Functional Analysis, Approximation and Computation
6 (1) (2014), 41-48

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Biquasitriangularity and derivations

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Abstract. A Banach space operator is biquasitriangular if its essential spectrum has no holes or pseudo holes. Biquasitriangular Banach space operators A, B have a biquasitriangular tensor product, a biquasitriangular left-right multiplication operator L_AR_B, and a biquasitriangular generalised derivation $L_A - R_B$. Moreover, the Weyl spectral identity: $\sigma_w(A \otimes B) = \sigma(A) \cdot \sigma_w(B) \cup \sigma_w(A) \cdot \sigma(B)$, the a-Weyl spectral identity: $\sigma_{aw}(A \otimes B) = \sigma(A) \cdot \sigma_{aw}(B) \cup \sigma_{aw}(A) \cdot \sigma(B)$, the δ-Weyl spectral identity: $\sigma_w(L_A - R_B) = (\sigma(A) - \sigma_w(B)) \cup (\sigma_w(A) - \sigma(B))$, and the a-$\delta$-Weyl spectral identity: $\sigma_{aw}(L_A - R_B) = (\sigma(A) - \sigma_{aw}(B^*)) \cup (\sigma_{aw}(A) - \sigma(B))$ hold.

1. Introduction

Let $\mathcal{X}$ (resp., $\mathcal{H}$) denote an infinite dimensional complex Banach space (resp., separable Hilbert space), and let $B[\mathcal{X}]$ (resp., $B[\mathcal{H}]$) denote the algebra of operators (equivalently, bounded linear transformations) on $\mathcal{X}$ (resp., $\mathcal{H}$). An operator $A \in B[\mathcal{H}]$ is quasitriangular if there is a sequence $\{P_n\}$ of finite-rank projections that converges strongly to the identity operator I and $\{(I - P_n)AP_n\}$ converges uniformly to the null operator [6, Section 2]. If both A and A^* are quasitriangular, then A is biquasitriangular ($\mathcal{BQT}$). Biquasitriangular operators are equivalently described as follows:

$$A \text{ is } \mathcal{BQT} \text{ if and only if } \sigma_{\ell e}(A) = \sigma_{re}(A) = \sigma_e(A) = \sigma_w(A) \tag{1}$$

[2, Theorem 5.4] and [3, Theorem 2.1] (also see [17, p.37]), where for an operator $A \in B[\mathcal{X}]$ with *index* $\mathrm{ind}(A)$, $\sigma_{\ell e}(A) = \{\lambda \in \sigma(A) : \dim(\lambda I - A)^{-1}(0) = \infty$ or $(\lambda I - A)(\mathcal{X})$ is not closed or $(\lambda I - A)(\mathcal{X})$ is not complemented in $\mathcal{X}\}$, $\sigma_{re}(A) = \{\lambda \in \sigma(A) : \dim(\mathcal{X} \setminus (\lambda I - A)(\mathcal{X})) = \infty$ or $(\lambda I - A)^{-1}(0)$ is not complemented in $\mathcal{X}\}$, $\sigma_e(A) = \sigma_{\ell e}(A) \cup \sigma_{re}(A)$ and $\sigma_w(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_e(A)$ or $\mathrm{ind}(\lambda I - A) \neq 0\}$ denote (respectively) the *left essential spectrum*, *the right essential spectrum*, *the (Fredholm) essential spectrum* and *the Weyl spectrum* of A [16].

Let $\mathbb{Z}$ denote the set of all integers, and set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, the set of all extended integers. Let $\mathcal{SF}$ denote the set of operators $A \in B[\mathcal{H}]$ which are either left semi–Fredholm or right semi–Fredholm, $\mathcal{F}$ the set of operators $A \in B[\mathcal{H}]$ which are Fredholm, $\sigma_k(A) = \{\lambda \in \mathbb{C}\colon \; \lambda I - A \in \mathcal{SF}$ and $\mathrm{ind}(\lambda I - A) = k\}$ for every $k \in \overline{\mathbb{Z}} \setminus \{0\}$, and let $\sigma_0(A) = \sigma(A) \setminus \sigma_w(A) = \{\lambda \in \sigma(A)\colon A - \lambda I \in \mathcal{F}$ and $\mathrm{ind}(A - \lambda I) = 0\}$. Then

2010 *Mathematics Subject Classification*. Primary 47A80; Secondary 47A53.
Keywords. Banach space; Biquasitriangular operators; Derivations; Tensor product; Weyl spectral identity; Perturbation.
Received: 1 January, 2014; Accepted: 3 February, 2014
Communicated by Dragan S. Djordjević
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$\sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma(A)$, and $\sigma_k(A)$ is a subset of the point spectrum of A for every extended integer $k \in \overline{\mathbb{Z}}$. Also recall that (see e.g., [17, p.3] and [11, p.147]), for each non-zero integer $k \in \mathbb{Z}\backslash\{0\}$, $\sigma_k(A)$ is a *hole* of $\sigma_e(A)$, and $\sigma_{+\infty}(A) = \sigma_{\ell e}(A)\backslash\sigma_{re}(A)$ and $\sigma_{-\infty}(A) = \sigma_{re}(A)\backslash\sigma_{\ell e}(A)$ are the *pseudo holes* of $\sigma_e(A)$. Furthermore,

$$\sigma_w(A) \;=\; \sigma_e(A) \cup \bigcup\nolimits_{k\in\mathbb{Z}\backslash\{0\}} \sigma_k(A) \;=\; \sigma(A)\backslash\sigma_0(A),$$

so that the Weyl spectrum is the union of the essential spectrum with all its holes. This is the Schechter Theorem (see, e.g., [11, Theorem 5.24]), which when applied to the definition of $\mathcal{BQT}$ in (1) implies:

