Functional Analysis,
Approximation and
Computation
6 (2) (2014), 9–22

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

On the Kato, semi-regular and essentially semi-regular spectra

Mohammed Benharrat[a], **Aymen Ammar**[b], **Aref Jeribi**[b], **Bekkai Messirdi**[c]

[a]*Département de Mathématiques et Informatique, Ecole National Polythecnique d'Oran (Ex. ENSET d'Oran); B.P. 1523 El M'Naouar-Oran, Oran, Algérie*
[b]*Département de Mathématiques, Université de Sfax, Faculté des sciences de Sfax, Route de soukra Km 3.5, B. P 1171, 3000, Sfax, Tunisie*
[c]*Département de Mathématiques, Université d'Oran, 31000 Oran, Algérie*

Abstract. In this paper, we give some properties of the semi-regular, essentially semi-regular and the operators of Kato type on a Banach space. We also show that the essentially semi-regular spectrum of closed, densely defined linear operator is stable under commuting compact perturbation and its Kato spectrum is stable subjected to additive commuting nilpotent perturbations.

1. Introduction

The concept of semi-regularity and essentially semi-regularity amongst the various concepts of regularity originated by the classical treatment of perturbation theory owed to Kato and its flourishing has greatly benefited from the work of many authors in the last years, in particular from the work of Mbekhta and Ouahab [24], Müller [26], Rakocevič [29], Mbekhta and Ouahab [5]. Recall that an operator A is said to be semi-regular if $R(A)$ is closed and $N(A^n) \subseteq R(A)$, for all $n \geq 0$ (see [24]), where $R(A)$ and $N(A)$ denote the range and the null space of A respectively. This concept leads in a natural way to the semi-regular spectrum $\sigma_{se}(A)$, an important subset of the ordinary spectrum which is defined as the set of all $\lambda \in \mathbb{C}$ for which $\lambda - A$ is not semi-regular and its essential version $\sigma_{es}(A)$ the set of all $\lambda \in \mathbb{C}$ for which $\lambda - A$ is not essentially semi-regular. The semi-regular spectrum was first introduced by Apostol [3] for operators on Hilbert spaces and successively studied by several authors mentioned above in the more general context of operators acting on Banach spaces. An operator A is called a Kato type operator if we can write $A = A_1 \oplus A_0$ where A_0 is a nilpotent operator and A_1 is a semi-regular one. In 1958 Kato proved that a closed semi-Fredholm operator is of Kato type. J. P. Labrousse [22] studied and characterized a new class of operators named quasi-Fredholm operators, in the case of Hilbert spaces and he proved that this class coincide with the set of Kato type operators and the Kato decomposition becomes a characterization of the quasi-Fredholm operators . But in the case of Banach spaces the Kato type operator is also quasi-Fredholm, the converse is not true. The study of such class of operators gives a new important part of the ordinary

2010 *Mathematics Subject Classification*. Primary 47A10, 47A55.
Keywords. Semi-regular operators; Kato type operators; Essential spectrum, Kato spectrum; Nilpotent operators.
Received: 17 February 2014; Accepted: 1 May 2014
Communicated by Bhagwatti Duggal
Email addresses: mohammed.benharrat@gmail.com (Mohammed Benharrat), Aref.Jeribi@fss.rnu.tn (Aref Jeribi), messirdi.bekkai@univ-oran.dz (Bekkai Messirdi)

spectrum called the Kato spectrum $\sigma_k(A)$ which is the set of all complex λ such that $\lambda - A$ is not of Kato type operator.

The aim of this paper is to investigate the classes of semi-regular, essentially semi-regular and the operators of Kato type. We show, under some assumptions, that the product of two commuting semi-regular (resp. essentially semi-regular) operators A and B is semi-regular and we prove that if A and B are closed densely defined linear operators and if for some $\lambda \in \rho(A) \cap \rho(S)$, the operator $(A - \lambda)^{-1} - (B - \lambda)^{-1}$ is a compact operator commuting with A or B then $\sigma_{es}(A) = \sigma_{es}(B)$. Moreover, if $\sigma(A) = \sigma_c(A)$ then $\sigma_{ei}(A) \subseteq \sigma_{ei}(B)$, $i = 3, 4, 5, 6$, eap, eδ, where $\sigma_c(A)$ is the continuous spectrum and $\sigma_{ei}(A)$ $i = 3, 4, 5, 6$, eap, eδ are some different definitions of the essential spectrum of A originated from the Fredholm theory. We give some interesting relationships between the Kato, semi-regular and essentially semi-regular spectra of two bounded linear operators and the corresponding spectra of their sum. Finally, we prove that if A is a closed operator and Q is nilpotent operator such that $QA = AQ$ then $\sigma_k(A + Q) = \sigma_k(A)$.

We organize our paper in the following way: In the next Section we give some preliminary results in which our investigation will be need. In Section 3, we give a case when the product of two commuting semi-regular operators is also semi-regular one, we establish many important properties of $\sigma_{se}(A)$, $\sigma_{es}(A)$ and $\sigma_k(A)$ and we present some relationships between those spectra and others essential spectra founded in the Fredholm theory. We also prove that the essentially semi-regular spectrum of closed densely defined operator is stable under commuting compact perturbation. Finally, in Section 4, we show that the Kato spectrum of unbounded operators is invariant under commuting nilpotent perturbations.

2. Preliminary Results

Let X be a Banach space. We denote by $\mathcal{L}(X)$ $\big($resp. $\mathcal{C}(X)\big)$ the set of all bounded (resp. closed, densely defined) linear operators from X into X and we denote by $\mathcal{K}(X)$ the subspace of compact operators from X into X. For $A \in \mathcal{C}(X)$, we write $\mathcal{D}(A) \subset X$ for the domain, $N(A) \subset X$ for the null space and $R(A) \subset X$ for the range of A. Let $\sigma(A)$ (resp. $\rho(A)$) denote the spectrum (resp. the resolvent set) of A.

Definition 2.1. *Let $A \in \mathcal{C}(X)$,*

(i) A is said to be semi-regular if $R(A)$ is closed and $N(A) \subseteq R(A^n)$, for all $n \geq 0$.

(ii) A is said to be essentially semi-regular if $R(A)$ is closed and there exists a finite dimensional subspace F such that $N(A) \subseteq R(A^n) + F$, for all $n \geq 0$.

Now, set

$$\mathcal{V}_0(X) := \{A \in \mathcal{C}(X) \text{ such that } A \text{ is semi-regular}\}$$

and

$$\mathcal{V}(X) := \{A \in \mathcal{C}(X) \text{ such that } A \text{ is essentially semi-regular}\}.$$

Trivial examples of semi-regular operators are surjective operators as well as injective operators with closed range, Fredholm operators and semi-Fredholm operators with jump equal zero. Some other examples of semi-regular operators may be found in Mbekhta and Ouahab [24] and Labrousse [22]. A semi-regular operator A has a closed range. It is evident that the reduced minimum modulus of A is useful to find conditions which ensure that $R(A)$ is closed. Recall that the reduced minimum modulus of a non-zero operator A is defined by

$$\gamma(A) = \inf_{x \notin N(A)} \frac{\|Ax\|}{\operatorname{dist}(x,\ N(A))},$$

where $\operatorname{dist}(x,\ N(A)) = \inf_{y \in N(A)} \big\|x - y\big\|$. If $A = 0$ then we take $\gamma(A) = \infty$. Note that (see [19]):

$$\gamma(A) > 0 \Leftrightarrow R(A) \text{ is closed.}$$

The following theorem gives several equivalent definitions of the semi-regular operators.

