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On pairs of generalized and hypergeneralized projections in a Hilbert space

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Abstract. We characterize generalized and hypergeneralized projection i.e. bounded linear operators which satisfy conditions $A^2 = A^*$, or $A^2 = A^\dagger$, respectively. We establish their matrix representations and examine conditions which imply that the product, difference and sum of these operators belongs to same class of operators.

1. Introducton

Let H be a Hilbert space and let $\mathcal{L}(H)$ be a space of all bounded linear operators on H. The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^* and $\sigma(A)$, respectively, will denote the range, the null space, the adjoint operator, and the spectrum of $A \in \mathcal{L}(H)$. Recall that $A \in \mathcal{L}(H)$ is a projection if $A^2 = A$, while it is an orthogonal projection if $A^* = A = A^2$. An operator A is hermitian (self adjoined) if $A = A^*$, normal if $AA^* = A^*A$, and unitary if $AA^* = A^*A = I$.

The Moore-Penrose inverse of $A \in \mathcal{L}(H)$, denoted by $A^\dagger$, is the unique solution to the equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

Notice that $A^\dagger$ exists if and only if $\mathcal{R}(A)$ is closed. Then $AA^\dagger$ is the orthogonal projection onto $\mathcal{R}(A)$ parallel to $\mathcal{N}(A^*)$, and $A^\dagger A$ is orthogonal projection onto $\mathcal{R}(A^*)$ parallel to $\mathcal{N}(A)$. Consequently, $I - AA^\dagger$ is the orthogonal projection onto $\mathcal{N}(A^*)$, and $I - A^\dagger A$ is the orthogonal projection onto $\mathcal{N}(A)$.

For $A \in \mathcal{L}(H)$, an element $B \in \mathcal{L}(H)$ is the Drazin inverse of A, if the following hold:

$$BAB = B, \quad BA = AB, \quad A^{n+1}B = A^n,$$

for some non-negative integer n. The smallest such n is called the Drazin index of A, denoted by $ind(A)$. By A^D we denote Drazin inverse of A. Recall that A^D is unique if it exsits. Also, if A^D exists then 0 is not the accumulation point of $\sigma(A)$.

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An operator A is invertible if and only if $ind(A) = 0$.

If $ind(A) \leq 1$, then A is group invertible and A^D is the group inverse of A, usually denoted by $A^\#$.

An operator $A \in \mathcal{L}(H)$ is EP if $AA^\dagger = A^\dagger A$, or equivalently, if $A^\dagger = A^D = A^\#$. The set of all EP operators on H will be denoted by $\mathcal{EP}(H)$. Self-adjoint and normal operators with closed range are important subset of set of all EP operators. However, converse is not true even in a finite dimensional case.

Recall that an operator $A \in \mathcal{L}(H)$ is a partial isometry, if and only if $A^* = A^\dagger$.

In this paper we will consider pairs of generalized and hypergeneralized projections on a Hilbert space, whose concept was introduced in [7]. These operators extend the idea of orthogonal projections by removing the idempotency requirement. Namely, we have the following definition:

Definition 1.1. *An operator $A \in \mathcal{L}(H)$ is*

(a) *a generalized projection if $A^2 = A^*$,*
(b) *a hypergeneralized projection if $A^2 = A^\dagger$.*

The set of all generalized projectons on H is denoted by $\mathcal{GP}(H)$, and the set of all hypergeneralized projectons is denoted by $\mathcal{HGP}(H)$.

We rely upon operator matrix representations whenever it is possible, which makes our proofs much simpler in several occasions.

2. Characterization of generalized and hypergeneralized projections

We begin this section by giving characterizations of generalized and hypergeneralized projection. The following result is in [4]. For the sake of completeness, we give a proof which is shorter than the one in [4].