(1′) $\quad A$ is $\mathcal{BQT}$ if and only if $\sigma_e(A)$ has no holes and no pseudo holes

(see [14]). This version of the definition of BQT has a natural extension to Banach space operators.

For an $A \in B[\mathcal{X}]$, let $\mathcal{N}(A) = A^{-1}(0)$ and $\mathcal{R}(A) = A(\mathcal{X})$ denote, respectively, the kernel and the range of A. We say [16, Definition III.16.1] that A is
upper semi-Fredholm, $A \in \Phi_{SF_+}(\mathcal{X})$, if $\mathcal{R}(A)$ is closed and $\dim \mathcal{N}(A) < \infty$,
lower semi-Fredholm, $A \in \Phi_{SF_-}(\mathcal{X})$, if $\operatorname{codim} \mathcal{R}(A) < \infty$,
semi-Fredholm, if $A \in \Phi_{SF}(\mathcal{X}) = \Phi_{SF_+}(\mathcal{X}) \cup \Phi_{SF_-}(\mathcal{X})$,
Fredholm, if $A \in \Phi(\mathcal{X}) = \Phi_{SF_+}(\mathcal{X}) \cap \Phi_{SF_-}(\mathcal{X})$.
Corresponding to these classes of operators we have the following spectra.
$\sigma_{SF_+}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \Phi_{SF_+}(\mathcal{X})\}$, the *upper semi-Fredholm spectrum*,
$\sigma_{SF_-}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \Phi_{SF_-}(\mathcal{X})\}$, the *lower semi-Fredholm spectrum*,
$\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \Phi(\mathcal{X})\} = \sigma_{SF_+}(A) \cup \sigma_{SF_-}(A)$, the *Fredholm spectrum*,
$s_k(A) = \{\lambda \in \mathbb{C} : A - \lambda I \in \Phi_{SF}(\mathcal{X})$ and $\operatorname{ind}(A - \lambda I) = k\}$ for each $k \in \overline{\mathbb{Z}}\backslash\{0\}$.

Recall that a *hole* of a set in a topological space is any bounded component of its complement. It is seen that $s_k(A)$ with $k \in \mathbb{Z} \setminus \{0\}$ is a hole of $\sigma_e(A)$. The set $s_k(A)$ with $k = +\infty$ is a hole of $\sigma_{SF_-}(A)$ which lies in $\sigma_{SF_+}(A)$ and the set $s_k(A)$ with $k = -\infty$ is a hole of $\sigma_{SF_+}(A)$ which lies in $\sigma_{SF_-}(A)$: Sets $s_k(A)$ with $k = \pm\infty$ are *pseudo holes* of $\sigma_e(A)$. *We say in the following that a Banach space operator $A \in B[\mathcal{X}]$ is biquasitriangular, $A \in \mathcal{BQT}$, if $\sigma_e(A)$ has no holes or pseudo holes.* It is immediate from the definition that $\sigma_e(A) = \sigma_w(A)$ for every $A \in \mathcal{BQT}$.

This paper considers Banach space $\mathcal{BQT}$ operators. We start by considering some elementary properties of $\mathcal{BQT}$ operators in Section 2, including preservation under similarity, as well as compact and commuting Riesz perturbations (see Theorem 1). Main result are proved in Section 3. These focus on tensor products $A \otimes B$ and derivations $\delta_{A,B} = L_A - R_B$; $A, B \in B[\mathcal{X}]$. Theorem 2 deals with the tensor product $A \otimes B$ of BQT operators A and $B \in B[\mathcal{X}]$. Tensor products preserve biquasitriangulaity and the biquasitriangularity property of tensor products implies that the Weyl spectral identity holds: Thus, if A and B are biquasitriangular, then so is their tensor product $A \otimes B$, and the Weyl spectral identity

$$\sigma_w(A \otimes B) = \sigma(A) \cdot \sigma_w(B) \cup \sigma_w(A) \cdot \sigma(B)$$

holds. These results (as well as their extensions) are carried from tensor products to the left-right multiplication operator $L_A R_B$ in Corollary 1, and then to derivations $\delta_{A,B} = L_A - R_B$ in Theorems 3 and 4 (where the tensor product $A \otimes B$ is replaced by the derivation $\delta_{A,B}$).