Theorem 2.2. *[24, Theorem 4.1] Let A be a closed opertor and $\lambda_0 \in \mathbb{C}$, the following statements are equivalent:*

1. *$\lambda_0 I - A$ is semi-regular.*
2. *$\gamma(\lambda_0 I - A) > 0$ and the mapping $\lambda \to \gamma(\lambda I - A)$ is continuous at λ_0*
3. *$\gamma(\lambda_0 I - A) > 0$ and the mapping $\lambda \to N(\lambda I - A)$ is continuous at λ_0 in the gap topology.*
4. *$R(\lambda_0 I - A)$ is closed in a neighborhood of λ_0 and the mapping $\lambda \to R(\lambda I - A)$ is continuous at λ_0 in the gap topology.*

We define the generalized range of a closed operator A by

$$R^\infty(A) := \bigcap_{n\in\mathbb{N}} R(A^n).$$

Lemma 2.3. *[24, Lemma 2.4] Let A be a closed operator. If A is semi-regular then $A(R^\infty(A) \cap D(A)) = R^\infty(A)$ and $R^\infty(A)$ is closed.*

Lemma 2.4. *Let A be a closed operator. If A is semi-regular then A^n is semi-regular for every $n \in \mathbb{N}$.*

Proof. Since A is regular we have by [24, Lemma 2.5] that $\gamma(A^n) \geq \gamma(A)^n > 0$, so that $B = A^n$ has closed range. Furthermore, $R^\infty(B) = R^\infty(A)$ and by [24, Lemma 2.1] $N(B) \subset R^\infty(A) = R^\infty(B)$. We conclude A^n is semi-regular. □

Theorem 2.5 ([26]). *Let $T,\ S \in \mathcal{L}(X)$, $TS = ST$. If TS is semi-regular (resp. essentially semi-regular), then both T and S are semi-regular (resp. essentially semi-regular).*

The product of two commuting semi-regular operators need not be semi-regular in general (see [26]). The following two theorems gives some case whence the converse of Theorem 2.5 is true.

Theorem 2.6 ([26]). *Let $T,\ S,\ C,\ D \in \mathcal{L}(X)$ be mutually commuting operators such that $TC + SD = I$. Then, TS is semi-regular if and only if both T and S are semi-regular.*

Theorem 2.7. *Let $T,\ S \in \mathcal{L}(X)$ such that $TS = ST$ and S is invertible. If T is semi-regular then TS is semi-regular.*

In the sequel let us denote by X/V the quotient space induced by a closed subspace V of X. Recall the following nice characterization of the bounded semi-regular (resp. the essentially semi-regular) operators.

Theorem 2.8. *[20] $T \in \mathcal{L}(X)$ is semi-regular (resp. essentially semi-regular) operator if and only if there exists a closed subspace V of X such that $TV = V$ and the operator $\hat{T} : X/V \to X/V$ induced by T is bounded below (resp. upper semi-Fredholm).*

Let (M, N) a pair of closed subspaces of X, A is said to be decomposed according to $X = M \oplus N$ if

$$P\mathcal{D}(A) \subset \mathcal{D}(A),\ AM \subset M,\ AN \subset N$$

where P is the projection on M along N. When A is decomposed as above, the pairs A_M, A_N of A in M, N, respectively can be defined, A_M is an operator in the Banach space M with $\mathcal{D}(A_M) = \mathcal{D}(A) \cap M$ such that $A_M x = Ax \in M$, A_N is defined similarly. In this case we write $A = A_M \oplus A_N$. Note that if A is closed the same is true for A_M and A_N.

Definition 2.9. *An operator $A \in C(X)$, is said to be of Kato type of order d, if there exists $d \in \mathbb{N}$ and a pair of closed subspaces (M, N) of X such that $A = A_M \oplus A_N$, with A_M is semi-regular and A_N is nilpotent of order d (i.e $(A_N)^d = 0$).*

An operator A is said to be of Kato type if a Kato type of order d, for some $d \in \mathbb{N}$.

Clearly, every semi-regular operator is of Kato type with $M = X$ and $N = \{0\}$ and a nilpotent operator has a decomposition with $M = \{0\}$ and $N = X$.

Every essentially semi-regular operator admits a decomposition (M, N) such that N is finite-dimensional vector space, so is of Kato type.

Theorem 2.10. *Let $A \in C(X)$ and assume that A is of Kato type of order d with a pair (M, N) of closed subspaces of X. Then:*

(i) $R^\infty(A) = AR^\infty(A) = R^\infty(A_M)$. Further, $R^\infty(A)$ is closed.

(ii) for every nonnegative integer $n \geq d$, we have $N(A) \cap R(A^n) = N(A) \cap M = N(A) \cap R(A^d)$.

(iii) for every nonnegative integer $n \geq d$, we have $R(A) + N(A^n) = A(M) \oplus N$ is closed.

Proof. (i) Since $A = A_M \oplus A_N$ it is clear that $A^n = A_M^n \oplus A_N^n$ for every $n \in \mathbb{N}$ and thus as A_N is nilpotent of degree d we obtain that $R(A^n) = R(A_M^n)$ for $n \geq d$ and hence $R^\infty(A) = R^\infty(A_M)$. On the other hand, since A_M is semi-regular we infer from Lemma 2.4 that A_M^n is semi-regular, in particular $R(A_M^n)$ is closed for all $n \in \mathbb{N}$ and hence $R^\infty(A_M)$ is closed.

(ii) Let $n \geq d$. Then

$$N(A) \cap R(A^n) = N(A) \cap R(A_M^n) \subseteq N(A) \cap R(A_M) \subseteq N(A) \cap M = N(A_M),$$

since A_M is semi-regular, we have $N(A_M) \subseteq N(A) \cap R(A_M^n) = N(A) \cap R(A^n)$. Hence (ii) holds.

(iii) Let $n \geq d$. Clearly $N \oplus N(A_M^n) = N(A^n)$ so that $N \subset N(A^n)$ and hence $R(A_M) \oplus N \subseteq R(A) + N(A^n)$. Conversely,

$$N(A^n) = N(A) = N(A_M^n) \oplus N(A_N^n) = N(A_M^n) \oplus N \subseteq R(A_M) \oplus N,$$

and from the semi-regularity of A_M it follows that $R(A) = R(A_M) \oplus R(A_N) \subset R(A_M) \oplus N$. Hence $R(A) + N(A^n) \subseteq R(A_M) \oplus N$, consequently, $R(A) + N(A^n) = A(M) \oplus N$ if $n \geq d$. Let now $\Psi : (m, n) \in M \times N \to \Psi(m, n) = m + n \in E$, clearly Ψ is a topological isomorphism and $\Psi(R(A_M), N) = R(A_M) \oplus N$ with $R(A_M)$ closed in M and hence $(R(A_M), N)$ is a closed, as desired. □

Note that by results of J.P. Labrousse [22], in the case of Hilbert spaces, the set of quasi-Fredholm operators coincides with the set of all Kato type operators. But in the case of Banach spaces the Kato type operator is also quasi-Fredholm, according to [22, Theorem 3.2.2] the converse is true when $R(A^d) \cap N(A)$ and $R(A) + N(A^d)$ are complemented in the Banach space X.

