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Non commutative Müller regularity

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Abstract. Both the joint spectrum of Joseph Taylor and the single variable spectrum of Tosio Kato are based on the concept of exactness, leading to the idea of Müller regularity.

Just occasionally, several variable spectral theory has something to say about spectral theory in one variable. Specifically, the joint spectrum of Joseph L. Taylor impinges on the single variable spectrum associated with Tosio Kato.

Whether in one or several variables, a "spectrum" is defined by a concept of *invertibility*, or more general *non singularity*. For example the classical spectrum $\sigma_A(a) \subseteq \mathbf{C}$ of a linear algebra element $a \in A$ consists of the (complex) numbers λ for which the scalar perturbations $a - \lambda$ are outside the invertible group $A^{-1} \subseteq A$; if we relax membership of A^{-1} to include either left invertibles or right invertibles then we reduce the spectrum to the left or the right spectrum. Seymour Goldberg tried to make a spectrum out of the *relatively regulars* $A^{\cap} \subseteq A$, but he found no spectral mapping theorems: there isn't any semigroup structure associated with the possession $a = aca$ of a generalized inverse $c \in A$. It turns out however that a certain strengthening of this property gives a perfectly sound spectral theory. There are two equivalent conditions: in a Banach algebra A we can ask for *holomorphic regularity* in the sense that there is a holomorphic function $a^\wedge : U \to A$, defined on an open neighbourhood $U \in \mathrm{Nbd}(0)$, for which

0.1 $$a - z \equiv (a - z)a^\wedge(z)(a - z) : U \to A\ .$$

Alternatively, specializing to $A = B(X)$ the operators on a Banach space X, combine relative regularity with an *exactness* condition going back to Pierre Saphar [24]. The joint invertibility, for a tuple of linear operators, of Joseph L. Taylor [**?**], is also based on "exactness". It is appropriate, therefore, to give a definition of exactness:

1. Definition *If A is an additive category and the ordered pair $(b, a) \in A^2$ is mutually compatible, in the sense that*

1.1 $$\exists\ ba \in A\ ,$$

the product $ba \in A$ is defined, then we declare the pair (b, a) to be weakly exact provided there is implication, for arbitrary (appropriately compatible) u and v in A,

1.2 $$bu = 0 = va \Longrightarrow vu = 0\ .$$

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We shall say that the pair $(b, a) \in A^2$ is splitting exact if there is another pair $(a', b') \in A^2$ for which

1.3 $$b'b + aa' = 1 .$$

We shall also say that the pair (b, a) is regular if each term has a generalized inverse

1.4 $$a \in aAa \ ; \ b \in bAb .$$

The mutual compatibility condition (1.1) says that $a : X \to Y$ and $b : Y \to Z$: the arrival object of a is the same as the departure object of b. For example if $a = 0$ is zero (1.2) says that $b \in A$ is a *monomorphism*; if instead $b = 0$ then it makes $a \in A$ an *epimorphism*. In the category of linear mappings between vector spaces, exactness (1.2) says precisely, with $a : X \to Y$ and $b : Y \to Z$, that ("linear exactness")

1.5 $$b^{-1}(0) \subseteq a(X) \subseteq Y .$$

To the extent that we are not requiring the opposite inclusion, which of course says ("chain condition")

1.6 $$ba = 0 ,$$

the conditions (1.2) and (1.3) might be thought of as a sort of "non commutative exactness".

In the category of linear mappings between vector spaces, this implication can also be reversed, by fiddling with *Hamel bases*; generally however *relative regularity* offers a bridge between splitting and weak exactness:

2. Theorem *In an additive category A,*

2.1 *splitting exactness implies weak exactness*

and

2.2 *weak exactness and regularity together imply splitting exactness ,*

while

2.3 *splitting exactness and chain condition together imply regularity .*

Proof. Implication (1.3)$\Longrightarrow$(1.2) is visible:

$$vu = (vb')(bu) + (va)(a'u) .$$

Conversely if the pair $(b, a) \in A^2$ is relatively regular in the sense (1.4), with

$$b = bb^\wedge b \ , \ a = aa^\wedge a \ ,$$

then weak exactness (1.2) gives

2.4 $$(1 - aa^\wedge)(1 - b^\wedge b) = 0 \in A \ ,$$

giving two candidates for the splitting material $(a', b') \in A^2$ •

For "chains", therefore, the three conditions (1.2), (1.3) and (1.4) are in a "love knot". The Saphar condition incorporates "self exactness":

3. Definition *We shall say that $a \in A$ is self exact provided*

3.1 *(a, a) is exact ,*

n exact provided

3.2 $$(a^n, a) \text{ is exact },$$

and hyper exact if it is n exact for every $n \in \mathbf{N}$.