2. Elementary properties of $\mathcal{BQT}$ operators

In the following we gather together some basic properties of $\mathcal{BQT}$ operators in $B[\mathcal{X}]$. Therefore, form now on, throughout the paper, suppose $A \in B[\mathcal{X}]$, where $\mathcal{X}$ is a Banach space. Let A be such that $A - \lambda I \in \Phi_{SF_+}(\mathcal{X})$. Then either $\operatorname{ind}(A - \lambda I) > 0$ or $\operatorname{ind}(A - \lambda I) \leq 0$. Since $\operatorname{ind}(A - \lambda I) > 0$ implies $\lambda \in \Phi(\mathcal{X})$, we conclude that $\lambda \notin \sigma_e(A)$ and the set $s_k(A) = \{\lambda \in \mathbb{C} : \lambda \in \Phi_{SF_+}(\mathcal{X}),\ 0 < \operatorname{ind}(A - \lambda I) = k\}$ for $k \neq 0$ is either a hole or a pseudo hole of $\sigma_e(A)$. Again, if $\operatorname{ind}(A - \lambda I) < 0$, then either $\lambda \notin \sigma_e(A)$ and $\operatorname{ind}(A - \lambda I) = k < 0$ or $\operatorname{ind}(A - \lambda I) =$

$-\infty$; thus either $\lambda \notin \sigma_e(A)$ and $\operatorname{ind}(A - \lambda I) = 0$, or, $s_k(A)$ ($k \neq 0$) is either a hole or a pseudo hole of $\sigma_e(A)$. Conclusion: If $A \in \mathcal{BQT}$, then $\sigma_{SF_+}(A) = \sigma_e(A)$. A similar argument works for when $A - \lambda I \in \Phi_{SF_-}(\mathcal{X})$. This yields a proof of the equivalence between (1) and (1′) (for Banach space $\mathcal{BQT}$ operators).

Proposition 1. *$A \in \mathcal{BQT}$ if and only if $\sigma_{SF_+}(A) = \sigma_{SF_-}(A) = \sigma_e(A) = \sigma_w(A)$.*

Let $\sigma_a(A)$ and $\sigma_s(A)$ denote, respectively, the approximate point spectrum and the surjectivity spectrum of $A \in B(\mathcal{X})$. That is let,

$\sigma_a(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not bounded below}\}$,

$\sigma_s(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not surjective}\}$.

If $A \in \mathcal{BQT}$ and $\lambda \notin \sigma_s(A)$, then $\lambda \notin \sigma_{SF_-}(A) = \sigma_w(A) \iff \lambda \notin \sigma(A)$. Hence, $\sigma_s(A) = \sigma(A)$. A similar argument shows that $A \in \mathcal{BQT}$ implies $\sigma_a(A) = \sigma(A)$. So:

Proposition 2. *[14] If $A \in \mathcal{BQT}$, them $\sigma(A) = \sigma_a(A) = \sigma_s(A)$.*

An operator $A \in B[\mathcal{X}]$, has *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0 for short, if for every open disc $\mathbb{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathbb{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(A - \lambda I)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathbb{D}_{\lambda_0}$$

is the function $f \equiv 0$. *A has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$*. Evidently, A has SVEP at points in the resolvent set and the boundary $\partial\sigma(A)$ of $\sigma(A)$. Also, A (resp., A^*) has SVEP at 0 if $\operatorname{asc}(A) < \infty$ (resp., $\operatorname{dsc}(A) < \infty$), where $\operatorname{asc}(A)$ (resp., $\operatorname{dsc}(A)$), the ascent of A (resp., the descent of A), is the least non-negative integer p such that $A^{-p}(0) = A^{-(p+1)}(0)$ (resp., $A^p(\mathcal{X}) = A^{p+1}(\mathcal{X})$). Let

$\sigma_b(A) = \{\lambda \in \sigma(A) : A - \lambda I \notin \Phi(\mathcal{X}) \text{ or } \operatorname{asc}(A - \lambda I) \neq \operatorname{dsc}(A - \lambda I)\}$

denote the *Browder spectrum of A*. Recall that $\sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A) \subseteq \sigma(A)$.

Proposition 3. *If $A \in \mathcal{BQT}$, then the following statements are equivalent.*

(i) $\sigma_w(A) = \sigma_b(A)$.

(ii) *A has SVEP on* $\sigma(A)\backslash\sigma_{SF_+}(A)$.

(iii) *A has SVEP on* $\sigma(A)\backslash\sigma_{SF_-}(A)$.

(iv) *A has SVEP on* $\sigma(A)\backslash\sigma_e(A)$.

(v) *A has SVEP on* $\sigma(A)\backslash\sigma_w(A)$.

Proof. Since $A \in \mathcal{BQT}$, it would suffice to prove (i) $\iff$ (v). If $\sigma_b(A) = \sigma_w(A)$, then $\lambda \notin \sigma_w(A)$ implies $\operatorname{asc}(A - \lambda I) < \infty$, and this in turn implies that A has SVEP on $\sigma(A)\backslash\sigma_w(A)$. Conversely, if A has SVEP at every $\lambda \notin \sigma_w(A)$, then $\operatorname{asc}(A - \lambda I) = \operatorname{dsc}(A - \lambda I) < \infty$, i.e., $\lambda \notin \sigma_b(A)$ (see [1, Theorems 3.4 and 3.16]). □

Operators $A \in B[\mathcal{X}]$ satisfying $\sigma_w(A) = \sigma_b(A)$ have been described in the literature as *satisfying Browder's theorem* (see, e.g., [1]).

