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Essentially $(\lambda, \mu)-$Hankel operators

Gopal Datt[a], Ritu Aggarwal[b]

[a] *Department of Mathematics, PGDAV College, University of Delhi, Delhi - 110065 (INDIA).*
[b] *Department of Mathematics, University of Delhi, Delhi - 110007 (INDIA).*

Abstract. The notion of essentially $(\lambda, \mu)-$Hankel operators has been introduced and some of its properties have been discussed. We also form a connection bridge between the classes of essentially $(\lambda, \mu)-$Hankel operators and essentially $\lambda-$Hankel operators.

1. Introduction

We denote the Hardy space of analytic functions on the unit disc by H^2. The set $e_n(z) = z^n$ for all $n \geq 0$ is an orthonormal basis for H^2 and is called the canonical basis of H^2. The unilateral shift U on the space H^2 is an isometry, it is just multiplication by z, that is, $U(f(z)) = zf(z)$ for all $f \in H^2$. Also, U is essentially unitary, that is, it is invertible in the Calkin algebra $\mathfrak{B}(H^2)/\mathcal{K}(H^2)$, where $\mathcal{B}(H^2)$ is the space of all operators on H^2 and $\mathcal{K}(H^2)$ is the set of all compact operators on H^2.

The classes of Hankel and Toeplitz operators form two important classes of operators. In terms of operator equations, Hankel and Toeplitz operators are characterized as solutions of the operator equations $U^*H = HU$ and $U^*TU = T$ respectively.

Mathematicians had always a keen interest in the essential commutant of the unilateral shift and it has sometimes been referred to as the set of essentially Toeplitz operators denoted by *essToep*. In [4], Barría and Halmos studied some of the properties of essential commutant of the unilateral shift. Obviously, *essToep* is the set of all those operators T satisfying $U^*TU - T = K$ for some compact operator K on H^2. A generalization of the operator equation $U^*TU = T$ was studied by S.Sun [14], where he completely solved the operator equation $U^*TU = \lambda T$, for complex number λ.

In [2], Avendaño came out with a generalization of the operator equation $U^*H = HU$ and considered the operator equation $U^*X - XU = \lambda X$, for arbitrary complex number λ. He called the solutions of this equation to be $\lambda-$Hankel operators. In reference to the Calkin algebra $\mathfrak{B}(H^2)/\mathcal{K}(H^2)$, *Avendaño* [3] studied another generalization of Hankel operators named as essentially Hankel operators. The class of all essentially Hankel operators on H^2 is denoted by *essHank* and consists of the operators X satisfying $U^*X - XU = K$ for

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Email addresses: gopal.d.sati@gmail.com (Gopal Datt), rituaggawaldu@rediffmail.com (Ritu Aggarwal)

some compact operator K on H^2. In [1], operators X satisfying $(U^* - \lambda I)X - XU = K$ for some compact operator K, are discussed and named as essentially $\lambda-$Hankel operators and this class is denoted by $essHank_\lambda$.

Motivated by the work of these mathematicians, the class of (λ, μ)-Hankel operators that can be characterized as solutions to the operator equation $(\mu U^* - \lambda I)X = XU$, $\lambda, \mu \in \mathbb{C}$ was introduced and discussed in [7].

For the last few years, many interesting results have been obtained about various generalizations of Hankel operators. We refer [5, 6, 11, 12, 15] and the references therein to provide a nice survey over the historical growth, details and applications of these operators. This paper extends the study further and introduce the notion of essentially $(\lambda, \mu)-$Hankel operators on the space H^2 and investigate some of its properties. For fixed λ, μ in complex plane, we denote the set of all essentially $(\lambda, \mu)-$Hankel operators on the space H^2 by $essHank_{(\lambda,\mu)}$. At the end, we present a bridge connecting this class with the class of essentially $\lambda-$Hankel operators. Throughout the paper, operator is used in reference to a bounded linear transformation on a Hilbert space.

2. Essentially $(\lambda, \mu)-$Hankel Operators

We recall the definition of the $(\lambda, \mu)-$Hankel operator, where $\lambda, \mu \in \mathbb{C}$.