Note ([13] Lemma 1) that there is implication

3.3 $$(cb, a) \text{ exact}, \ (c, b) \text{ exact } \Longrightarrow (c, ba) \text{ exact}$$

and

3.4 $$(c, ba) \text{ exact}, \ (b, a) \text{ exact } \Longrightarrow (cb, a) \text{ exact} ;$$

in particular

3.5 $$(a^n, a) \text{ exact} \Longleftrightarrow (a, a^n) \text{ exact} .$$

There are of course at least three versions of Definition 3, according as "exactness" means weak exactness or splitting exactness, or the linear exactness of (1.5); the implications (3.3) and (3.4) hold both for splitting exactness and for linear exactness, although not [8] for weak exactness. It is also necessary, for $a : X \to Y$ to be self exact, that it is *self compatible* in the sense that a^2 is defined: $Y = X$.

Exactness imposes a certain discipline on relative regularity, and hence opens the door to the polynomial spectral mapping theorems which eluded Goldberg:

4. Definition *A non commutative regularity is $H \subseteq A$ for which if $(b, a) \in A^2$ is splitting exact there is equivalence*

4.1 $$ba \in H \Longleftrightarrow \{a, b\} \subseteq H .$$

For a *Müller regularity* $H \subseteq A$ there is for arbitrary $n \in \mathbf{N}$ equivalence

4.2 $$a \in H \Longleftrightarrow a^n \in H ,$$

while the stronger condition [23]

4.3 $$\exists \, (b', a') \in A^2 : \ (a, b, a', b') \in A^4 \text{ commutative and } b'b - aa' = 1$$

is required for two way implication (4.1).

Most of the familiar "invertibilities" are non commutative regularities, including "the relative regularity $A^\cap$ of Goldberg":

5. Theorem $A^\cap \subseteq A$ *is a non commutative regularity.*

Proof. This is not rocket science ([13] Theorem 3): if $ba = bacba$ and $b'b + aa' = 1$ then

5.1 $$(1 - aa')a(1 - cba) = 0 = (1 - bac)b(1 - b'b) ;$$

conversely (1.4) and (2.4) give

5.2 $$baa^\wedge b^\wedge ba = b(aa^\wedge + b^\wedge b - 1)a = ba \bullet$$

Theorem 5 explains why adding Saphar hyperexactness to regularity contributes to spectral mapping theorems.

Each of A^{-1}, A^{-1}_{left} and A^{-1}_{right} are non-commutative regularities. For example, recall

5.3 $$\{a, b\} \subseteq A^{-1}_{left} \Longrightarrow ba \in A^{-1}_{left} \Longrightarrow a \in A^{-1}_{left} ;$$

conversely

5.4 $$ba \in A^{-1}_{left}, a \in A^{-1}_{right} \Longrightarrow b \in A^{-1}_{left} .$$

The proof that A^{-1}_{left} is a "non commutative regularity" is just a beefed-up version of the proof of (5.4). In the category $A = BL$ of bounded linear operators between Banach spaces, relative regularity $a \in aAa$ implies that $a : X \to Y$ has *closed range*

5.5 $$a(X) = \mathrm{cl}\, a(X)$$

The "holomorphic regularity" (0.1) is equivalent [21],[13] to relative regularity together with *hyperexactness*. We shall also describe holomorphic regularity as *Kato invertibility*. Theorem 10 of [13] says that Kato invertibility is a Müller regularity. Slightly more generally, we shall describe $a \in BL$ as *Kato non singular* if is hyperexact with closed range. We claim

6. Theorem *Kato non singularity is a Müller regularity, and closed range a non commutative regularity.*
Proof. This is Theorem 1 of [14]: if $a : X \to Y$ and $b : Y \to Z$ then