Recall from [8, Proposition 6.16] that the class of Hilbert space $\mathcal{BQT}$ operators is stable under similarities and under perturbation by compact operators. This result has a natural extension to Banach space $\mathcal{BQT}$ operators: Indeed, along with being invariant under similarities, the class $\mathcal{BQT}$ is invariant under perturbations by commuting Riesz operators. But before we go on to prove this, we introduce some complementary notation and results. Recall that an $A \in B[\mathcal{X}]$ is a Riesz operator, $A \in R[\mathcal{X}]$, if every of its non-zero spectral points is a finite rank pole of the (resolvent of the) operator. (Thus, if $A \in R[\mathcal{X}]$, then $\sigma_e(A) = \{0\}$.) Let $\ell^\infty(\mathcal{X})$

denote the Banach space of all bounded sequences of elements of $\mathcal{X}$(with its natural "supremum norm"), let $m(\mathcal{X})$ denote the space of all precompact sequences of $\mathcal{X}$, and let $\mathcal{X}_q = \ell^\infty(\mathcal{X})/m(\mathcal{X})$. The *unital homomorphism* T_q, with kernel the ideal of compact operators $K[\mathcal{X}]$, effecting the "essential enlargement"

$$T_q : B[\mathcal{X}] \to B[\mathcal{X}_q] \quad \text{so that} \quad T_q : A \mapsto A_q$$

is then a norm decreasing monomorphism from $B[\mathcal{X}]/K[\mathcal{X}] \to B[\mathcal{X}_q]$ such that T_q maps upper semi-Fredholm (resp., lower semi-Fredholm) operators in $B[\mathcal{X}]$ onto bounded below (resp., surjective) operators in $B[\mathcal{X}_q]$ (see [4] and [16], Theorems 17.6 and 17.9, respectively). We show that $\mathcal{BQT}$ is similarity invariant. Let $\alpha(A) = \dim \mathcal{N}(A)$ and $\beta(A) = \operatorname{codim} \mathcal{R}(A)$ denote the deficiency indices of $A \in B[\mathcal{X}]$.

Proposition 4. *Let $A, B, S \in B[\mathcal{X}]$ be such that $A \in \mathcal{BQT}$, S is invertible, and $AS = SB$. Then $B \in \mathcal{BQT}$.*

Proof. Using the notation above, the hypotheses imply $A_qS_q = S_qB_q$, where S_q is invertible in $B[X_q]$. Since similar operators have the same spectra, $\sigma_x(B) = \sigma_x(A)$ for $\sigma_x = \sigma_{SF_+}$ or σ_{SF_-} or σ_e. It being evident that $\alpha(A - \lambda I) = \alpha(B - \lambda I)$ and $\beta(A - \lambda I) = \beta(B - \lambda I)$ for all complex λ, the proof is complete. □

Let $\delta_{A,B} \in B[B[\mathcal{X}]])$ denote the generalized derivation $\delta_{A,B}(X) = AX - XB$. We say that the operators $A, B \in B[\mathcal{X}]$ are *quasinilpotent equivalent* if

$$d(A, B) = \lim_{n\to\infty} \|\delta^n_{A,B}(I)\|^{\frac{1}{n}} = \lim_{n\to\infty} \|\delta^n_{B,A}(I)\|^{\frac{1}{n}} = 0.$$

Quasinilpotent equivalent operators have the same approximate point and surjectivity spectrum [15, Proposition 3.4.11]. Clearly, if $A, B \in B[\mathcal{X}]$ are such that $d(T_qA, T_qB) = 0$, that is if A_q and B_q are quasinilpotent equivalent, then $\sigma_a(A_q) = \sigma_a(B_q)$ and $\sigma_s(A_q) = \sigma_s(B_q)$, and hence $\sigma_x(A) = \sigma_x(B)$, where $\sigma_x = \sigma_{SF_+}$ or σ_{SF_-} or σ_e. This implies that if $A \in \mathcal{BQT}$, and A_q and B_q are quasinilpotent equivalent, then $\sigma_{SF_+}(B) = \sigma_{SF_-}(B) = \sigma_e(B) = \sigma_w(A)$. Does $\sigma_e(B) = \sigma_w(B)$? The following theorem says that the answer is in the affirmative in the case in which B is a perturbation of A by a compact operator or by a commuting Riesz operator.

Theorem 1. *The class of $\mathcal{BQT}$ operators is stable under perturbation by* (i) *compact operators and* (ii) commuting Riesz operators.

Proof. Let $A, B \in B[\mathcal{X}]$, where $A \in \mathcal{BQT}$. Let T_q be the homomorphism defined above.

(i) If $B \in \mathcal{K}[\mathcal{X}]$, then $T_q(A - tB) = A_q$ for all $0 \le t \le 1$, and so $\sigma_x(T_q(A - tB)) = \sigma_x(T_qA)$ for $\sigma_x = \sigma_a$ or σ_s or σ and all $0 \le t \le 1$. Hence $\sigma_x(A - tB) = \sigma_x(A)$ for $\sigma_x = \sigma_{SF_+}$ or σ_{SF_-} or σ_e and all $0 \le t \le 1$. The local constancy of the index implies $\operatorname{ind}(A - B) = \operatorname{ind}(A)$; hence we have also that $\sigma_w(A - B) = \sigma_w(A)$.