Definition 2.1. *Let $\lambda, \mu \in \mathbb{C}$ be fixed. A bounded linear operator X on H^2 is said to be $(\lambda, \mu)-$Hankel operator if it satisfies*

$$\mu U^* X - XU = \lambda X.$$

A $(0, 1)-$Hankel operator is a Hankel operator and a $(\lambda, 1)-$Hankel operator is just a $\lambda-$Hankel operator. We now introduce the notion of essentially $(\lambda, \mu)-$Hankel operators on the space H^2 as

Definition 2.2. *For fixed complex numbers λ and μ, a bounded linear operator X on H^2 is said to be an essentially $(\lambda, \mu)-$Hankel operator if $(\mu U^* - \lambda I)X - XU \in \mathcal{K}(H^2)$.*

We denote the set of all essentially $(\lambda, \mu)-$Hankel operators as $essHank_{(\lambda,\mu)}$. Thus, every Hankel operator is in $essHank_{(0,1)}$. Also, $essHank_{(0,1)} = essHank$ and $essHank_{(\lambda,1)} = essHank_\lambda$. Further, we have the following facts about $essHank_{(\lambda,\mu)}$, which follow directly from the definition.

Proposition 2.3. *For $\lambda, \mu \in \mathbb{C}$, we have the following*

1. *Every compact operator is in $essHank_{(\lambda,\mu)}$.*
2. *Every $(\lambda, \mu)-$Hankel operator is in $essHank_{(\lambda,\mu)}$.*
3. *$I \notin essHank_{(\lambda,\mu)}$.*

As U is essentially unitary, it is evident to prove that if X is a $(\lambda, \mu)-$Hankel operator and K is a compact operator then $X + K$ is an essentially $(\lambda, \mu)-$Hankel operator. However, we find that converse may fail to hold, i.e. if $X \in essHank_{(\lambda,\mu)}$ then X may not be a compact perturbation of a (λ, μ) Hankel operator. For $\mu = 1$ and $\lambda = 0$, the Cesaro operator is an essentially Hankel operator and hence essentially $(0, 1)-$Hankel operator but it is not a compact perturbation of $(0, 1)-$Hankel operator (= Hankel operator)[3].

It is known that $essToep$ forms a C^*-algebra, however $essHank$ is not even an algebra [3]. Evidentally, $essHank_{(0,1)} = essHank$ is not an algebra implies that $essHank_{(\lambda,\mu)}$ is not always an algebra. We provide a result, which gives a necessary and sufficient condition for the product of two essentially $(\lambda, \mu)-$Hankel operators to be an essentially $(\lambda, \mu)-$Hankel operator.

Theorem 2.4. *Let $X_1, X_2 \in essHank_{(\lambda,\mu)}$. Then $X_1X_2 \in essHank_{(\lambda,\mu)}$ if and only if $X_1(UX_2 - X_2U)$ is compact.*

Proof. If $X_1, X_2 \in essHank_{(\lambda,\mu)}$. Then a simple and straight forward computation shows that

$$(\mu U^* - \lambda I)X_1X_2 - X_1X_2U = (X_1UX_2 - X_1X_2U) + K$$

for some compact operator K on H^2. As a consequence of this, we get the result. □

In the next result we show that no non-zero Toeplitz operator is in $essHank_{(\lambda,\mu)}$. For this, we first prove the following.

Lemma 2.5. *A non-zero Toeplitz operator can not be $(\lambda, \mu)-$Hankel operator.*

Proof. Let T be a non-zero Toeplitz operator. Then $U^*TU = T$. Let, if possible, T be a $(\lambda, \mu)-$Hankel operator. Then $(\mu U^* - \lambda I)T = TU$. This provides $(\mu U^* - \lambda I)TU - TUU = 0$, which gives that $T(\mu I - \lambda U - U^2) = 0$. Since both T and $(\mu I - \lambda U - U^2)$ are Toeplitz so we get that either $T = 0$ or $(\mu I - \lambda U - U^2) = 0$ [9]. T being non-zero we get $(\mu I - \lambda U - U^2) = 0$. Thus $\mu e_n = \lambda e_{n+1} + e_{n+2}$ for all $n \geq 0$. This is a contradiction. Hence T is not a $(\lambda, \mu)-$Hankel operator. □

It is evident that the zero operator is a Toeplitz operator and is also in $essHank_{(\lambda,\mu)}$ for every $\lambda, \mu \in \mathbb{C}$. Using Lemma 2.5, we have the following.