Theorem 2.1. *Let $A \in \mathcal{L}(H)$. Then the following conditions are equivalent:*

(a) *A is a generalized projection,*
(b) *A is a normal operator and $A^4 = A$,*
(c) *A ia a partial isometry and $A^4 = A$.*

Proof. (a) $\Longrightarrow$ (b): Since

$$AA^* = AA^2 = A^3 = A^2A = A^*A,$$
$$A^4 = (A^2)^2 = (A^*)^2 = (A^2)^* = (A^*)^* = A,$$

the implication is obvious.

(b) $\Longrightarrow$ (a): If $AA^* = A^*A$, recall that then there exists a unique spectral measure E on the Borrel subsets of $\sigma(A)$ such that A has the following spectral representation

$$A = \int_{\sigma(A)} \lambda dE_\lambda.$$

From $A^4 = A$ we conclude $\sigma(A) \subset \{0, 1, e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}\}$. Now,

$$A = 0E(0) \oplus 1E(1) \oplus e^{\frac{2\pi i}{3}} E(e^{\frac{2\pi i}{3}}) \oplus e^{\frac{-2\pi i}{3}} E(e^{\frac{-2\pi i}{3}}),$$

where $E(\alpha)$ is the spectral projection of the normal operator A associated with spectral point α, $E(\alpha) \neq 0$ if $\alpha \in \sigma(A)$, $E(\alpha) = 0$ if $\alpha \in \{0, 1, e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}\} \backslash \sigma(A)$ and $\sum_{\alpha \in \sigma(A)} \oplus E(\alpha) = I$. It is easy to see that $A^2 = A^*$.

(a) $\Longrightarrow$ (c): If $A^* = A^2$, then we know $A = A^4 = AA^2A = AA^*A$. Multiplying from the left side (or from the right side) by A^*, we get $A^*AA^*A = A^*A$ (or $AA^*AA^* = AA^*$), which proves that A^*A (or AA^*) is the orthogonal projection onto $\mathcal{R}(A^*A) = \mathcal{R}(A^*) = \mathcal{N}(A)^\perp$ (or $\mathcal{R}(AA^*) = \mathcal{R}(A) = \mathcal{N}(A^*)^\perp$) i.e. $A^* = A^\dagger$ and A is a partial isometry.

(c) $\Longrightarrow$ (a): If A is a partial isometry, we know that $A^* = A^\dagger$ and AA^* is the orthogonal projection onto $\mathcal{R}(AA^*) = \mathcal{R}(A)$. Thus, $AA^*A = P_{\mathcal{R}(A)}A = A$. Now, $A^4 = AA^2A = A$ implies $A^2 = A^*$. $\square$

Now we prove a similar result for hypergeneralized projections.

Theorem 2.2. *Let $A \in \mathcal{L}(H)$. Then the following conditions are equivalent:*

(a) *A is a hypergeneralized projecton,*
(b) *A^3 is an orthogonal projection onto $\mathcal{R}(A)$,*
(c) *A is an EP operator and $A^4 = A$*

Proof. (a) $\implies$ (b): If $A^2 = A^\dagger$, then from $A^3 = AA^\dagger = P_{\mathcal{R}(A)}$ conclusion follows.

(b) $\implies$ (a): If $A^3 = P_{\mathcal{R}(A)}$, a direct verification of the Moore-Penrose equations shows that $A^2 = A^\dagger$.

(a) $\implies$ (c): Since

$$AA^\dagger = AA^2 = A^3 = A^2A = A^\dagger A,$$

we conclude that A is EP, $A^\dagger = A^\#$, $(A^\dagger)^n = (A^n)^\dagger$ and

$$A^4 = (A^2)^2 = (A^\dagger)^2 = (A^2)^\dagger = (A^\dagger)^\dagger = A.$$

(c $\Rightarrow$ a) If A is an EP operator, then $A^\dagger = A^\#$ and $ind(A) = 1$ or, equivalently, $A^2A^\dagger = A$. Since $A^4 = A^2A^2 = A$, from uniqueness of $A^\dagger$ follows $A^2 = A^\dagger$. □

We can give matrix representatons of generalized and hypergeneralized projections based upon previous characterizatons.

