(a, b) = \|a - b\|$ for all $a, b \in \mathcal{A}$. □

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