Theorem 2.6. *The only Toeplitz operator in $essHank_{(\lambda,\mu)}$ is the zero operator.*

Avendaño in [3], has shown that *essToep* ∩ *essHank* forms an algebra without identity. We generalize this result and show that the intersection of $essHank_{(\lambda,\mu)}$ and *essToep* is an algebra of operators on H^2 without identity.

Theorem 2.7. *essToep* ∩ $essHank_{(\lambda,\mu)}$ *is an algebra without identity.*

Proof. Let $X, T \in essToep \cap essHank_{(\lambda,\mu)}$. Then $U^*XU - X$, $(\mu U^* - \lambda I)X - XU$, $U^*TU - T$ and $(\mu U^* - \lambda I)T - TU$ are compact. These facts can be used to get that $(\mu U^* - \lambda I)XT - XTU$ is compact, which means that $XT \in essHank_{(\lambda,\mu)}$. Since *essToep* is an algebra we get *essToep* ∩ $essHank_{(\lambda,\mu)}$ is an algebra.

As identity is a non-zero Toeplitz operator and if it is in *essToep* ∩ $essHank_{(\lambda,\mu)}$, we get a contradiction in reference to Lemma 2.5. This completes the proof. □

It is easy to prove that $essHank_{(\lambda,\mu)}$ is a norm closed vector subspace of $\mathfrak{B}(H^2)$.

Theorem 2.8. $essHank_{(\lambda,\mu)}$ *is a norm closed vector subspace of* $\mathfrak{B}(H^2)$.

Proof. Proof follows by applying the same techniques as in case of *essHank* or $essHank_\lambda$ in [3, 1]. □

Since $\mathcal{K}(H^2)$ is self adjoint and contained in $essHank_{(\lambda,\mu)}$ for $\lambda, \mu \in \mathbb{C}$ so for a compact operator X on H^2, $essHank_{(\lambda,\mu)}$ contains both X and X^*. Next lemma deals with the case of non-compact operators and helps to provide information regarding the self adjoint nature of $essHank_{(\lambda,\mu)}$.

Lemma 2.9. *Let $0 \neq \mu$ and X be a non-compact operator on H^2. Then $X \in essHank_{(\lambda,\mu)}$ if and only if $X^* \in essHank_{(\alpha,\beta)}$, where $\alpha = -\overline{\lambda}/\overline{\mu}$, $\beta = 1/\overline{\mu}$.*

Proof. Let X be a non-compact operator on H^2. Then

$$\begin{aligned} X \in essHank_{(\lambda,\mu)} &\Leftrightarrow (\mu U^* - \lambda I)X - XU \text{ is compact} \\ &\Leftrightarrow X^*(\overline{\mu}U - \overline{\lambda}I) - U^*X^* \text{ is compact} \\ &\Leftrightarrow (1/\overline{\mu})U^*X^* - (-\overline{\lambda}/\overline{\mu})X^* - X^*U \text{ is compact} \\ &\Leftrightarrow X^* \in essHank_{(\alpha,\beta)}\ , \end{aligned}$$

where $\alpha = -\overline{\lambda}/\overline{\mu}$, $\beta = 1/\overline{\mu}$. □

It is clear from here that if $\lambda = 0$ and $\mu = 1$ then $(\alpha, \beta) = (\lambda, \mu) = (0, 1)$ so that $essHank_{(0,1)}$ is self adjoint, which is proved by *Avendaño* in [3].