Theorem 2.3. *Let $A \in \mathcal{L}(H)$ be a generalized projection. Then A is a closed range operator, $H = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Restriction $A_1 = A|_{\mathcal{R}(A)}$ is unitary on $\mathcal{R}(A)$ and A^3 is an orthogonal projection on $\mathcal{R}(A)$. Moreover, A has the following matrix representaton with the respect to decomposition of the space*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}.$$

Proof. If $A^2 = A^*$, A is a partial isometry (i.e. orthogonal projection) onto $\mathcal{R}(A) = \mathcal{R}(A^*) = \mathcal{N}(A)^\perp$. Thus, $\mathcal{R}(A)$ is a closed subset in H as a range of an orthogonal projection on a Hilbert space and we have the following decomposition of the space $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$.

Now, A has the following matrix representation in accordance with decomposition $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

where $A_1^2 = A_1^*$, $A_1^4 = A_1$ and $A_1A_1^* = A_1^*A_1 = A_1^3 = I_{\mathcal{R}(A)}$.
□

Theorem 2.4. *Let $A \in \mathcal{L}(H)$ be a hypergeneralized projection. Then A is a closed range operator, $H = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = \mathcal{R}(A) \oplus \mathcal{N}(A)$. Restriction $A_1 = A|_{\mathcal{R}(A)}$ satisfies $A_1^3 = I_{\mathcal{R}(A)}$, $A_1^2 = A_1^\dagger$ and A has the following matrix representaton with the respect to decomposition of the space*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}.$$

Proof. If A is hypergeneralized projecton, A is EP and we have the following decomposition of the space $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$ and A has the required representation. □

Notice that since $\mathcal{R}(A)$ is closed for both generalized and hypergeneralized projections, these operators have the Moore-Penrose and Drazin inverses. Besides, they are EP operators, which implies that $A^\dagger = A^D = A^\# = A^2 = A^4$. For generalized projections we can be more precise:

$$A^\dagger = A^D = A^\# = A^2 = A^* = A^4.$$

We can also write

$$\mathcal{GP}(H) \subseteq \mathcal{HGP}(H) \subseteq \mathcal{EP}(H).$$

Theorem 2.5. *Let $A \in \mathcal{L}(H)$. Then the following holds:*

(a) *$A \in \mathcal{GP}(H)$ if and only if $A^* \in \mathcal{GP}(H)$;*
(b) *$A \in \mathcal{GP}(H)$ if and only if $A^\dagger \in \mathcal{GP}(H)$;*
(c) *If $ind(A) \leq 1$, then $A \in \mathcal{GP}(H)$ if and only if $A^\# \in \mathcal{GP}(H)$.*

Proof. (a) If $A \in \mathcal{GP}(H)$, then $(A^*)^2 = (A^2)^* = (A^*)^* = A$ meaning that $A^* \in \mathcal{GP}(H)$. Conversely, if $A^* \in \mathcal{GP}(H)$, then $A^2 = ((A^*)^*)^2 = ((A^*)^2)^* = A*$ and $A \in \mathcal{GP}(H)$.

(b) If $A \in \mathcal{GP}(H)$, then $A^\dagger = A^\# = A^* = A^2$ and $(A^\dagger)^2 = (A^2)^\dagger = (A^*)^\dagger = (A^\dagger)^*$ implying $A^\dagger \in \mathcal{GP}$.

If $A^\dagger \in \mathcal{GP}(H)$, then A and $A^\dagger$ have the representation

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} A_1^*B & 0 \\ A_2^*B & 0 \end{bmatrix},$$

where $B = (A_1A_1^* + A_2A_2^*)^{-1}$. From $(A^\dagger)^2 = (A^\dagger)^*$, we get

$$\begin{bmatrix} A_1^*BA_1^*B & 0 \\ A_2^*BA_1^*B & 0 \end{bmatrix} = \begin{bmatrix} BA_1 & BA_2 \\ 0 & 0 \end{bmatrix},$$

which implies $A_2^* = 0$, $B = (A_1A_1^*)^{-1}$ and

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $(A_1^{-1})^2 = (A_1^{-1})^*$, the same equality is also satisfied for A_1 and $A \in \mathcal{GP}$.