For every operator $A \in C(X)$, let us define the Kato spectrum, the semi-regular spectrum and the essentially semi-regular spectrum as follows respectively:

$\sigma_k(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not of Kato type}\}$,

$\sigma_{se}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not semi-regular}\}$,

$\sigma_{es}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not essentially semi-regular}\}$.

For every bounded operator A on X, the sets $\sigma_k(A)$, $\sigma_{se}(A)$ and $\sigma_{es}(A)$ are a compact subset of the complex plane, and ordered by :

$\sigma_k(A) \subseteq \sigma_{es}(A) \subseteq \sigma_{se}(A)$.

Note that the Kato spectrum is not necessarily non-empty, for example, each nilpotent operator has empty Kato spectrum, and differs from the semi-regular spectrum on at most countably many isolated points, more precisely the sets $\sigma_{se}(A) \setminus \sigma_k(A)$ and $\sigma_{es}(A) \setminus \sigma_k(A)$ are at most countable.

3. Main Results

In this section we present some results concerning the semi-regular spectrum, essential semi-regular spectrum and the Kato spectrum of an operator. We know that the product of two commuting semi-regular operators need not be semi-regular in general, see [26]. The Theorem 2.6 and Theorem 2.7 gives some cases whence the converse of Theorem 2.5 is true. In the following we continue the investigation of this question and we give others cases when the product of two commuting semi-regular operators is also semi-regular operator. We begin by the following definition.

Definition 3.1. *Let X be a Banach space and $A \in \mathcal{C}(X)$.*

1. *An operator $B \in \mathcal{C}(X)$ is called g_1-inverse of A if*

$$R(A) \subset \mathcal{D}(B),\ R(B) \subset \mathcal{D}(A) \text{ and } Au = ABAu \text{ for all } u \in \mathcal{D}(A),$$

we denote by

$$\mathcal{G}_1(A) := \Big\{B \in \mathcal{C}(X) \text{ such that } B \text{ is } g_1\text{-inverse of } A\Big\}.$$

2. *An operator $B \in \mathcal{C}(X)$ is called g_2-inverse (generalized inverse)of A if*

$$\begin{cases} R(A) \subset \mathcal{D}(B),\ R(B) \subset \mathcal{D}(A) \\ Au = ABAu, \text{ for all } u \in \mathcal{D}(A) \\ Bv = BABv, \text{ for all } v \in \mathcal{D}(B), \end{cases}$$

we denote by

$$\mathcal{G}_2(A) := \Big\{B \in \mathcal{C}(X) \text{ such that } B \text{ is } g_2\text{-inverse of } A\Big\}.$$

Remark 3.2. *(i) The relation (g_2-inverse) is symmetric.*

(ii) It is easy to see that if A is a one-sided inverse of B then B is a generalized inverse of A.

(iii) $\mathcal{G}_2(A) \subset \mathcal{G}_1(A)$.

Lemma 3.3. *[21, Lemma 1.3] Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{G}_2(A)$. Then*

(i) AB is a projection of $\mathcal{D}(B)$ onto $R(A)$ and $N(AB) = N(A)$.

(ii) BA is a projection of $\mathcal{D}(A)$ onto $R(B)$ and $N(BA) = R(A)$.

Remark 3.4. *Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{G}_2(A)$. Then*

$$\mathcal{D}(B) = N(B) \oplus R(A) \text{ and } \ \mathcal{D}(A) = N(A) \oplus R(B).$$

Corollary 3.5. *[21, Corollary 1.7] Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{G}_1(A)$. Then*

$$AB \in \mathcal{L}(X) \text{ if and only } N(B) \oplus R(A) = X.$$

An operator $A \in \mathcal{C}(X)$ is said to commute with $T \in \mathcal{L}(X)$ (T commute with A) if $TA \subset AT$. It means that whenever $x \in \mathcal{D}(A)$, Tx also belongs to $\mathcal{D}(A)$ and $TAx = ATx$.

Proposition 3.6. *Let $A \in \mathcal{C}(X)$, $B \in \mathcal{G}_1(A)$ with $AB \in \mathcal{L}(X)$ and $T \in \mathcal{L}(X)$ commuting with A and B. If $R(T)$ is closed then $R(TA)$ is closed.*

Proof. Let $(y_n) \subset R(TA)$ such that $y_n \to y$, there exists $x_n \in \mathcal{D}(A)$, with $y_n = TAx_n$. Since $A = ABA$, $TABAx_n = AB(TAx_n)$ and AB is a bounded operator we obtain $ABy = y$. Using Lemma 3.3 we infer that there exists $x \in \mathcal{D}(A)$ such that $y = Ax$. Let

$$z_n = BAx_n - BABTAx_n,$$

then $Tz_n = BABTAx_n - TBABTAx_n = BABy_n - TBABy_n$, on the other hand, AB is bounded by Lemma 3.3, then $(Tz_n)_n$ converge to $By - TBy$, since $R(T)$ is closed then there exists $z \in X$ such that $Tz = By - TBy$, which implies that $AT(z + BAx) = y$. Hence, $y \in R(TA)$. □

Theorem 3.7. *Let $A \in \mathcal{C}(X)$, $B \in \mathcal{G}_1(A)$ with $AB \in \mathcal{L}(X)$ and T is essentially semi-regular commuting with A and B. If $N(TA) \subset N(T)$ and A is surjective then TA is essentially semi-regular.*

Proof. $R(T)$ is closed , then by Proposition 3.6 $R(TA)$ is closed. T is essentially semi-regular implies that there exists a subspace F with finite dimensional such that

$$N(TA) \subset N(T) \subset \bigcap_{n\in\mathbb{N}} R(T^n) + F,$$

since A is surjective, $\bigcap_{n\in\mathbb{N}} R(T^n) \subset \bigcap_{n\in\mathbb{N}} R((TA)^n)$ and hence $N(TA) \subset \bigcap_{n\in\mathbb{N}} R((TA)^n)$. □

Corollary 3.8. *Let $A \in C(X)$, $B \in \mathcal{G}_1(A)$ with $AB \in \mathcal{L}(X)$ and T is semi-regular commuting with A and B. If $N(TA) \subset N(T)$ and A is surjective then TA is semi-regular.*

Corollary 3.9. *Let $A \in C(X)$, $B \in \mathcal{G}_1(A)$ with $AB \in \mathcal{L}(X)$ and T is semi-regular (resp. essentially semi-regular) commuting with A and B. If $0 \in \rho(A)$ then TA (resp. essentially semi-regular).*

In the following, we consider some perturbations of a semi-regular (resp. essentially semi-regular) operator T and their effect on the semi-regular (resp. essentially semi-regular) spectrum.