Remark 2.10. *If λ and μ are complex numbers such that $\lambda \neq 0$ and $\mu = -\lambda/\bar{\lambda}$ then the class $essHank_{(\lambda,\mu)}$ is self adjoint.*

Remark 2.11. *For any $\mu \in \mathbb{C}$ with $|\mu| = 1$, there exists a line L passing through origin in the complex plane such that essHank$_{(\lambda,\mu)}$ is self adjoint for $\lambda \in L$.*

Remark 2.12. *For any $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we find unique μ on the unit circle such that essHank$_{(\lambda,\mu)}$ is self adjoint.*

It is known that $\mathcal{K}(H^2) \subseteq essHank_{(\lambda,\mu)}$ for each $\lambda, \mu \in \mathbb{C}$. We find that if $|\lambda - \alpha| \neq |\mu - \beta|$ then the set of compact operators becomes the common portion between $essHank_{(\lambda,\mu)}$ and $essHank_{(\alpha,\beta)}$.

Lemma 2.13. *If (α, β) and (λ, μ) are distinct pairs of complex numbers satisfying $|\lambda-\alpha| \neq |\mu-\beta|$ then $essHank_{(\lambda,\mu)} \cap essHank_{(\alpha,\beta)} = \mathcal{K}(H^2)$.*

Proof. We only need to show that $essHank_{(\lambda,\mu)} \cap essHank_{(\alpha,\beta)} \subseteq \mathcal{K}(H^2)$. Let $X \in essHank_{(\lambda,\mu)} \cap essHank_{(\alpha,\beta)}$ then $(\mu U^* - \lambda I)X - XU$ and $(\beta U^* - \alpha I)X - XU$ are compact. As a consequence we find that $((\mu - \beta)U^* - (\lambda - \alpha)I)X$ is compact. Now we divide the proof into two cases.
Case (1): $|\lambda - \alpha| > |\mu - \beta|$.
Subcase(1) : Let $\mu = \beta$. Then $\lambda \neq \alpha$ and $-(\lambda - \alpha I)X$ is compact, which implies X is compact.
Subcase(2) : Let $\mu \neq \beta$. As $|\lambda - \alpha| > |\mu - \beta|$, so $(U^* - (\lambda - \alpha)/(\mu - \beta)I)$ is invertible. Hence the compactness of $((\mu - \beta)U^* - (\lambda - \alpha)I)X$ yields the compactness of X.
Case (2): $|\lambda - \alpha| < |\mu - \beta|$.
Subcase(1) : If $\lambda = \alpha$ then $\mu \neq \beta$ and $(\mu - \beta)U^*X$ is compact. Since U^* is essentially unitary, we get that X is compact.
Subcase(2) : Let $\lambda \neq \alpha$. Now using the facts that U is essentially unitary and $((\mu - \beta)U^* - (\lambda - \alpha)I)X$ is compact, we get that $((\mu - \beta)I - (\lambda - \alpha)U)X$ is compact. The condition $|\lambda - \alpha| < |\mu - \beta|$ provides that $((\mu - \beta)I - (\lambda - \alpha)U)$ is invertible, which yields that X is compact.
In each case we have seen that X is compact and hence we have the desired conclusion. □

It is immediate from Lemma 2.13 that for a given $\lambda \neq 0$, if we take any complex number μ not lying in the circle centered at 1 and radius $|\lambda|$, then $essHank \cap essHank_{(\lambda,\mu)} = \mathcal{K}(H^2)$. Also, if $\mu \neq 1$, then $essHank_\lambda \cap essHank_{(\lambda,\mu)} = \mathcal{K}(H^2)$ for any complex number λ. Existence of some non-compact (λ, μ)−Hankel operators is shown for some specific choices of λ and μ in [7]. Therefore, we have some classes $essHank_{(\lambda,\mu)}$, for $\lambda, \mu \in \mathbb{C}$, containing non-compact operators. Without any extra efforts, we obtain the following results using Lemma 2.13.

Theorem 2.14. *If $essHank_{(\lambda,\mu)}$ contains a non-compact operator X for $\lambda, \mu \in \mathbb{C}$ then $X \notin essHank_{(\alpha,\beta)}$, for α, β satisfying $|\lambda - \alpha| \neq |\mu - \beta|$.*

Theorem 2.15. *If $\lambda, \mu \in \mathbb{C}$ are such that $|\lambda + \frac{\overline{\lambda}}{\overline{\mu}}| \neq |\mu - \frac{1}{\overline{\mu}}|$ and $essHank_{(\lambda,\mu)}$ contains a non-compact operator X then neither X is self adjoint nor $essHank_{(\lambda,\mu)}$ is self adjoint.*

In the next theorem we see that an essentially (λ, μ)−Hankel operator can not be essentially invertible.