(ii) Assume now that $B \in R[\mathcal{X}]$, and $AB = BA$. Then $T_q(A - tB) = A_q - tB_q$ for all $0 \le t \le 1$, where B_q is quasinilpotent and $A_qB_q = B_qA_q$. Since

$$\delta^n_{A_q - tB_q, A_q}(I) = (-1)^n t^n B_q^n = (-1)^n \delta^n_{A_q, A_q - tB_q}(I),$$

it follows that $d(A_q - tB_q, A_q) = 0$ and hence $A_q - tB_q$ and A_q are quasinilpotent equivalent for all $0 \le t \le 1$. Thus, as before, $\sigma_x(A - tB) = \sigma_x(A)$ for $\sigma_x = \sigma_{SF_+}$ or σ_{SF_-} or σ_e and all $0 \le t \le 1$. Once again the local constancy of the index implies that we also have $\sigma_w(A - B) = \sigma_w(A)$. □

3. Main results: Tensor products and the operator $\delta_{A,B} = L_A - R_B$.

A pair $\langle \mathcal{X}, \tilde{\mathcal{X}} \rangle$ of Banach spaces is a *dual pairing* if either $\tilde{\mathcal{X}} = \mathcal{X}^*$, the dual space of $\mathcal{X}$, or $\mathcal{X} = \tilde{\mathcal{X}}^*$. Let

$$\mathcal{X} \times \tilde{\mathcal{X}} \to \mathbb{C}, \qquad (x, u) \mapsto \langle x, u \rangle,$$

denote the canonical bilinear mapping (in both cases), and let $L[\mathcal{X}]$ denote the subalgebra of $B[\mathcal{X}]$ consisting of operators $T \in B[\tilde{\mathcal{X}}]$ for which there exists an operator $T' \in B[\tilde{\mathcal{X}}]$ with $\langle Tx, u \rangle = \langle x, T'u \rangle$ for all $x \in \mathcal{X}$ and

$u \in \tilde{\mathcal{X}}$. It is then clear that (i) if the dual pairing is $\langle \mathcal{X}^*, \mathcal{X} \rangle$, then $L[\mathcal{X}^*] = B[\mathcal{X}^*]$, and (ii) each rank one operator $f_{y,v} : \mathcal{X} \to \mathcal{X},\ x \mapsto \langle x, v \rangle y,\ y \in \mathcal{X}$ and $v \in \tilde{\mathcal{X}}$, is contained in $L[\mathcal{X}]$. Following Eschmeier [7, p.50], we say that a tensor product of Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ relative to the dual pairings $\langle \mathcal{X}, \tilde{\mathcal{X}} \rangle$, $\langle \mathcal{Y}, \tilde{\mathcal{Y}} \rangle$ is a Banach space $\mathcal{X} \tilde{\otimes} \mathcal{Y}$ together with continuous bilinear mappings

$$\mathcal{X} \times \mathcal{Y} \to \mathcal{X} \tilde{\otimes} \mathcal{Y}, \qquad (x, y) \mapsto x \otimes y,$$
$$L[\mathcal{X}] \times L[\mathcal{Y}] \to B[\mathcal{X} \tilde{\otimes} \mathcal{Y}], \qquad (T, S) \mapsto T \otimes S,$$

which satisfy the following conditions.

(i) $\|x \otimes y\| = \|x\| \, \|y\|$,

(ii) $(T \otimes S)(x \otimes y) = Tx \otimes Ty$,

(iii) $(T_1 \otimes S_1) \circ (T_1 \otimes S_2) = T_1 T_2 \otimes S_1 S_2, \quad I \otimes I = I$,

(iv) $\mathcal{R}(f_{x,u}) \otimes I) \subset \{x \otimes y \colon y \in \mathcal{Y}\}, \quad \mathcal{R}(I \otimes f_{y,v}) \subset \{x \otimes y \colon x \in \mathcal{X}\}$.

The completion $\mathcal{X} \tilde{\otimes} \mathcal{Y}$ $(= \mathcal{X} \tilde{\otimes}_\alpha \mathcal{Y})$ of the algebraic tensor product of $\mathcal{X}$ and $\mathcal{Y}$ with respect to a quasi-uniform norm α then defines in a natural way a tensor product relative to the dual pairings $\langle \mathcal{X}, \mathcal{X}^* \rangle$ and $\langle \mathcal{Y}, \mathcal{Y}^* \rangle$. Given operators $A \in B[\mathcal{X}]$ and $B \in B[\mathcal{Y}]$, let $A \otimes B \in B[\mathcal{X} \tilde{\otimes} \mathcal{Y}]$ denote the tensor product of A and B. Then, see [7] and [9, 10],

$$\begin{aligned}
&\sigma_{SF_+}(A \otimes B) = \sigma_a(A) \cdot \sigma_{SF_+}(B) \cup \sigma_{SF_+}(A) \cdot \sigma_a(B),\\
&\sigma_{SF_-}(A \otimes B) = \sigma_s(A) \cdot \sigma_{SF_-}(B) \cup \sigma_{SF_-}(A) \cdot \sigma_s(B),\\
&\sigma_e(A \otimes B) = \sigma(A) \cdot \sigma_e(B) \cup \sigma_e(A) \cdot \sigma(B),\\
&\sigma_b(A \otimes B) = \sigma(A) \cdot \sigma_b(B) \cup \sigma_b(A) \cdot \sigma(B),\\
&\sigma_w(A \otimes B) \subseteq \sigma(A) \cdot \sigma_w(B) \cup \sigma_w(A) \cdot \sigma(B).
\end{aligned}$$