Proposition 3.10. *Let $A \in C(X)$ and $\lambda \in \rho(A)$. Then*

$$\mu \in \sigma_{se}(A) \text{ if and only if } \mu \neq \lambda \text{ and } (\mu-\lambda)^{-1} \in \sigma_{se}((\lambda - A)^{-1}).$$

Proof. We start from the identity

$$(\lambda - A)^{-1} - (\mu - \lambda)^{-1} = -(\mu - \lambda)^{-1}(\mu - A)(\lambda - A)^{-1}.$$

Since $(\lambda - A)^{-1}$ is a bounded invertible operator commute with A, it follows from Theorems 2.5 and 2.7 together that $(\lambda - A)^{-1} - (\mu - \lambda)^{-1}$ is semi-regular if and only if $(\mu - A)$ is semi-regular. This is equivalent to the statement of the theorem. □

Proposition 3.11. *Let $A \in C(X)$ and $\lambda \in \rho(A)$. Then*

$$\mu \in \sigma_{es}(A) \text{ if and only if } \mu \neq \lambda \text{ and } (\mu-\lambda)^{-1} \in \sigma_{es}((\lambda - A)^{-1}).$$

Recall that the nullity, $\alpha(A)$ of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$ of A is defined as the codimension of $R(A)$ in X. An operator $A \in C(X)$ is said to be upper semi-Fredholm if $\alpha(A) < \infty$ and $R(A)$ is closed. Now, we give some interesting characterization of essentially semi-regular operators by means of the upper semi-Fredholm operators.

Proposition 3.12. *Let $A \in C(X)$ is essentially semi-regular if and only if there exists a closed subspace $V \subset X$ such that $AV = V$ and the operator $\hat{A} : X/V \to X/V$ induced by A is upper semi-Fredholm.*

Proof. Let $A \in C(X)$ is essentially semi-regular and set $V = R^\infty(A)$. Then there exists $d \in \mathbb{N}$ and a pair of closed subspaces (M, N) of X such that $A = A_M \oplus A_N$, with A_M is semi-regular and A_N is nilpotent of order d with $\dim N < \infty$. We deduce that $V = R^\infty(A_M) \subset M$ and $AV = A_M V = V$. If $x = m + n$ satisfies $Ax \in V$, then $A_M m \in V$ so that $m \in V$. Thus $x \in N + V$ and $N(\hat{A}) \subset N + V$. Hence $\dim N(\hat{A}) < \infty$. Let $Q : X \to X/V$ be the canonical projection. Since $V \subset R(A)$ and

$$R(\hat{A}) = \left\{ Ax + V \text{ such that } x \in V \right\} = QR(A)$$

is closed. Thus $\hat{A}$ is upper semi-Fredholm.

Conversely, let $V \subset X$ a closed subspace such that $AV = V$ and the operator $\hat{A} : X/V \to X/V$ induced by A is upper semi-Fredholm. We first prove that $R(A)$ is closed. Let $Q : X \to X/V$ be the canonical projection. If $y \in X$ and $Qy \in R(\hat{A})$, then $y \in R(A) + V \subset R(A) + F$ since $V \subset R(A)$ Thus $R(A)$ is a subspace of finite codimension of the closed space $Q^{-1}R(\hat{A})$, so is closed. Further, $V \subset R^\infty(A)$. If $Ax = 0$, then $\hat{A}(x + V) = 0$, i.e. $Qx \in N(\hat{V})$. Thus $N(A) \subset Q^{-1}N(\hat{A}) \subset V + F \subset R^\infty(A) + F$. □

Remark 3.13. *Proposition 3.12 generalize Theorem 2.8 to the unbounded operators case.*

Theorem 3.14. *Let $A \in \mathcal{C}(X)$ is essentially semi-regular and $K \in \mathcal{K}(X)$ commute with A, then $A + K$ is essentially semi-regular.*

Proof. Let $A \in \mathcal{C}(X)$ is essentially semi-regular and let K be a compact operator commuting with A. Let $V = R^{\infty}(A)$, since $AV = V$, by Lomonosov's theorem, $KV \subset V$, hence we can define the operators

$$\hat{A} : X/V \to X/V \quad \text{and } \hat{K} : X/V \to X/V$$

induced by A and K respectively. Then both $\hat{K}$ and $\hat{A}$ have the same property and consequently, $\hat{A} + \hat{K}$ is upper semi-Fredholm. Thus, by Proposition 3.12, $A + K$ is essentially semi-regular. □

The set of upper semi-Fredholm operators is defined by

$$\Phi_{+}(X) = \{A \in \mathcal{C}(X) \text{ such that } \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } X\},$$

the set of lower semi-Ferdholm operators defined by

$$\Phi_{-}(X) = \{A \in \mathcal{C}(X) \text{ such that } \beta(A) < \infty \text{ and } R(A) \text{ is closed in } X\},$$

the set of semi-Fredholm operators defined by

$$\Phi_{\pm}(X) := \Phi_{+}(X) \cup \Phi_{-}(X),$$

and the set of Fredholm operators is defined by

$$\Phi(X) := \Phi_{+}(X) \cap \Phi_{-}(X).$$

If $A \in \Phi(X)$, the number $i(A) = \alpha(A) - \beta(A)$ is called the index of A. It is clear that if $A \in \Phi(X)$ then $i(A) < \infty$. If $A \in \Phi_{+}(X) \setminus \Phi(X)$ then $i(A) = -\infty$ and if $A \in \Phi_{-}(X) \setminus \Phi(X)$ then $i(A) = +\infty$. A complex number λ is in Φ_{+A}, Φ_{-A}, $\Phi_{\pm A}$ or Φ_{A} if $\lambda - A$ is in $\Phi_{+}(X)$, $\Phi_{-}(X)$, $\Phi_{\pm}(X)$ or $\Phi(X)$ respectively. An operator is said to be a Riesz operator if $\Phi_{A}(X) = \mathbb{C}\backslash\{0\}$.

There are several, and in general, non-equivalent definitions of the essential spectrum of a closed operator on a Banach space. For a self-adjoint operator in a Hilbert space, there seems to be only one reasonable way to define the essential spectrum: the set of all points of the spectrum that are not isolated eigenvalues of finite algebraic multiplicity.

By the help of above set classes, for $A \in \mathcal{C}(X)$, we can define the following essential spectra:

$$\begin{aligned}
\sigma_{e1}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_{+A}(X)\} := \mathbb{C} \setminus \Phi_{+A},\\
\sigma_{e2}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_{-A}(X)\} := \mathbb{C} \setminus \Phi_{-A},\\
\sigma_{e3}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_{\pm A}(X)\} := \mathbb{C} \setminus \Phi_{\pm A},\\
\sigma_{e4}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_{A}(X)\} \;\; := \mathbb{C} \setminus \Phi_{A},\\
\sigma_{e5}(A) &:= \mathbb{C} \setminus \rho_{5}(A),\\
\sigma_{e6}(A) &:= \sigma(A) \setminus \sigma_{d}(A),\\
\sigma_{eap}(A) &:= \mathbb{C} \setminus \rho_{eap}(A),\\
\sigma_{e\delta}(A) &:= \mathbb{C} \setminus \rho_{e\delta}(A),
\end{aligned}$$

where $\rho_{5}(A) := \{\lambda \in \Phi(A) \text{ such that } i(\lambda - A) = 0\}$ and $\sigma_{d}(A)$ is the set of isolated points λ of the spectrum such that the corresponding Riesz projectors P_{λ} is finite dimensional.

$$\rho_{eap}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi_{+}(X) \text{ and } i(\lambda - A) \leq 0\}$$

and

$$\rho_{e\delta}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi_{-}(X) \text{ and } i(\lambda - A) \geq 0\}.$$

We call $\sigma_{e1}(.)$, $\sigma_{e2}(.)$ the Gustafson and Weidmann essential spectra [10]. $\sigma_{e3}(.)$ is the Kato essential spectrum [19]. $\sigma_{e4}(.)$ is the Wolf essential spectrum [32]. $\sigma_{e5}(.)$ the Schechter essential spectrum [13, 30].