Theorem 2.16. *Let $X \in essHank_{(\lambda,\mu)}$. Then $0 \in \sigma_e(X)$, where $\sigma_e(X)$ denotes the essential spectrum of the operator X.*

Proof. Let $X \in essHank_{(\lambda,\mu)}$. Then $(\mu U^* - \lambda I)X - XU$ is compact. Now we prove the result by considering the following cases.
Case (1): $\lambda = 0, \mu = 0$. This case is trivial to prove.
Case (2): $\lambda = 0, \mu = 1$. In this case X is an essentially Hankel operator. Using [3, Lemma 3.1], $0 \in \sigma_e(X)$.
Case (3): $\lambda = 1, \mu = 0$. In this case X satisfies that $-X - XU$ is compact. Now, if possible, $0 \notin \sigma_e(X)$ then X is essentially invertible operator and this yields that $(I + U)$ is compact. This provides a contradiction as $(I + U)$ is a non-zero Toeplitz operator. Therefore $0 \in \sigma_e(X)$.
Case(4): $\lambda = 1, \mu = 1$. Then $0 \notin \sigma_e(X)$ implies that $(\mu U^* - \lambda I) = XUX^{-1} + K$ for some compact operator K. This provides that $(\mu U^* - \lambda I)$ and U are essentially similar which is not true. Therefore $0 \in \sigma_e(X)$.
Case(5): For any complex numbers $\lambda, \mu \notin \{0, 1\}$, the proof follows along the lines of proof of case (4). □

It has been shown that a non-zero Toeplitz operator can not be in $essHank_{(\lambda,\mu)}$. However, the invariance of $essHank_{(\lambda,\mu)}$ operators under multiplication by a Toeplitz operator can be proved.

Theorem 2.17. *If X is in $essHank_{(\lambda,\mu)}$ and T is any Toeplitz operator on H^2 then XT and TX both are in $essHank_{(\lambda,\mu)}$.*

Proof. Let T be a Toeplitz operator. Therefore $U^*TU = T$, which on premultiplying by U and using the fact that U is essentially unitary yields that $UT - TU$ is a compact operator. Now if X is an operator in $essHank_{(\lambda,\mu)}$ then $(\mu U^* - \lambda I)X - XU$ is a compact operator on H^2. A simple computation shows that $(\mu U^* - \lambda I)XT - XTU$ is a compact operator so that $XT \in essHank_{(\lambda,\mu)}$. Therefore $XT \in essHank_{(\lambda,\mu)}$.

On similar lines, we can show that $TX \in essHank_{(\lambda,\mu)}$. □

It is easy to observe the following.

Theorem 2.18. *If X is in $essHank_{(\lambda,\mu)}$ and T is in $essToep$ then XT and TX both are in $essHank_{(\lambda,\mu)}$.*

It is known that the Rhaly matrix R_a induced by a sequence $a = \{a_n\}$ of scalars, represents a bounded linear operator on the Hardy space H^2 if $\{na_n\}$ is bounded [10,13].

In [3], R.A.M. *Avendaño* proved that if R_a is bounded then $R_a \in essToep$ if and only if $R_a \in essHank = essHank_{(0,1)}$. This may not be true for general $\lambda, \mu \in \mathbb{C}$. However, it is immediate to see that $R_a \in essToep \cap essHank_{(\lambda,\mu)}$ if and only if $R_a \in essHank \cap essHank_{(\lambda,\mu)}$.

With simple computations we can prove the following.