(c) If $A \in \mathcal{GP}(H)$, then A is EP and " $\Rightarrow$ " part is established in (b) of this theorem.

To prove " $\Leftarrow$ ", assume that $H = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$ and $ind(A) \leq 1$. Then

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad A^\# = \begin{bmatrix} A_1^\# & (A_1^\#)^2A_2 \\ 0 & 0 \end{bmatrix}.$$

Since $(A\#)^2 = (A^\#)^*$, we get $A_2 = 0$ and $(A_1^\#)^2 = (A_1^\#)^*$. Fron the fact that A_1 is surjective on $\mathcal{R}(A)$ and $\mathcal{R}(A_1) \cap \mathcal{N}(A_1) = \{0\}$, we have $A_1^\# = A_1^{-1}$. Consequently, $(A_1^{-1})^2 = (A_1^{-1})^*$ and $A_1^2 = A_1^*$. □

Theorem 2.6. *Let $A \in \mathcal{L}(H)$. Then the following holds:*

(a) *$A \in \mathcal{HGP}(H)$ if and only if $A^* \in \mathcal{HGP}(H)$;*
(b) *$A \in \mathcal{HGP}(H)$ if and only if $A^\dagger \in \mathcal{HGP}(H)$;*
(c) *If $ind(A) \leq 1$, then $A \in \mathcal{HGP}(H)$ if and only if $A^\# \in \mathcal{HGP}(H)$.*

Proof. Proofs of (a) and (b) are similar to proofs of Theorem 2.5 (a) and (b).

(c) We should only prove that $A^\# \in \mathcal{HGP}(H)$ implies $A \in \mathcal{HGP}(H)$, since the " $\Rightarrow$ " is analogous to the sema part of Theorem 2.5.

Let $H = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$ and $ind(A) \leq 1$. Then

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad A^\# = \begin{bmatrix} A_1^{-1} & (A_1^{-1})^2A_2 \\ 0 & 0 \end{bmatrix}, \quad (A^\#)^\dagger = \begin{bmatrix} (A_1^{-1})^*B & 0 \\ (A_2^{-1})^*B & 0 \end{bmatrix},$$

where $B = (A_1^{-1}(A_1^{-1})^* + A_2^{-1}(A_2^{-1})^*)^{-1}$. From $(A^\#)^\dagger = (A^\#)^2$, we get $A_2 = 0$ and $A_1 = A_1^{-2}$. Multiplying with A_1^2, the last equation becomes $A_1^3 = I_{\mathcal{R}(A)}$ and $A \in \mathcal{HGP}(H)$. □

As we know, if A is a projection (orthogonal projection), $I - A$ is also a projection (orthogonal projection). It is of interest to examine whether generalized and hypergeneralized projections keep the same property.

Example 2.7. *If* $H = C^2$ *and* $A = \begin{bmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 \end{bmatrix}$, *then* $A^2 = A^*$, *but* $I - A = \begin{bmatrix} 1 - e^{\frac{2\pi i}{3}} & 0 \\ 0 & 1 \end{bmatrix}$ *and, clearly,* $I - A \neq (I - A)^4$ *implying that* $I - A$ *is not a generalized projection.*

Thus, we have the following theorem.

Theorem 2.8. *Let* $A \in \mathcal{L}(H)$ *be a generalized projection. Then* $I - A$ *is a normal operator. Moreover,* $I - A$ *is a generalized projection if and only if* A *is an orthogonal projection.*

If $I - A$ *is a generalized projection, then* A *is a normal operator and* A *is a generalized projection if and only if* $I - A$ *is an orthogonal projection.*