6.1 $$b^{-1}(0) \cap a(X) = \{0\}, b^{-1}(0) + a(X) \text{ closed} \Longrightarrow a(X) \text{ closed}$$

and

6.2 $$b(Y), b^{-1}(0) + a(X) \text{ closed} \Longrightarrow ba(X) \text{ closed} \Longrightarrow b^{-1}(0) + a(X) \text{ closed} .$$

Obviously there are "weak" and "splitting" versions of self and hyper exactness. In a *normed linear category* there is another kind of exactness, intermediate between weak and splitting: we can ask of $(b, a) \in A^2$ that [11] there are $k > 0$ and $h > 0$ for which, for arbitrary, suitably compatible, u,v,

6.3 $$\|vu\| \leq k\|v\|\,\|bu\| + h\|va\|\,\|u\| .$$

In the category BL of bounded operators between Banach spaces, (6.3) holds, at least for chains (1.6), provided (1.5) is satisfied while each of a and b have closed range.

Both monomorphism and epimorphism define non commutative regularities:

7. Theorem *If* (b, a) *is splitting exact in the sense* (1.3) *then*

7.1 $$ba \text{ monomorphic} \Longrightarrow b \text{ monomorphic}$$

and

7.2 $$ba \text{ epimorphic} \Longrightarrow a \text{ epimorphic} .$$

Proof. For (7.1) argue that, analogous to (5.4),

$$bv = 0 \Longrightarrow v = b'(bv) + a(a'v) ;$$

but now, with $w = a'v$,

$$bv = 0 \Longrightarrow (ba)w = 0 \Longrightarrow w = 0 \Longrightarrow v = aw = 0 .$$

The argument for (7.2) is identical ("reverse products") •

Note [8] that we cannot replace the splitting exactness (1.3) here by weak exactness (1.2). In a variant, in a "normed linear category" A suppose that a product ba is "strongly monomorphic" in the sense that there is $k > 0$ for which, for arbitrary compatible u,

7.3 $$\|(ba)u\| \geq k\|u\| ,$$

so that also a is strongly monomorphic,

7.4 $$\|au\| \geq (k/\|b\|)\|u\| \ ,$$

then splitting exactness (1.3) says that also b satisfies (7.4):

$$v = b'(bv) + a(a'v) \ , \implies bv = bb'bv + ba(a'v) \ ,$$

giving $ba(a'v) = (1 - bb')bv$ and hence

$$\|v\| \leq (\|b'\| + \|a\|/k)\|k(1 + \|b\| \ \|b'\|)\|bv\| \ .$$

Thus "strong monomorphism" also defines a "non commutative regularity". We have not settled whether or not we can replace splitting exactness (1.3) by "normed linear exactness" (6.3) in either Theorem 7 or this argument.

There is a "linear" analogue of (7.1) from Theorem 7:

7.5 $$b^{-1}(0) \subseteq a(X) \Longrightarrow b^{-1}(0) \subseteq a \ (ba)^{-1}(0) \ .$$

Taylor invertibility for n tuples $a \in A^n$ is also defined by exactness:

8. Definition *We shall say that* $(a, b) \in A^2$ *is weakly left non singular if there is implication*

8.1 $$bu = au = 0 \Longrightarrow u = 0 \ ;$$

weakly right non singular if there is implication

8.2 $$vb = va = 0 \Longrightarrow v = 0 \ ,$$

and weakly middle non singular if

8.3 $$\left(\begin{pmatrix}-b & a\end{pmatrix}, \begin{pmatrix}a \\ b\end{pmatrix}\right) \text{ is weakly exact} \ .$$

Also we shall say that (a, b) *is weakly Taylor non singular if the sequence*

8.4 $$\left(0, \begin{pmatrix}-b & a\end{pmatrix}, \begin{pmatrix}a \\ b\end{pmatrix}, 0\right)$$

is weakly exact, and Taylor invertible if it is splitting exact.

We refer to the sequence (8.4) as the *Koszul complex* of the pair (a, b). We make this definition without assuming commutivity

8.5 $$ba = ab \ ;$$

of course commutivity (7.5) is necessary and sufficient for the sequence (8.4) to satisfy the chain condition (1.6) (and hence be a "complex"). Of course "exactness" in (8.4) means the exactness of each of the obvious three pairs.