Consider the above inclusion involving the Weyl spectrum of tensor products. Following the terminology introduced in [12], when this inclusion becomes an identity we say that A and B *satisfy the Weyl spectral identity* (WSI) (see [12, 13] for a detailed account on the WSI). It has been proved in [14, Theorem 1] that the biquasitriangular property transfers from $A \in B[\mathcal{X}]$ and $B \in B[\mathcal{Y}]$ to $A \otimes B \in B[\mathcal{X} \tilde{\otimes} \mathcal{Y}]$, and also that, in this case, A and B satisfy the Weyl spectral identity.

Theorem 2. *[14] A and B biquasitriangular implies $A \otimes B$ biquasitriangular* (i.e., *if $A, B \in \mathcal{BQT}$, them $A \otimes B \in \mathcal{BQT}$*). *Furthermore, if A and B are biquasitriangular, then $A \otimes B$ satisfies the Weyl spectral identity. That is, if $A, B \in \mathcal{BQT}$, then*

$$\sigma_w(A \otimes B) = \sigma(A) \cdot \sigma_w(B) \cup \sigma_w(A) \cdot \sigma(B).$$

The theorem implies that if $A, B \in \mathcal{BQT}$, then

$$\sigma(A \otimes B) = \sigma_a(A \otimes B) = \sigma_s(A \otimes B),$$

$$\begin{aligned}
\sigma_{aw}(A \otimes B) &= \sigma_{sw}(A \otimes B) = \sigma_w(A \otimes B)\\
&= \sigma(A) \cdot \sigma_w(B) \cup \sigma_w(A) \cdot \sigma(B)\\
&= \sigma_{aw}(A) \cdot \sigma_s(B) \cup \sigma_a(A) \cdot \sigma_{sw}(B)\\
&= \sigma_{sw}(A) \cdot \sigma_a(B) \cup \sigma_a(A) \cdot \sigma_{sw}(B),
\end{aligned}$$

where

$$\sigma_{aw}(A) = \{\lambda \in \sigma_a(A) \colon \text{ either } \lambda \in \sigma_{SF_+}(A)$$

$$\begin{aligned}&\text{or } A-\lambda I \text{ is upper semi-Fredholm of positive index}\},\\ \sigma_{sw}(A) = \{\lambda\in\sigma_s(A):\ &\text{either } \lambda\in\sigma_{SF_-}(A)\\ &\text{or } A-\lambda I \text{ is lower semi-Fredholm of negative index}\}.\end{aligned}$$

An *operator ideal* J between Banach spaces $\mathcal{Y}$ and $\mathcal{X}$ is a linear subspace of $B(\mathcal{Y},\mathcal{X})$ equipped with a Banach norm α such that

(i) $x\otimes y'\in J$ and $\alpha(x\otimes y') = \|x\|\,\|y\|$,

(ii) $\triangle_{ST}(A) = L_S R_T(A) = SAT$ and $\alpha(SAT)\le \|S\|\alpha(A)\|T\|$,

for all $x\in\mathcal{X}$, $y'\in\mathcal{Y}^*$, $A\in J$, $S\in B[\mathcal{X}]$ and $T\in B[\mathcal{Y}]$ [7, p.51]. Thus defined, each J is a tensor product relative to the dual pairings $\langle\mathcal{X},\mathcal{X}^*\rangle$ and $\langle\mathcal{Y}^*,\mathcal{Y}\rangle$ and the bilinear mappings

$$\mathcal{X}\times\mathcal{Y}^*\to J,\qquad (x,y')\to x\otimes y',$$
$$B[\mathcal{X}]\times B[\mathcal{Y}^*]\to B(J),\qquad (S,T^*)\to S\otimes T^*,$$

where $S\otimes T^*(A) = SAT$ $(= L_S R_T(A) = \triangle_{S,T}(A))$. The next result is immediate from the above and Theorem 2.