Proof. Let us assume that A has representation

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}.$$

Then

$$I - A = \begin{bmatrix} I_{\mathcal{R}(A)} - A_1 & 0 \\ 0 & I_{\mathcal{N}(A)} \end{bmatrix}$$

and it is obvious that normality of A implies normality of $I - A$. Also,

$$(I - A)^2 = \begin{bmatrix} (I_{\mathcal{R}(A)} - A_1)^2 & 0 \\ 0 & I_{\mathcal{N}(A)} \end{bmatrix} = \begin{bmatrix} (I_{\mathcal{R}(A)} - A_1)^* & 0 \\ 0 & I_{\mathcal{N}(A)} \end{bmatrix} = (I - A)^*$$

holds if and only if $(I_{\mathcal{R}(A)} - A_1)^2 = (I_{\mathcal{R}(A)} - A_1)^*$. Since $A^2 = A^*$, we get

$$I_{\mathcal{R}(A)} - 2A_1 + A_1^2 = I_{\mathcal{R}(A)} - 2A_1 + A^* = I_{\mathcal{R}(A)} - A_1^*,$$

which is satisfied if and only if $A_1 = A_1^*$. Hence, $A = A^* = A^2$. □

Next example shows that Theorem 2.6 does not hold for hypergeneralized projections.

Example 2.9. *Let* $H = C^2$ *and* $A = \begin{bmatrix} 1 & 1 \\ 0 & e^{\frac{2\pi i}{3}} \end{bmatrix}$. *Then* $A^2 = \begin{bmatrix} 1 & 1 + e^{\frac{2\pi i}{3}} \\ 0 & e^{\frac{-2\pi i}{3}} \end{bmatrix}$, $A^3 = I_{\mathcal{R}(A)}$, $A^4 = A$ *and* A *is a hypergeneralizes projection. However,* $I - A = \begin{bmatrix} 0 & -1 \\ 0 & 1 - e^{\frac{2\pi i}{3}} \end{bmatrix}$ *and it is not normal.*

3. Properties of product, difference and sum of generalized and hypergeneralized projections

In this section we will examine under what conditions product, difference and sum of generalized (hypergeneralized) projections is a generalized (hypergeneralized) projection. Next theorem gives very useful matrix representations of generalized projections.

Theorem 3.1. *Let* $A, B \in \mathcal{GP}(H)$ *and* $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$. *Then* B *has the following representation with respect to decomposition of the space:*

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

where

$$\begin{aligned} B_1^* &= B_1^2 + B_2B_3, \\ B_2^* &= B_3B_1 + B_4B_3, \\ B_3^* &= B_1B_2 + B_2B_4, \\ B_4^* &= B_3B_2 + B_4^2. \end{aligned}$$

Proof. Let B has a representation

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then, if

$$B^2 = \begin{bmatrix} B_1^2 B_2 B_3 & B_1B_2 + B_2B_4 \\ B_3B_1 + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix} = \begin{bmatrix} B_1^* & B_3^* \\ B_2^* & B_4^* \end{bmatrix} = B^*,$$

conclusion follows directly. □

Theorem 3.2. *Let $A, B \in \mathcal{GP}(H)$. Then the following conditions are equivalent:*

(a) $AB \in \mathcal{GP}(H)$
(b) $AB = BA$;
(c) AB *is normal.*

Proof. ((a) ⇒ (b) and (c)) Assume that A, B have representations given in Theorem 3.1. Then

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} B_1A_1 & B_1A_2 \\ B_3A_1 & B_3A_2 \end{bmatrix}.$$

It is easy to see that

$$(AB)^2 = \begin{bmatrix} (A_1B_1)^2 & A_1B_1A_1B_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (A_1B_1)^* & 0 \\ (A_1B_2)^* & 0 \end{bmatrix} = (AB)^*$$

if and only if $A_1B_1 = B_1A_1$, $A_1B_1A_1B_2 = 0$ and $(A_1B_2)^* = 0$, if and only if A_1 and B_1 commute and $B_2 = 0$. Again form Theorem 3.1 we conclude that $B_3 = 0$, $B_1^* = B_1^2$ and $B_4^* = B_4^2$. Now, B has the form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_4 \end{bmatrix}$$

and $AB = BA$. Moreover,

$$AB(AB)^* = \begin{bmatrix} A_1B_1(A_1B_1)^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (A_1B_1)^*A_1B_1 & 0 \\ 0 & 0 \end{bmatrix} = (AB)^*AB.$$

((b) ⇒ (a)) If $AB = BA$, Theorem 3.1 implies $B_2 = 0$, $B_3 = 0$, $A_1B_1 = B_1A_1$. Direct calculation shows that $(AB)^2 = (AB)^*$.