$\sigma_{eap}(.)$ is the essential approximate point spectrum [18]. $\sigma_{e\delta}(.)$ is the essential defect spectrum [1, 18]. $\sigma_{e6}(.)$ is the Browder spectrum [2, 27]. In the 2000s, A. Jeribi and their collaborators are continued the research on the essential spectra and they applied the results to transport operators (see [11, 12, 14–17]). Recall that this various notions of essential spectrum, generally non equivalent, appear in the applications of spectral theory (see, for example [24, 26, 32]). Evidently can by ordered as:

$$\sigma_k(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{e3}(T) \subseteq \sigma_{e4}(T) \subseteq \sigma_{e5}(T) \subseteq \sigma_{e6}(T),$$

$$\sigma_{e5}(T) = \sigma_{eap}(T) \cup \sigma_{e\delta}(T),\ \sigma_{e1}(T) \subseteq \sigma_{eap}(T) \text{ and } \sigma_{e2}(T) \subseteq \sigma_{e\delta}(T).$$

A very detailed and far-reaching account of these notations can be seen in [2, 15, 19, 26]. It is well known that $\Phi_+(A) \cup \Phi_-(A) \subset \mathcal{V}(X)$, $\mathcal{V}_0(X)$ and $\mathcal{V}(X)$ are neither semi-groups nor open or closed subset of $\mathcal{L}(X)$. From the paper of C. Shomoeger [31] we get

$$\mathrm{int}(\mathcal{V}(X)) := \Phi_+(X) \cup \Phi_-(X)$$

and

$$\mathrm{int}(\mathcal{V}_0(X)) := \Big\{A \in \Phi_\pm(X) \text{ such that } \alpha(A) = 0 \text{ or } \beta(A) = 0\Big\}.$$

One of the central questions in the study of essential spectra of closed densely defined linear operators on Banach spaces consists in showing when different notions of essential spectrum coincide and is the invariance of the different essential spectra under additive perturbation. The mathematical literature devoted to this subject is considerable. Among the works in this direction we can quote, for example, [10–12, 32] (see also the references therein).

Remark 3.15. *If λ in the continuous spectrum $\sigma_c(A)$ of a closed operator A then $R(\lambda - A)$ is not closed. Therefore $\lambda \in \sigma_i(A)$, $i \in \Lambda = \{1, 2, 3,\ 4,\ 5,\ 6,\ ap,\ \delta, se, es\}$. Consequently we have*

$$\sigma_c(A) \subset \bigcap_{i \in \Lambda} \sigma_i(A).$$

Corollary 3.16. *For a closed operator A, if $\sigma(A) = \sigma_c(A)$ then*

$$\sigma(A) = \sigma_i(A) \quad \text{for all } i \in \{1, 2, 3,\ 4,\ 5,\ 6,\ ap,\ \delta, se, es.\}.$$

In the following we give some relationships of the semi-regular spectrum, essentially semi-regular spectrum and the Kato spectrum and some essential spectra defined above.

Theorem 3.17. *Let $A,\ B \in \mathcal{C}(X)$ and let $\lambda \in \rho(A) \cap \rho(B)$. If $(\lambda - A)^{-1} - (\lambda - B)^{-1}$ is a compact operator commuting with A or B, then*

$$\sigma_{es}(A) = \sigma_{es}(B).$$

If further, $\sigma(A) = \sigma_c(A)$, then

$$\sigma_{ei}(A) \subseteq \sigma_{ei}(B),\ \ i = \ 3,\ 4,\ 5,\ 6,\ ap,\ \delta.$$

Proof. Using Theorem 3.14 we infer that

$$\sigma_{es}((\lambda - A)^{-1}) = \sigma_{es}((\lambda - B)^{-1})$$

and by Proposition 3.11 we have $\sigma_{es}(A) = \sigma_{es}(B)$. If further, $\sigma(A) = \sigma_c(A)$ then from Corollary 3.16 we deduce that $\sigma_{es}(A) = \sigma_{es}(B) \subseteq \sigma_{ei}(B)$, $i = \ 3,\ 4,\ 5,\ 6,$ eap, eδ. □

Proposition 3.18. *Let $A \in \mathcal{C}(X)$. If $0 \in \rho(A)$, then for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$ we have*

$$\lambda \in \rho_k(A) \text{ if and only if } \lambda^{-1} \in \rho_k(A^{-1}),$$

where $\rho_k(A) = \mathbb{C} \setminus \sigma_k(A)$.

Proof. Let $0 \in \rho(A)$, the resolvent identity implies that

$$\lambda - A = -\lambda(A^{-1} - \lambda^{-1})A. \tag{1}$$

If $\lambda \in \rho_k(A)$, then there exists a pair of closed and $(\lambda - A)$-invariant subspaces (M, N) of X such that $(\lambda - A)_M$ is semi-regular and $(\lambda - A)_N$ is nilpotent. Hence $((A^{-1} - \lambda^{-1})A))_M$ is semi-regular and $((A^{-1} - \lambda^{-1})A)_N$ is nilpotent. This shows that $(A^{-1} - \lambda^{-1})_M$ is semi-regular and $(A^{-1} - \lambda^{-1})_N$ is nilpotent.

Conversely, if $\lambda^{-1} \in \rho_k(A^{-1})$, then $A^{-1} - \lambda^{-1}$ is of Kato type commute with A invertible, it follows from Eq. (1) that $\lambda - A$ is of Kato type. □

Note that the semi-regular bounded operators are stable also under quasi-nilpotent perturbation and small perturbations, see also [25], in this case we have the following results by virtue of the Propositions 3.10 and 3.11, we have

Theorem 3.19. *Let $T, S \in \mathcal{L}(X)$ and let $\lambda \in \rho(T) \cap \rho(S)$. Suppose that one of the following conditions holds*

(i) $(\lambda - T)^{-1} - (\lambda - S)^{-1}$ is a quasi-nilpotent operator commuting with T or S.

(ii) If there exist $\varepsilon > 0$ such that $\left\|(\lambda - T)^{-1} - (\lambda - S)^{-1}\right\| < \varepsilon$.

Then

$$\sigma_i(T) = \sigma_i(S), i = se, es.$$

If further, $\sigma(T) = \sigma_c(T)$ then

$$\sigma_{ei}(T) \subseteq \sigma_{ei}(S), \;\; i = \; 3, \; 4, \; 5, \; 6, \; eap, e\delta.$$

where $\sigma_c(T)$ is the continuous spectrum of T.

An operator $A \in \mathcal{L}(X)$ is said to be weakly compact if $A(M)$ is relatively weakly compact in X for every bounded subset $M \subset X$.

