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Strong convergence of SP-iteration scheme for generalized Z-type condition

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Abstract. In this paper, we study the convergence of SP-iterative scheme for generalized Z-type condition introduced by Bosede in [4] which is more general than Zamfirescu operator and establish strong convergence theorems for above said iteration scheme and condition in the framework of normed linear spaces.

1. Introduction and Preliminaries

There is a close relationship between the problem of solving a nonlinear equation and that of approximating fixed points of a corresponding contractive type operator. Consequently, there is a theoretical and practical interest in approximating fixed points of different contractive type operators. Let (X, d) be a complete metric space and $T: X \to X$ a self mapping of X. Suppose that $F(T) = \{p \in X : Tp = p\}$ is the set of fixed points of T. There are several iteration schemes in the literature for which the fixed points of operators have been approximated over the years by various authors. In a complete metric space, the Picard iteration scheme for $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots. \tag{1.1}$$

has been employed to approximate the fixed point of mappings satisfying the inequality

$$d(Tx, Ty) \leq a\, d(x, y) \tag{1.2}$$

for all $x, y \in X$ and $a \in [0, 1)$. Condition (1.2) is called the Banach's contraction condition.

The mapping T is called Kannan mapping [8] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)] \tag{1.3}$$

for all $x, y \in X$.

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The mapping T is called Chatterjea mapping [5] if there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)] \tag{1.4}$$

for all $x,\ y \in X$.

In 1953, W.R. Mann defined the Mann iteration [9] as

$$u_{n+1} = (1 - a_n)u_n + a_n T u_n, \tag{1.5}$$

where $\{a_n\}$ is a sequence of positive numbers in [0,1].

In 1974, S. Ishikawa defined the Ishikawa iteration [7] as

$$\begin{aligned} s_{n+1} &= (1 - a_n)s_n + a_n T t_n, \\ t_n &= (1 - b_n)s_n + b_n T s_n, \end{aligned} \tag{1.6}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1].

In 2009, S. Thianwan defined the new two step iteration [13] as

$$\begin{aligned} v_{n+1} &= (1 - a_n)w_n + a_n T w_n, \\ w_n &= (1 - b_n)v_n + b_n T v_n, \end{aligned} \tag{1.7}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1].

In 2001, M.A. Noor defined the three step Noor iteration [10] as

$$\begin{aligned} p_{n+1} &= (1 - a_n)p_n + a_n T q_n, \\ q_n &= (1 - b_n)p_n + b_n T r_n, \\ r_n &= (1 - c_n)p_n + c_n T p_n, \end{aligned} \tag{1.8}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in [0,1].

Recently, Phuengrattana and Suantai defined the *SP* iteration [11] as

$$\begin{aligned} x_{n+1} &= (1 - a_n)y_n + a_n T y_n, \\ y_n &= (1 - b_n)z_n + b_n T z_n, \\ z_n &= (1 - c_n)x_n + c_n T x_n, \end{aligned} \tag{1.9}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in [0,1].

Remark 1.1. *(1) If $c_n = 0$, then (1.8) reduces to the Ishikawa iteration (1.6).*

(2) If $b_n = c_n = 0$, then (1.8) reduces to the Mann iteration (1.5).

(3) If $b_n = 0$, then (1.7) reduces to the Mann iteration (1.5).

(4) If $b_n = c_n = 0$, then (1.9) reduces to the Mann iteration (1.5).

(5) If $c_n = 0$, then (1.9) reduces to the new two step iteration (1.7).

In 1972, Zamfirescu [15] obtained the following interesting fixed point theorem.

Theorem Z. Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping for which there exists the real number a, b and c satisfying $a \in (0,1)$, $b, c \in (0, \frac{1}{2})$ such that for any pair $x, y \in X$, at least one of the following conditions holds:

(Z_1) $d(Tx, Ty) \leq a\, d(x, y)$

(Z_2) $d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)]$

(Z_3) $d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \; n = 0, 1, 2, \dots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

The conditions $(Z_1) - (Z_3)$ can be written in the following equivalent form

$$d(Tx, Ty) \leq h \max\Big\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\Big\} \tag{1.10}$$

for all $x, y \in X$ and $0 < h < 1$, has been obtained by Ciric [6] in 1974.