The Müller conditions (4.3) are sufficient for Taylor invertibility, and in particular for the splitting version of middle non singularity (8.3). For the category of bounded linear operators between Banach spaces, middle non singularity has been characterized by Gonzalez [7]:

9. Theorem *Necessary and sufficient for* (a, b) *to be middle non singular is that*

9.1 $$b^{-1}(0) \subseteq a \ b^{-1}(0) \ ;$$

9.2 $$a^{-1}(0) \subseteq b \ a^{-1}(0) \ ;$$

9.3 $$b(X) \cap a(X) \subseteq (ba)(ab - ba)^{-1}(0) \equiv \big((ab) \wedge (ba)\big)(X)\ .$$

If (9.1) *and* (9.2) *both hold then also*

9.4 $$(ba)^{-1}(0) + (ab)^{-1}(0) \subseteq b^{-1}(0) + a^{-1}(0)\ .$$

Proof. This is Theorem 4 of [13] •

Note in Theorem 9 that while a and b need not commute, they need to be both mutually compatible and self compatible. The notation at the end of (9.3) is from [9].

$S \wedge T : (S - T)^{-1}(0) \to Y$ is the common restriction of $T : X \to Y$ and $S : X \to Y$ to the subspace on which they agree

For linear operators, "ascent one" and "descent one" define Müller regularities:

10. Theorem *If* $(a, b) \in A^2$ *is (linearly) middle non singular then*

10.1 $$(b^{-1}(0) \cap b(X)) + (a^{-1}(0) \cap a(X)) = ((ba)^{-1}(0) + (ab)^{-1}(0)) \cap (ba)(X) \cap (ab)(X)$$

and

10.2 $$(b^{-1}(0) + b(X)) \cap (a^{-1}(0) + a(X)) = ((ba)^{-1}(0) + (ab)^{-1}(0)) + ((ab)(X) \cap (ba)(X))\ .$$

Proof. Inclusion $b^{-1}(0) + a^{-1}(0) \subseteq (ab)^{-1}(0) + (ba)^{-1}(0)$ always holds, while (9.1) and (9.3) give

$$b^{-1}(0) \cap b(X) \subseteq a(X) \cap b(X) \subseteq (ba)(X) \cap (ab)(X)\ ,$$

and similarly for $a^{-1}(0) \cap a(X)$; thus the first term on the left hand side of (10.1), and similarly the second, is included in the right. Conversely inclusion $(ba)X \subseteq b(X)$ and $(ab)X \subseteq a(X)$ always hold, while (9.4) finishes the proof of (10.1).

Towards (10.2), (9.1) and (9.2), and then (9.3), imply

$$(b^{-1}(0) + b(X)) \cap (a^{-1}(0) + a(X)) = b(X) \cap a(X) = (ba)(X) \cap (ab)(X)\ ,$$

while always

$$b^{-1}(0) + a^{-1}(0) \subseteq (ab)^{-1}(0) + (ba)^{-1}(0)\ ;$$

the opposite inclusion is (9.4) •

If $(a, b) \in A^2$ is middle non singular it follows that

10.5 $$b^{-1}(0) \cap b(X) = \{0\} = a^{-1}(0) \cap a(X) \Longleftrightarrow ((ba)^{-1}(0) + (ab)^{-1}(0)) \cap (ba)X \cap (ab)(X) = \{0\}$$

and

10.6 $$((ba)^{-1}(0) + (ab)^{-1}(0)) + ((ba)X \cap (ab)X) = X \Longleftrightarrow b^{-1}(0) + b(X) = X = a^{-1}(0) + a(X)\ .$$

Theorem 10 looks much simpler when $ab = ba$. The conditions in (10.5) and (10.6) say that the operators a,b and hence also $ab = ba$ are of "ascent ≤ 1", respectively "descent ≤ 1". Thus Theorem 10 indeed shows that the ascent one and descent one conditions define Müller regularities. By considering powers a^k and b^k it follows that the same is true for "finite ascent" and "finite descent". To see that also "SVEP at 0" is a Müller regularity we should [2] replace the ranges $a(X)$. $b(X)$ and $ba(X)$ in Theorem 10 by "holomorphic ranges" $a^\omega(x)$, $b^\omega(X)$ and $(ba)^\omega(X)$.

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