Theorem 2.19. *If $\lambda \neq 1, \mu \neq 0$ and $|\lambda| \neq |\mu - 1|$ then $R_a \in essHank \cap essHank_{(\lambda,\mu)}$ if and only if R_a is compact.*

Corollary 2.20. *Let $R_a \in essToep$ be a Rhaly operator with determining sequence $a = \{a_n\} \in \ell^2$. Then $R_a \in essHank_{(\lambda,\mu)}$, $\lambda \neq 1, \mu \neq 0$ and $|\lambda| \neq |\mu - 1|$, if and only if $lim_{n\to\infty}(n+1)|a_n| = 0$.*

Proof. Proof follows by using the fact [10] that R_a is compact if and only if $lim_{n\to\infty}(n+1)|a_n| = 0$. □

We have seen that for $\lambda \in \mathbb{C}$ and $\mu \neq 1 \in \mathbb{C}$, the two classes $essHank_{(\lambda,\mu)}$ and $essHank_\lambda$, share only compact operators. By the following theorems we try to construct a two way path between the two classes so that an operator of one class can generate an operator of the other class. Consider the operator D_μ, $\mu \in \mathbb{C}$, on the space H^2 given by $D_\mu e_n = \mu^n e_n$ for $n \geq 0$. For the boundedness of D_μ, we need $|\mu| \leq 1$. D_μ satisfies $D_\mu U^* = \frac{1}{\mu} U^* D_\mu$ and $UD_\mu = \frac{1}{\mu} D_\mu U$. Now we have the following.

Theorem 2.21. *Let $\mu \in \mathbb{C}$ and $0 < |\mu| \leq 1$.*

1. *If X is in $essHank_{(\lambda,\mu)}$ then $D_\mu X$ is in $essHank_\lambda$.*
2. *If Y is in $essHank_{\frac{\lambda}{\mu}}$ then YD_μ is in $essHank_{(\lambda,\mu)}$.*

Proof. If $(\mu U^* - \lambda)X - XU = K_1$ and $(U^* - \frac{\lambda}{\mu} I)Y - YU = K_2$ for compact operators K_1, K_2, then on multiplying these equations on left and right by D_μ respectively and using the properties of D_μ we get the proof of (1) and (2). □

Analogously, we can show the following.

Theorem 2.22. *Let $\mu \in \mathbb{C}$ and $|\mu| \geq 1$.*

1. *If X is in $essHank_{(\lambda,\mu)}$ then $XD_{\frac{1}{\mu}}$ is in $essHank_{\frac{\lambda}{\mu}}$.*
2. *If Y is in $essHank_\lambda$ then $D_{\frac{1}{\mu}} Y$ is in $essHank_{(\lambda,\mu)}$.*

References

[1] S.C. Arora and Jyoti Bhola, Essentially λ-Hankel operators, Journal of Mathematics, vol. 2013, Article ID 865396, 5 pages, 2013.
[2] Ruben $Martínez - Avendaño$, A generalization of Hankel operators, J. Func. Anal., Vol. 190, 2002, 418–446.
[3] Ruben $Martínez - Avendaño$, Essentially Hankel operators, J. London Math. Soc., Vol.66(2) (2002) 741-752.
[4] *José Barría* and P.R. Halmos, Asymptotic Toeplitz operators, Trans. Amer. Math. Soc., 273, 1982, 621–630.
[5] Arlen Brown and P.R. Halmos, Algebraic properties of Toeplitz operators, J. Reigne Angew. Math., 213, 1963, 89–102.
[6] Gopal Datt and Deepak Kumar Porwal, On weighted slant Hankel operators, Filomat 27:2 (2013), 227 - 243.
[7] Gopal Datt and Ritu aggarwal,On a generalization of Hankel operators via operator equations, to appear in Extracta Mathematica.
[8] Ronald G. Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1952.
[9] P.R. Halmos, A Hilbert Space Problem Book, Springer-Verlag, New York, 1982.
[10] G. Leibowitz, Rhaly matrices, J. Math. Anal. Appl., Vol.128 (1987), 272–286.
[11] Vladimir Peller, Hankel operators and applications, Springer-Verlag, New York, 2003.
[12] S.C. Power, Hankel operators on Hilbert space, Pitman Publishing, Boston, 1982.
[13] H. C. Rhaly, Jr. Terraced matrices, Bull. London math Soc., Vol. 21 (1989), 399–406.
[14] S. Sun, On the operator equation $U^*TU = \lambda T$, Kexue Tongbao (English ed.), 29, 1984, 298–299.
[15] O. Toeplitz, Zur theorie der quadratischen und bilinearan Formen von unendlichvielen, Veranderlichen, Math. Ann., Vol. 70, 1911, 351–376.