Corollary 1. *If $A,B\in\mathcal{BQT}$ (so also $B^*\in\mathcal{BQT}$), then $L_AR_B\in\mathcal{BQT}$ is such that*

$$\sigma_w(L_AR_B) = \sigma(A)\cdot\sigma_w(B)\cup\sigma_w(A)\cdot\sigma(B).$$

Note that the above is the analogous of WSI (which was defined for tensor products), replacing $A\otimes B$ with L_AR_B. The corollary implies, in particular, that $A\in\mathcal{BQT}$ implies $L_A, R_A\in\mathcal{BQT}$. We prove below that $A,B\in\mathcal{BQT}$ implies $\delta_{A,B} = L_A - R_B\in\mathcal{BQT}$, and also that $\delta_{A,B} = L_A-R_B$ satisfies the analogous of WSI. Observe (from an application of the spectral mapping theorem) that

$$\begin{aligned}&\sigma_a(\delta_{A,B}) = \sigma_a(A)-\sigma_s(B),\qquad \sigma_s(\delta_{A,B}) = \sigma_s(A)-\sigma_a(B),\\ &\sigma(\delta_{A,B}) = \sigma(A)-\sigma(B),\qquad \sigma_e(\delta_{A,B}) = \big(\sigma(A)-\sigma_e(B)\big)\cup\big(\sigma_e(A)-\sigma(B)\big),\\ &\sigma_{SF_+}(\delta_{A,B}) = \big(\sigma_a(A)-\sigma_{SF_-}(B)\big)\cup\big(\sigma_{SF_+}(A)-\sigma_s(B)\big),\\ &\sigma_{SF_-}(\delta_{A,B}) = \big(\sigma_s(A)-\sigma_{SF_+}(B)\big)\cup\big(\sigma_{SF_-}(A)-\sigma_a(B)\big),\\ &\sigma_b(\delta_{A,B}) = \big(\sigma(A)-\sigma_b(B)\big)\cup\big(\sigma_b(A)-\sigma(B)\big),\end{aligned}$$

where $\sigma_x(A)-\sigma_y(B)$ means (symmetrical) numerical difference (not set difference).

The relationship between the various Weyl spectra of $\delta_{A,B}$ is a bit more delicate. Let $H_0(A)$ denote the quasinilpotent part $H_0(A) = \{x\in\mathcal{X} : \lim_{n\to\infty}\|A^nx\|^{\frac{1}{n}} = 0\}$ of $A\in B[\mathcal{X}]$ [1, Page 43].

Theorem 3. *We claim that*

$$\sigma_w(\delta_{A,B})\subseteq\big(\sigma(A)-\sigma_w(B)\big)\cup\big(\sigma_w(A)-\sigma(B)\big),$$
$$\sigma_{aw}(\delta_{A,B})\subseteq\big(\sigma_a(A)-\sigma_{sw}(B)\big)\cup\big(\sigma_{aw}(A)-\sigma_s(B)\big),$$

for every $A\in B[\mathcal{X}]$ and $B\in B[\mathcal{Y}]$.

Proof. If $\lambda\notin\big(\sigma(A)-\sigma_w(B)\big)\cup\big(\sigma_w(A)-\sigma(B)\big)$ and $\lambda\in\sigma(A)-\sigma(B)$, then there are finite sequences $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$, of points $\mu_i\in\sigma(A)$ and $\nu_i\in\sigma(B)$, and an integer $m\ge 1$ such that $\lambda=\mu_i-\nu_i$, $\nu_i\notin\sigma_w(B)\supseteq\sigma_e(B)$ and $\mu_i\notin\sigma_w(A)\supseteq\sigma_e(A)$, $1\le i\le n$, $\mu_i\in\mathrm{iso}\sigma(A)$ for all $1\le i\le m$ and $\nu_i\in\mathrm{iso}\sigma(B)$ for all $m+1\le i\le n$. Evidently, $\lambda\notin\sigma_e(\delta_{A,B})$. We prove that $\mathrm{ind}(\delta_{A,B}-\lambda I) = 0$. Recall from [7, Theorem 4.2] that

$$\begin{aligned}\operatorname{ind}(\delta_{A,B}-\lambda I) &= \sum_{i=m+1}^{n} \dim H_0(B-\nu_i I)\operatorname{ind}(A-\mu_i I)\\ &-\sum_{i=1}^{m} \dim H_0(A-\mu_i I)\operatorname{ind}(B-\nu_i I).\end{aligned}$$

As seen above, $\mu_i \notin \sigma_e(A)$ and $\mu_i \in \operatorname{iso}\sigma(A)$ for all $1 \le i \le m$; hence $A-\mu_i I$ has finite ascent (and descent), $H_0(A-\mu_i I) = (A-\mu_i I)^{-s}(0)$ for some positive integer s with $\dim H_0(A-\mu_i I) < \infty$ for all $1 \le i \le m$. Similarly, $A-\mu_i I$ has finite ascent (and descent), $H_0(B-\nu_i I) = (B-\nu_i I)^{-t}(0)$ for some positive integer t with $\dim H_0(B-\nu_i I) < \infty$ for all $m+1 \le i \le n$. Since already $\operatorname{ind}(A-\mu_i I) = 0$ for all $m+1 \le i \le n$ ($\mu_i \notin \sigma_w(A)$) and $\operatorname{ind}(B-\nu_i I) = 0$ for all $1 \le i \le m$ ($\nu_i \notin \sigma_w(B)$), we conclude that $\operatorname{ind}(\delta_{A,B}-\lambda I) = 0$. So $\lambda \notin \sigma_w(\delta_{A,B})$, and the inclusion is proved.