A mapping satisfying (1.10) is called Ciric quasi-contraction. It is obvious that each of the conditions $(Z_1) - (Z_3)$ implies (1.10).

An operator T satisfying the contractive conditions $(Z_1) - (Z_3)$ in the theorem Z is called Z-operator.

In 2004, Berinde [1] proved the strong convergence of Ishikawa iterative process defined by: for $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \end{aligned}$$

to approximate fixed points of Zamfirescu operator in an arbitrary Banach space E. While proving the theorem, he made use of the condition,

$$\|Tx - Ty\| \leq \delta \, \|x - y\| + 2\delta \, \|x - Tx\| \tag{1.11}$$

which holds for any $x, y \in E$ where $0 \leq \delta < 1$.

In this paper, inspired and motivated by [1, 15], we employ a condition introduced in [4] which is more general than condition (1.11) and establish some fixed point theorems in normed linear space. The condition is defined as follows:

Let C be a nonempty, closed, convex subset of a normed space E and $T\colon C \to C$ a self map of C. There exists a constant $L \geq 0$ such that for all $x, y \in C$, we have

$$\|Tx - Ty\| \leq e^{L\,\|x - Tx\|}\big(\delta \, \|x - y\| + 2\delta \, \|x - Tx\|\big) \tag{1.12}$$

where $0 \leq \delta < 1$ and e^x denotes the exponential function of $x \in C$.

Throughout this paper, we call this condition as generalized Z-type condition.

Remark 1.2. *If $L = 0$, in the above condition, we obtain*

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\|,$$

which is the Zamfirescu condition used by Berinde [1] where

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}, \quad 0 \leq \delta < 1,$$

while constants a, b and c are as defined in Theorem Z.

Example 1.3. *Let X be the real line with the usual norm $\|.\|$ and suppose $K = [0, 1]$. Define $T: K \to K$ by $\mathrm{Tx} = \frac{x+1}{2}$ for all $x, y \in K$. Obviously T is self-mapping with a unique fixed point 1. Now we check that condition (1.12) is true. If $x, y \in [0, 1]$, then $\|Tx - Ty\| \leq e^{L\|x-Tx\|}\left[\delta \|x - y\| + 2\delta \|x - Tx\|\right]$ where $0 \leq \delta < 1$. In fact*

$$\|Tx - Ty\| = \left\|\frac{x-y}{2}\right\|$$

and

$$e^{L\|x-Tx\|}\left[\delta \|x - y\| + 2\delta \|x - Tx\|\right] = e^{L\|\frac{x-1}{2}\|}\left[\delta \|x - y\| + \delta \|x - 1\|\right].$$

Clearly, if we chose $x = 0$ and $y = 1$, then contractive condition (1.12) is satisfied since

$$\|Tx - Ty\| = \left\|\frac{x-y}{2}\right\| = \frac{1}{2}$$

and for $L \geq 0$, we chose $L = 0$, then

$$\begin{aligned} &e^{L\|x-Tx\|}\left[\delta \|x - y\| + 2\delta \|x - Tx\|\right] = e^{L\|\frac{x-1}{2}\|}\left[\delta \|x - y\| + \delta \|x - 1\|\right] \\ &= e^{0(1/2)}(2\delta) = 2\delta, \quad \textit{where} \quad 0 < \delta < 1. \end{aligned}$$

Therefore

$$\|Tx - Ty\| \leq e^{L\|x-Tx\|}\left[\delta \|x - y\| + 2\delta \|x - Tx\|\right].$$

Hence T is a self mapping with unique fixed point 1 and satisfying the contractive condition (1.12).