((c) ⇒ (a)) If we use representations given in Theorem 3.1, then condition

$$\begin{aligned} AB(AB)^* &= \begin{bmatrix} A_1B_1(A_1B_1)^* + A_1B_2(A_1B_2)^* & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (A_1B_1)^*A_1B_1 & (A_1B_1)^*A_1B_2 \\ (A_1B_2)^*A_1B_1 & (A_1B_2)^*A_1B_2 \end{bmatrix} = (AB)^*AB \end{aligned}$$

implies that $(A_1B_2)^*A_1B_2 = 0$, from where $B_2 = 0$ follows. Consequently, $B_3 = 0$ and $(AB)^2 = (AB)^*$. □

Theorem 3.3. *Let $A, B \in \mathcal{GP}(H)$. Then the following conditions are equivalent:*

(a) $A + B \in \mathcal{GP}(H)$
(b) $AB = BA = 0$.

Proof. ((a) ⇒ (b)) If A, B have representations given in Theorem 3.1, then

$$A + B = \begin{bmatrix} A_1 + B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

and if

$$\begin{aligned} (A+B)^2 &= \begin{bmatrix} (A_1 + B_1)^2 + B_2B_3 & (A_1 + B_1)B_2B_4 \\ B_3(A_1 + B_1) + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix} \\ &= \begin{bmatrix} (A_1 + B_1)^* & B_3^* \\ B_2^* & B_4^* \end{bmatrix} = (A+B)^*, \end{aligned}$$

it is clear that $(A_1 + B_1)^2 = (A_1 + B_1)^*$, $B_2 = B_3 = 0$, $B_4^2 = B_4^*$. Besides,

$$(A_1 + B_1)^2 = A_1^2 + A_1B_1 + B_1A_1 + B_1^2 = A_1^* + B_1^* = (A_1 + B_1)^*$$

is true if $A_1B_1 = B_1A_1 = 0$, $A_1^2 = A_1^*$ and $B_1^2 = B_1^*$. In this case we obtain $AB = BA = 0$.

((b) ⇒ (a)) If $AB = BA = 0$, then $A_1B_1 = B_1A_1 = 0$, $B_2 = B_3 = 0$, $B_1^2 = B_1^*$, $B_4^2 = B_4^*$ and, obviously, $(A + B)^2 = (A + B)^*$. □

Theorem 3.4. *Let $A, B \in \mathcal{GP}(H)$. Then $A - B \in \mathcal{GP}(H)$ if and only if $AB = BA = B^*$.*

Proof. If A, B have representations given in Theorem 3.1, then

$$A - B = \begin{bmatrix} A_1 - B_1 & -B_2 \\ -B_3 & -B_4 \end{bmatrix}.$$

From

$$\begin{aligned} (A-B)^2 &= \begin{bmatrix} (A_1 - B_1)^2 + B_2B_3 & -(A_1 - B_1) + B_2B_4 \\ -B_3(A_1 + B_1) + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix} \\ &= \begin{bmatrix} (A_1 - B_1)^* & -B_3^* \\ -B_2^* & -B_4^* \end{bmatrix} = (A-B)^*, \end{aligned}$$

$B_2 = 0$, $B_3 = 0$, $B_4^2 = -B_4^*$ and

$$(A_1 - B_1)^2 = A_1^2 - A_1B_1 - B_1A_1 + B_1^2 = A_1^* - B_1^*$$

follows. This is true if and only if $A_1B_1 = B_1A_1 = B_1^*$ and $B_4 = 0$, and in that case $AB = BA = B^*$. □