A Banach space X is said to have the Dunford-Pettis property if for each Banach space Y every weakly compact operator $A : X \to Y$ takes weakly compact sets in X into norm compact sets of Y.

It is well known that any L^1 space has the Dunford-Pettis property [7]. Also, if Ω is a compact Hausdorff space, $C(\Omega)$ has the DP property [9]. For further examples we refer to [6] or [8, p. 494, 497, 508, 511]. Note that the Dunford-Pettis property is not preserved under conjugation. However, if X is a Banach space whose dual has the Dunford-Pettis property then X has the Dunford-Pettis property (see [9]). For more information we refer to the paper of Diestel [6] which contains a survey and exposition of the Dunford-Pettis property and related topics.

In the following results we compare between the essentially semi-regular spectrum of A and $A + B$, where A is the generator of a one-parameter semi-group and B is a small perturbation. We denote by $r(A)$ the spectral radius of a bounded operator A.

Theorem 3.20. *Let X be a Banach space have the Dunford-Pettis property. Let $A \in \mathcal{C}(X)$ and B be a positif bounded operator on X. If for some $\lambda \in \rho(A)$, $r[(\lambda - A)^{-1}B] < 1$, and the operators $(\lambda - A)^{-1}B^{\frac{1}{2}}$ and $B^{\frac{1}{2}}(\lambda - A)^{-1}$ are weakly compact on X. Then*

$$\sigma_{es}(A + B) = \sigma_{es}(A).$$

If further, $\sigma(A) = \sigma_c(A)$ then

$$\sigma_{ei}(A) \subseteq \sigma_{ei}(A + B), \; i = 3, \; 4, \; 5, \; 6, \; eap, e\delta.$$

Proof. Let $\lambda \in \rho(A)$ such that $r[(\lambda - A)^{-1}B] < 1$ then $\lambda \in \rho(A + B)$ and

$$(\lambda - A - B)^{-1} - (\lambda - A)^{-1} = (\lambda - A)^{-1}\sum_{n=1}^{+\infty}[B(\lambda - A)^{-1}]^n.$$

All terms of this series contains the term $(\lambda - A)^{-1}B(\lambda - A)^{-1}$. On the other hand

$$(\lambda - A)^{-1}B(\lambda - A)^{-1} = (\lambda - A)^{-1}B^{\frac{1}{2}}B^{\frac{1}{2}}(\lambda - A)^{-1}$$

is a composition of two weakly operators on the Banach space X which posses Dunford-Pettis property, it follows from [23, Lemma 2.1] that $(\lambda - A)^{-1}B(\lambda - A)^{-1}$ is a compact operator commuting with $(\lambda - A)^{-1}$, hence $(\lambda - A - B)^{-1} - (\lambda - A)^{-1}$ is a compact operator. Theorem 3.17 implies that $\sigma_{es}(A + B) = \sigma_{es}(A)$. □

Theorem 3.21. *Let $A,\ B \in \mathcal{L}(X)$ such that A is generator of a C_0-semigroup $(T(t))_t$ on X. Then*

$$\sigma_{es}(A + B) = \sigma_{es}(A).$$

If further, $\sigma(A) = \sigma_c(A)$ Then

$$\sigma_{es}(A) \subseteq \sigma_{ei}(A + B),\ \ i = \ 3,\ 4,\ 5,\ 6,\ eap,\ e\delta.$$

Proof. Using [28, Lemma 1.5.1, p. 151] we infer that there exists a norm $|.|$ on X such that $\|x\| \leq |x| \leq M\|x\|$ for $x \in X$, $|T(t)| \leq e^{wt}$ and $\left\|(A - \lambda)^{-1}\right\| \leq \dfrac{1}{\lambda - w}$ for $\mathrm{Re}\lambda > w$. Thus, for $\lambda > w + |B|$ the bounded operator $B(A - \lambda)^{-1}$ satisfies $|B(A - \lambda)^{-1}| < 1$ therefore $I - B(A - \lambda)^{-1}$ is invertible for $\lambda > w + |B|$. Set

$$Q = (A - \lambda)^{-1}[I - B(A - \lambda)^{-1}] = (A - \lambda)^{-1}\sum_{n=0}^{+\infty}[B(A - \lambda)^{-1}]^n$$

then

$$(\lambda I - A - B)Q = [I - B(A - \lambda)^{-1})]^{-1} - B(A - \lambda)^{-1}[I - B(A - \lambda)^{-1})]^{-1} = I$$

and

$$\begin{aligned} Q(\lambda - A - B)x &= (A - \lambda)^{-1}(\lambda I - A - B)x + \sum_{n=1}^{+\infty}(A - \lambda)^{-1}[B(A - \lambda)^{-1}]^n(\lambda - A - B)x \\ &= x - (A - \lambda)^{-1}Bx + \sum_{n=1}^{+\infty}(A - \lambda)^{-1}[B(A - \lambda)^{-1}]^n x - \sum_{n=2}^{+\infty}(A - \lambda)^{-1}[B(A - \lambda)^{-1}]^n x. \end{aligned}$$

Then

$$Q(\lambda - A - B)x = x.$$

Therefore, the resolvent of $A + B$ exists for $\lambda > w + |B|$ and it given by Q. Moreover,

$$|(\lambda - A - B)^{-1}| = |(A - \lambda)^{-1}\sum_{n=1}^{+\infty}[B(A - \lambda)^{-1}]^n| \leq \frac{1}{(\lambda - w - |B|)}.$$

Since $|(A - \lambda)^{-1} - (B - A - \lambda)^{-1}| \leq \dfrac{1}{(\lambda - w)} + \dfrac{1}{(\lambda - w - |B|)}$, then

$$\lim_{\mathrm{Re}\lambda \to \infty} |(A - \lambda)^{-1} - (B - A - \lambda)^{-1}| = 0,$$

hence from Theorem 3.19 we get $\sigma_{es}(A + B) = \sigma_{es}(A)$. □

We study a class of bounded linear operators acting on a Banach space X called semi-regular perturbation. Among other things we characterize a relation between the union of the semi-regular spectrum of two operators and semi-regular spectrum of their sum.

Definition 3.22. *An operator $A \in \mathcal{L}(X)$ is called semi-regular perturbation if $T + A$ is semi regular for every essentially semi-regular operator commuting with A. We denote by*

$$\mathcal{F}_e(X) = \{T \in \mathcal{L}(X),\ T + K \in \mathcal{V}(X) \text{ for all } K \in \mathcal{V}(X),\ TK = KT\}.$$

Examples of semi-regular perturbation operators are the compact operators, operators with finite rank, Riesz operators, quasi-nilpotent operators, nilpotent operators, and sufficiently small perturbation of all semi-regular operators.