Example 1.4. *Let X be the real line with the usual norm $\|.\|$ and suppose $K = \{0, 1, 2\}$. Define $T: K \to K$ by*

$$\begin{cases} Tx = 1, & \textit{if} \quad x = 0 \\ \quad\;\; = 2, & \textit{otherwise.} \end{cases}$$

Let us take $x = 0$, $y = 1$ and $L = 0$. Then from condition (1.12), we have

$$\begin{aligned} 1 &\leq e^{0(1)}[\delta(1) + 2\delta(1)] \\ &\leq 1(3\delta) = 3\delta \end{aligned}$$

which implies $\delta \geq \frac{1}{3}$. Now if we take $0 < \delta < 1$, then condition (1.12) is satisfied and 2 is of course a unique fixed point of T.

Iteration procedures in fixed point theory are lead by the considerations in summability theory. For example, if a given sequence converges, then we don't look for the convergence of the sequence of its arithmetic means. Similarly, if the sequence of Picard iterates of any mapping T converges, then we don't look for the convergence of other iteration procedures.

We need the following useful lemma to prove our main results in this paper.

Lemma 1.5. *(See [3]) Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying the following condition:*

$$\alpha_{n+1} \leq (1-\delta_n)\alpha_n + \beta_n + \gamma_n, \quad \forall\, n \geq 0,$$

where $\{\delta_n\}_{n=0}^{\infty} \subset [0,1]$. If $\sum_{n=0}^{\infty} \delta_n = \infty$, $\lim_{n\to\infty} \beta_n = O(\delta_n)$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$, then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main Results

In this section, we establish strong convergence theorems of SP-iteration scheme (1.9) to converge to a fixed point of generalized Z-type condition in the framework of normed linear spaces.

Theorem 2.1. *Let C be a nonempty closed convex subset of a normed linear space E. Let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.12) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.9). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof. From the assumption $F(T) \neq \emptyset$, it follows that T has a fixed point in C, say u. Since T satisfies generalized Z-type condition given by (1.12). Now using the iterative sequence defined by (1.9) and (1.12), we have

$$\begin{aligned}
\|x_{n+1} - u\| &= \left\|(1-a_n)y_n + a_nTy_n - u\right\| \\
&= \left\|(1-a_n)(y_n - u) + a_n(Ty_n - u)\right\| \\
&\leq (1-a_n)\left\|y_n - u\right\| + a_n\left\|Ty_n - u\right\| \\
&= (1-a_n)\left\|y_n - u\right\| + a_n\left\|Tu - Ty_n\right\| \\
&\leq (1-a_n)\left\|y_n - u\right\| \\
&\quad + a_n\Big[e^{L\|u-Tu\|}(2\delta\,\|u - Tu\| + \delta\,\left\|u - y_n\right\|)\Big] \\
&= (1-a_n)\left\|y_n - u\right\| \\
&\quad + a_n\Big[e^{L\|u-u\|}(2\delta\,\|u - u\| + \delta\,\left\|u - y_n\right\|)\Big] \\
&= (1-a_n)\left\|y_n - u\right\| \\
&\quad + a_n\Big[e^{L(0)}(2\delta\,(0) + \delta\,\left\|u - y_n\right\|)\Big] \\
&= (1-a_n)\left\|y_n - u\right\| + a_n\delta\,\left\|y_n - u\right\|
\end{aligned}$$

which gives

$$\|x_{n+1} - u\| \leq (1 - a_n + a_n\delta)\left\|y_n - u\right\|. \tag{2.1}$$

Again from (1.9) and (1.12), we have

$$\begin{aligned}
\left\|y_n - u\right\| &= \|(1-b_n)z_n + b_n T z_n - u\| \\
&= \|(1-b_n)(z_n - u) + b_n(Tz_n - u)\| \\
&\leq (1-b_n)\|z_n - u\| + b_n\|Tz_n - u\| \\
&= (1-b_n)\|z_n - u\| + b_n\|Tu - Tz_n\| \\
&\leq (1-b_n)\|z_n - u\| \\
&\quad + b_n\Big[e^{L\|u - Tu\|}(2\delta\,\|u - Tu\| + \delta\,\|u - z_n\|)\Big] \\
&= (1-b_n)\|z_n - u\| \\
&\quad + b_n\Big[e^{L\|u - u\|}(2\delta\,\|u - u\| + \delta\,\|u - z_n\|)\Big] \\
&= (1-b_n)\|z_n - u\| \\
&\quad + b_n\Big[e^{L(0)}(2\delta\,(0) + \delta\,\|u - z_n\|)\Big] \\
&= (1-b_n)\|z_n - u\| + b_n\delta\,\|z_n - u\|
\end{aligned}$$