Theorem 3.5. *Let $A, B \in \mathcal{HGP}(H)$. Then $AB \in \mathcal{HGP}(H)$ if and only if $AB = BA$.*

Proof. Let $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$ and $A, B \in \mathcal{HGP}(H)$ have representations

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{bmatrix}, (AB)^2 = \begin{bmatrix} A_1B_1A_1B_1 & A_1B_1A_1B_2 \\ 0 & 0 \end{bmatrix}.$$

It is not difficult to see that

$$(AB)^\dagger = \begin{bmatrix} (A_1B_1)^*D^{-1} & 0 \\ (A_1B_2)^*D^{-1} & 0 \end{bmatrix},$$

where $D = A_1B_1(A_1B_1)^* + A_1B_2(A_1B_2)^* > 0$ is invertible.

If $(AB)^2 = (AB)^\dagger$, then $B_2 = 0$ which implies $D = A_1B_1(A_1B_1)^*$ is invertible and

$$(AB)^2 = \begin{bmatrix} (A_1B_1)^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (A_1B_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = (AB)^\dagger,$$

from where $A_1B_1 = B_1A_1$ follows.

We can rewrite B in form

$$B = \begin{bmatrix} B_1 & 0 \\ B_3 & B_4 \end{bmatrix},$$

while $B^\dagger = B^2$ is

$$B^\dagger = \begin{bmatrix} B_1^2 & 0 \\ B_3B_1 + B_4B_3 & B_4^2 \end{bmatrix}.$$

The Moore-Penrose equation in the matrix form is

$$\begin{aligned} B^\dagger BB^\dagger &= \begin{bmatrix} B_1^5 & 0 \\ (B_3B_1 + B_4B_3)B_1^3 + B_4^2(B_3B_1^2 + B_4(B_3B_1 + B_4B_3)) & B_4^5 \end{bmatrix} \\ &= \begin{bmatrix} B_1^2 & 0 \\ B_3B_1 + B_4B_3 & B_4^2 \end{bmatrix} = B^\dagger. \end{aligned}$$

Now, $B_1^5 = B_1^2$, $B_4^5 = B_4^2$ and

$$B_3B_1^4 + B_4B_3B_1^3 + B_4^2B_3B_1^2 + B_4^3B_3B_1 + B_4^4B_3 = B_3B_1 + B_4B_3,$$

which is equivalent to

$$B_4B_3B_1^3 + B_4^2B_3B_1^2 + B_4^3B_3B_1 = 0$$

and $B_3 = 0$.

Finally,

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_4 \end{bmatrix},$$

and $AB = BA$.

Conversely, assume that hypergeneralized projections A, B commute i.e. that

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1A_1 & 0 \\ B_3A_1 & 0 \end{bmatrix} = BA.$$

This implies $B_2 = 0$, $B_3 = 0$, $A_1B_1 = B_1A_1$ and it is easy to see that $(AB)^2 = (AB)^\dagger$. □

4. Additional results

Remark 1. Let A be a generalized projection. Then for an arbitrary $\alpha \in \mathbb{C}$, αA is not necessarily a generalized projection. Due to a condition $A^3 = I_{\mathcal{R}(A)}$, we have that $(\alpha A)^3 = I_{\mathcal{R}(A)}$ and $(\alpha\lambda)^3 = 1$, where $\lambda \in \sigma(A)$. Thus we get $\alpha \in \sigma(A)$.

Remark 2. Product of orthogonal projector P and generalized inverse A in general case does not keep any of the properties that either of these operators has. Observe the decomposition $H = L \oplus L^\perp$, where $L = \mathcal{R}(P)$. Then

$$P = \begin{bmatrix} I_L & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad PA = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}.$$

It is not difficult to see that PA is orthogonal projection if and only if $A_1 = I_L$. Then

$$A = \begin{bmatrix} I_L & 0 \\ 0 & A_4 \end{bmatrix}.$$

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