Theorem 3.23. *Let T and S be two bounded operators on a Banach space X. If $TS \in \mathcal{F}_e(X)$ then*

$$[\sigma_{es}(T) \cup \sigma_{es}(S)] \setminus \{0\} \subset \sigma_{es}(T + S) \setminus \{0\},$$

and

$$[\sigma_{se}(T) \cup \sigma_{se}(S)] \setminus \{0\} \subset \sigma_{se}(T + S) \setminus \{0\}.$$

Proof. If $\lambda \notin \sigma_{es}(T + S) \setminus \{0\}$, then $T + S - \lambda$ is essentially semi-regular on other hand we have

$$(T - \lambda)(S - \lambda) = TS - \lambda(T + S - \lambda).$$

Since, $TS \in \mathcal{F}_e(X)$, then $(T - \lambda)(S - \lambda)$ is essentially semi-regular. It follows Theorem 2.5 that $(T - \lambda)$ and $(S - \lambda)$ are both essentially semi-regular operators then $\lambda \notin [\sigma_{se}(T) \cup \sigma_{se}(S)] \setminus \{0\}$.
For the case semi-regular operators we use the same proof. □

Remark 3.24. *The converse of the inclusions not holds in generally, but the equality hold in the following case.*

Proposition 3.25. *Let $T,\ S,\ C,\ D \in \mathcal{L}(X)$ be mutually commuting operators such that $TC + SD = I$. If $TS \in \mathcal{F}_e(X)$ then*

$$[\sigma_{es}(T) \cup \sigma_{es}(S)] \setminus \{0\} = \sigma_{es}(T + S) \setminus \{0\},$$

and

$$[\sigma_{se}(T) \cup \sigma_{se}(S)] \setminus \{0\} = \sigma_{se}(T + S) \setminus \{0\}.$$

Proof. By Theorems 2.6 and 3.25. □

Recall that An operator $T \in \mathcal{L}(X)$ is called a left (right) divisor of zero if $TS = 0$ $(ST = 0)$ for some non-zero operator $S \in \mathcal{L}(X)$.

Proposition 3.26. . *Let $T \in \mathcal{L}(X)$ is a left (right) divisor of zero, i.e $TS = 0$ $(ST = 0)$ for $S \in \mathcal{L}(X)$, then*

$$\Big[\sigma_{es}(T) \cup \sigma_{es}(S)\Big] \setminus \{0\} = \sigma_{es}(T + S) \setminus \{0\},$$

$$\Big[\sigma_{se}(T) \cup \sigma_{se}(S)\Big] \setminus \{0\} = \sigma_{se}(T + S) \setminus \{0\},$$

$$\Big[\sigma_{k}(T) \cup \sigma_{k}(S)\Big] \setminus \{0\} = \sigma_{k}(T + S) \setminus \{0\},$$

$$\Big[\sigma_{e4}(T) \cup \sigma_{e4}(S)\Big] \setminus \{0\} = \sigma_{e4}(T + S) \setminus \{0\},$$

and

$$\Big(\Big[\sigma_{se}(T) \setminus \sigma_{e4}(S)\Big] \cup \Big[\sigma_{se}(S) \setminus \sigma_{e4}(T)\Big]\Big) \setminus \{0\}$$

is at most countable.

4. Invariance of the Kato spectrum by commuting nilpotent perturbation

We start by collecting together some results, which will be used to show that the Kato spectrum of an operator is stable by a commuting nilpotent perturbation. We begin this section by the following results:

Proposition 4.1. *Let $A \in C(X)$ and Q be a nilpotent operator commuting with A. Then $A+Q$ is a nilpotent operator if and only if A is a nilpotent operator.*

Proof. Assume that A is a nilpotent operator. Let r, s be the nonnegative integers such that $A^r = 0 \neq A^{r-1}$ and $Q^s = 0 \neq Q^{s-1}$. Let $m = \max(r, s)$. Then

$$(A+Q)^{2m} = C_{2m}^{0}A^{2m} + \ldots + C_{2m}^{m}Q^mA^m + C_{2m}^{m+1}Q^{m+1}A^{m-1} + \ldots + C_{2m}^{2m}Q^{2m} = 0.$$

Hence $A+Q$ is a nilpotent operator. For the converse statement we used the relation $A = (A+Q) - Q$. $\square$

Lemma 4.2. *$A \in C(X)$ is of Kato type operator if and only if there exists a closed subspace V of X such that $AV = V$ and the operator $\hat{A} : X/V \to X/V$ induced by A is a direct sum of bounded below operator and nilpotent operator.*

Proof. Let $A \in C(X)$. If A is semi-regular we play the lemma 2.8 by taking the nilpotent operator is the zero operator. If A is a nilpotent operator we take $V = \{0\}$. Now suppose that A is not semi-regular neither nilpotent with admits a Kato decomposition (M, N), then set $V = R^{\infty}(A)$. It well know by Theorem 2.10 that V is closed , $V \subseteq M$ and $AV = V$. Furthermore

$$X/V = M/V \oplus N/V, \quad \hat{A}(M/V) \subseteq M/V \quad \text{and} \quad \hat{A}(N/V) \subseteq N/V.$$

Denote $\hat{A}_1$ (resp. $\hat{A}_2$) the restriction of $\hat{A}$ on M/V (resp. N/V). Then we have $\hat{A} = \hat{A}_1 \oplus \hat{A}_2$. Since A_N is a nilpotent operator then $\hat{A}_2$ is a nilpotent operator and by Theorem 2.8, $\hat{A}_1$ is bounded below because A_M is a semi-regular.

Conversely, let V be a closed subspace of X with $AV = V$ and $\hat{A}$ is decomposed according to $X/V = M/V \oplus N/V$ and the parts $\hat{A}_1$ and $\hat{A}_2$ are bounded below and nilpotent respectively, where M, N are two closed subspaces of X. The fact that $AV = V$, we can easily proves that (M, N) is a Kato decomposition of A and hence T is of Kato type operator. $\square$

Denote by

$\sigma_{su}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not onto}\}$

$\sigma_{ap}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not bounded below }\}.$

The defect spectrum and the approximate point spectrum of A respectively.

We show now that the operators of Kato type are stable under commuting nilpotent perturbations.

Theorem 4.3. *Let $A \in C(X)$, $AQ = QA$, where Q is a nilpotent operator on X. Then*

$$\sigma_k(A+Q) = \sigma_k(A)$$

Proof. Let A be an operator of Kato type and Q be a nilpotent operator commuting with A. If A is semi-regular we apply the [20, Theorem 6.] and if A is a nilpotent we apply the Proposition 4.1. Now suppose that A is not semi-regular neither nilpotent. Denote $V = R^{\infty}(A)$, $A_1 = A_V$ and $\hat{A} : X/V \to X/V$ induced by A. Clearly $Q(V) \subseteq V$, so that we can defined the operators $Q_1 = Q_V$ and $\hat{Q} : X/V \to X/V$ induced by Q. Obviously, Q_1 and $\hat{Q}$ are nilpotent operators. Further, $A_1Q_1 = Q_1A_1$ and $\hat{A}\hat{Q} = \hat{Q}\hat{A}$. By the stability under nilpotent perturbation of $\sigma_{ap}(A)$ and $\sigma_{ap}(A)$ we have

$$\sigma_{su}(A_1 + Q_1) = \sigma_{su}(A_1)$$

$$\sigma_{ap}(\hat{A}+\hat{Q})=\sigma_{ap}(\hat{A})$$

and

$$\sigma(\hat{A}+\hat{Q})=\sigma(\hat{A}).$$

Thus $0\notin\sigma_{su}(A_1+Q_1)$, so $(A+Q)(V)=V$. By Lemma 4.2, $\hat{A}=\hat{A}_1\oplus\hat{A}_2$, with $\hat{A}_1$ is bounded below and $\hat{A}_2$ is a nilpotent operator. Hence

$$\sigma_{ap}(\hat{A}+\hat{Q})=\sigma_{ap}(\hat{A})=\sigma_{ap}(\hat{A}_1)\cup\sigma_{ap}(\hat{A}_2)$$

and

$$\sigma(\hat{A}+\hat{Q})=\sigma(\hat{A})=\sigma(\hat{A}_1)\cup\sigma(\hat{A}_2).$$