which gives

$$\left\|y_n - u\right\| \leq (1 - b_n + b_n\delta)\|z_n - u\|. \tag{2.2}$$

Similarly, using the same method as above, we can get

$$\|z_n - u\| \leq (1 - c_n + c_n\delta)\|x_n - u\|. \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain

$$\left\|y_n - u\right\| \leq (1 - b_n + b_n\delta)(1 - c_n + c_n\delta)\|x_n - u\|. \tag{2.4}$$

Now, substituting (2.4) into (2.1), we get

$$\begin{aligned}
\|x_{n+1} - u\| &\leq (1 - a_n + a_n\delta)(1 - b_n + b_n\delta)(1 - c_n + c_n\delta)\|x_n - u\| \\
&\leq (1 - a_n + a_n\delta)\|x_n - u\| \\
&= [1 - (1-\delta)a_n]\|x_n - u\|, \quad n = 0, 1, 2, \ldots. \qquad (2.5)
\end{aligned}$$

Since $0 \leq \delta < 1$, $a_n \in [0, 1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, setting $p_n = \|x_n - u\|$, $s_n = (1-\delta)a_n$ and by applying Lemma 1.5 in (2.5), it follows that

$$\lim_{n\to\infty}\|x_n - u\| = 0.$$

Thus $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.

To show uniqueness of the fixed point u, assume that u_1, $u_2 \in F(T)$ and $u_1 \neq u_2$.

Applying generalized Z-type condition given by (1.12) and using the fact that $0 \leq \delta < 1$, we obtain

$$\begin{aligned}
\|u_1 - u_2\| &= \|Tu_1 - Tu_2\| \\
&\leq e^{L\|u_1 - Tu_1\|}(2\delta\,\|u_1 - Tu_1\| + \delta\,\|u_1 - u_2\|) \\
&\leq e^{L\|u_1 - u_1\|}(2\delta\,\|u_1 - u_1\| + \delta\,\|u_1 - u_2\|) \\
&= e^{L(0)}(2\delta\,(0) + \delta\,\|u_1 - u_2\|) \\
&= \delta\,\|u_1 - u_2\| \\
&< \|u_1 - u_2\|, \quad \text{since} \quad 0 \leq \delta < 1,
\end{aligned}$$

which is a contradiction. Therefore $u_1 = u_2$. Thus $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T. This completes the proof. □

Theorem 2.2. *Let C be a nonempty closed convex subset of a normed linear space E. Let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.12) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.8). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1. This completes the proof. □

Theorem 2.3. *Let C be a nonempty closed convex subset of a normed linear space E. Let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.12) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.7). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof. The proof of Theorem 2.3 immediately follows by putting $c_n = 0$ in Theorem 2.1. This completes the proof. □

Theorem 2.4. *Let C be a nonempty closed convex subset of a normed linear space E. Let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.12) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.5). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof. The proof of Theorem 2.4 immediately follows by putting $b_n = c_n = 0$ in Theorem 2.1. This completes the proof. □

Corollary 2.5. *[[14], Theorem 2.1] Let E be an arbitrary Banach space, C a nonempty closed convex subset of E and $T\colon C \to C$ a Zamfirescu operator. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.7). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

Remark 2.6. *Our results extend and improve upon, among others, the corresponding results proved by Berinde [2], Yildirim et al. [14] and Bosede [4] to the case of SP-iteration scheme [11] and Noor iteration scheme [10] considered in this paper.*

Remark 2.7. *Our results also extend and improve the corresponding results of [12] to the case of generalized Z-type condition, SP-iteration scheme [11] and Noor iteration scheme [10] considered in this paper.*

3. Conclusion

The generalized Z-type condition (1.12) is more general than the condition (1.11) which is used by Berinde in [1]. Thus the results obtained in this paper are good improvement and generalization of several known results in the existing literature (see, e.g., [2, 4, 12, 14] and some others).

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