The other inclusion is similar. If $\lambda \notin \big(\sigma_a(A)-\sigma_{sw}(B)\big) \cup \big(\sigma_{aw}(A)-\sigma_s(B)\big)$ and $\lambda \in \sigma_a(A)-\sigma_s(B)$, then for every $\mu_i \in \sigma_a(A)$ and $\nu_i \in \sigma_s(B)$ such that $\lambda = \mu_i - \nu_i$ we must have that $\nu_i \notin \sigma_{sw}(B) \supseteq \sigma_{SF_-}(B)$ and $\mu_i \notin \sigma_{aw}(A) \supseteq \sigma_{SF_+}(A)$. Thus $\lambda \notin \sigma_{SF_+}(\delta_{A,B})$. We claim that $\operatorname{ind}(\delta_{A,B}-\lambda I) \le 0$. For if not, then $\operatorname{ind}(\delta_{A,B}-\lambda I) > 0$ implies $\lambda \notin \sigma_e(\delta_{A,B})$. Using the argument from the proof of the first inclusion it then follows that $\operatorname{ind}(\delta_{A,B}-\lambda I) = 0$, which is a contradiction. Hence $\lambda \notin \sigma_{aw}(\delta_{A,B})$, and the inclusion is proved. □

Let a-WSI denote the approximate point Weyl spectrum version of WSI: A and B satisfy the a-WSI (or the a-WSI holds for the tensor product $A \otimes B$) if [14]

$$\sigma_{aw}(A \otimes B) = \sigma_{aw}(A)\cdot\sigma_a(B) \cup \sigma_a(A)\cdot\sigma_{aw}(B)$$

(also see [5]). Let δ-WSI and δ-a-WSI, be the versions of WSI and a-WSI, respectively, with $A \otimes B$ replaced with $\delta_{A,B}$. The following theorem proves that, if A and B are biquasitriangular, then they satisfy the δ-WSI and δ-a-WSI,

$$\sigma_w(\delta_{A,B}) = \big(\sigma_w(A)-\sigma(B)\big) \cup \big(\sigma(A)-\sigma_w(B)\big),$$

and also

$$\begin{aligned}\sigma_{aw}(\delta_{A,B}) &= \big(\sigma_a(A)-\sigma_{aw}(B^*)\big) \cup \big(\sigma_{aw}(A)-\sigma_a(B^*)\big)\\ &= \big(\sigma(A)-\sigma_{aw}(B^*)\big) \cup \big(\sigma_{aw}(A)-\sigma(B)\big).\end{aligned}$$

Theorem 4. *If $A \in B[\mathcal{X}]$ and $B \in B[\mathcal{Y}]$ are $\mathcal{BQT}$ operators, then $\delta_{A,B} \in \mathcal{BQT}$, and satisfies both δ-WSI and δ-a-WSI.*

Proof. If $A, B \in \mathcal{BQT}$, then $\sigma_{SF_+}(T) = \sigma_{SF_-}(T) = \sigma_e(T) = \sigma_w(T)$, and $\sigma(T) = \sigma_a(T) = \sigma_s(T)$ and $\sigma_{aw}(T) = \sigma_{sw}(T) = \sigma_w(T) = \sigma_e(T)$, where $T = A$ or B. Hence

$$\begin{aligned}\sigma_{SF_+}(\delta_{AB}) &= \big(\sigma_a(A)-\sigma_{SF_-}(B)\big) \cup \big(\sigma_{SF_+}(A)-\sigma_s(B)\big)\\ &= \big(\sigma_s(A)-\sigma_{SF_+}(B)\big) \cup \big(\sigma_{SF_-}(A)-\sigma_a(B)\big) = \sigma_{SF_-}(\delta_{A,B})\\ &= \big(\sigma(A)-\sigma_e(B)\big) \cup \big(\sigma_e(A)-\sigma(B)\big) = \sigma_e(\delta_{A,B}),\end{aligned}$$

$$\begin{aligned}\sigma_w(\delta_{A,B}) &\subseteq \big(\sigma(A)-\sigma_w(B)\big) \cup \big(\sigma_w(A)-\sigma(B)\big)\\ &= \big(\sigma(A)-\sigma_e(B)\big) \cup \big(\sigma_e(A)-\sigma(B)\big) = \sigma_e(\delta_{A,B}) \subseteq \sigma_w(\delta_{A,B})\\ &\Rightarrow \sigma_e(\delta_{A,B}) = \sigma_w(\delta_{A,B}) = \big(\sigma(A)-\sigma_w(B)\big) \cup \big(\sigma_w(A)-\sigma(B)\big),\end{aligned}$$

and

$$\begin{aligned}\sigma_{aw}(\delta_{A,B}) &\subseteq \Big(\sigma_a(A) - \sigma_{sw}(B)\Big) \cup \Big(\sigma_{aw}(A) - \sigma_s(B)\Big)\\ &= \Big(\sigma_a(A) - \sigma_{SF_-}(B)\Big) \cup \Big(\sigma_{SF_+}(A) - \sigma_s(B)\Big) = \sigma_{SF_+}(\delta_{A,B}) \subseteq \sigma_{aw}(\delta_{A,B})\\ &\Rightarrow \sigma_{aw}(\delta_{A,B}) = \Big(\sigma_a(A) - \sigma_{sw}(B)\Big) \cup \Big(\sigma_{aw}(A) - \sigma_s(B)\Big),\end{aligned}$$

which leads to the claimed results. □

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