On the other hand, $\sigma_{ap}(\hat{A}_2)=\sigma(\hat{A}_2)=\{0\}$ and $0\notin\sigma_{ap}(\hat{A}_1)\subseteq\sigma(\hat{A}_1)$, this implies that $\sigma(\hat{A})$ and hence $\sigma(\hat{A}+\hat{Q})$ is separated in two disjoints parts $\sigma(\hat{A}_1)$ and $\sigma(\hat{A}_2)$. By [19, Theorem 6.17], we have a decomposition of $\hat{A}$ (and hence of $\hat{A}+\hat{Q}$) according to the decomposition of X/V in such way that

$$\sigma((\hat{A}+\hat{Q})_{M/V})=\sigma(\hat{A}_1) \quad \text{and} \quad \sigma((\hat{A}+\hat{Q})_{N/V})=\sigma(\hat{A}_2).$$

where M,N are two closed subspaces of X. Thus $(\hat{A}+\hat{Q})_{N/V}$ is a nilpotent operator and $\sigma_{ap}((\hat{A}+\hat{Q})_{M/V})=\sigma_{ap}(\hat{A}_1)$, i.e $(\hat{A}+\hat{Q})_{M/V}$ is bounded below. This shows that $\hat{A}+\hat{Q}$ is a direct sum of bounded below operator and a nilpotent operator. Then by Lemma 4.2, $A+Q$ is of Kato type operator. □

Theorem 4.4. *Let $A,B\in\mathcal{C}(X)$. If $\lambda\in\rho(A)\cap\rho(B)$, such that $(\lambda I-A)^{-1}-(\lambda I-B)^{-1}$ is a nilpotent operator commuting with A and B, then*

$$\sigma_k(A)=\sigma_k(B).$$

If, further, $\sigma(A)=\sigma_c(A)$ then

$$\sigma_{ei}(A)\subseteq\sigma_{ei}(B),\ \ i=3,\ 4,\ 5,\ 6,\ ap,\ \delta.$$

Proof. The assumptions of Theorem 4.3 implies that
$\sigma_k((\lambda I-A)^{-1})=\sigma_k((\lambda I-B)^{-1})$ and by Proposition 3.18 we have $\sigma_k(A)=\sigma_k(B)$. □

References

[1] F. Abdmouleh and A. Jeribi, Symmetric family of Fredholm operators of indices zero, stability of essential spectra and application to transport operators, J. Math. Anal. Appl. 358 (2010) 414–424.
[2] P. Aiena, Fredholm and local spectral theory, with applications to multipliers, Kluwer Academic Publishres, 2004.
[3] C. Apostol, The reduced minimum modulus, Mich. Math. J. 32 (1985) 279–294.
[4] M. Benharrat, B. Messirdi, Relationship between the Kato essential spectrum and a variant of essential spectrum, General Mathematics 20 (4) (2012) 71–88.
[5] M. Berkani , A. Ouahab, Opérateur essentiellement régulier dans les espaces de Banach, Rend. Circ. Math. Palermo Serie II XLVI (1997) 131–160.
[6] J. Diestel, Geometry of Banach spaces-Selected topics, Lecture Notes in Mathematics, 485, Springer, New-York, 1975.
[7] N. Dunford and Pettis, Linear operations on summable functions, Tran. Amer. Math. Soc. 47 (1940) 323–392.
[8] N. Dunford and J. T. Schwartz, Linear operators, Interscience Publishers Inc., New-York, Part 1, 1958.
[9] A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$, Canad. J. Math. 5 (1953) 129–173.
[10] K. Gustafson and J. Weidmann, On the essential spectrum, J. Math. Anal. Appl. 25 (1969) 121–127.
[11] A. Jeribi, Quelques remarques sur les opérateurs de Fredholm et application á l'équation de transport, C. R. Acad. Sci. Paris Srie I 325 (1997) 43–48.
[12] A. Jeribi, Quelques remarques sur le spectre de Weyl et applications, C. R. Acad. Sci. Paris Srie I 327 (1998) 485–490.
[13] A. Jeribi, A characterization of the essential spectrum and applications, Boll. dell. Unio. Mate. Ital. 8 B-5 (2002) 805-825.
[14] A. Jeribi, A characterization of the Schechter essential spectrum on Banach spaces and applications, J. Math. Anal. Appl. 271 (2002) 343-358.
[15] A. Jeribi, Some remarks on the Schechter essential spectrum and applications to transport equations, J. Math. Anal. Appl. 275 (2002) 222–237.
[16] A. Jeribi, Fredholm operators and essential spectra, Arch. Ineq. Appl., 2 (2004) 123–140.
[17] A. Jeribi and M. Mnif, Fredholm operators, essential spectra and application to transport equation, Acta Appl. Math. 89 (2006) 155-176.

[18] A. Jeribi and N. Moalla, A characterization of some subsets of Schechter's essential spectrum and application to singular transport equation, J. Math. Anal. Appl. 358 (2009) 434-444.
[19] T. Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, J. Anal. Math. 6 (1958) 261–322, .
[20] V. Kordula and V. Müller, The distance from the Apostol spectrum, Proc. Amer. Math. Soc. 124 (1996) 3055–3061.
[21] J-P. Labrousse, Inverses généralisés d'oprateurs non borns, Proc. Amer. Math. Soc. 115 (1) (1992) 125–129.
[22] J-P. Labrousse, Les opérateurs quasi-Fredholm une généralisation des opérateurs semi-Fredholm, Rend. Circ. Math. Palermo 29 (2) (1980) 161–258.
[23] K. Latrach, Essential spectra on spaces with the Dunford-Pettis property, J. Math. Anal. Appl. 233 (1999) 607–622.
[24] M. Mbekhta, A. Ouahab, Opérateur s-régulier dans un espace de Banach et théorie spectrale, Pub. Irma. Lille. Vol. 22 No XII, (1990).
[25] M. Mbekhta, V. Müller, On the axiomatic theory of spectrum II, Studia Math. 199 (1996) 129–147.
[26] V. Müller, On the regular spectrum, J. Operator Theory 31 (1994) 363-80.
[27] V. Müller, spectral theory of linear operators and spectral systems in Banach algebra, Birkhauser, 2007.
[28] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[29] V. Rakocevič, Generalized spectrum and commuting compact perturbations, Proc. Edinb. Math. Soc. 36 (1993) 197–209.
[30] A. Schechter, Principles of functional analysis, Academic Press, New york, 1971.
[31] C. Schmoeger, Perturbation properties of some class of operators, Rend. Math. Appl., 7 (1994) 533–541.
[32] F. Wolf, On the invariance of the essential spectrum under a change of the boundary conditions of partial differential operators, Indag. Math. 21 (1959) 142–147 .