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Weighted Drazin inverse of a modified matrix

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Abstract. We present conditions under which the weighted Drazin inverse of a modified matrix $A - CWD_{d,W}WB$ can be expressed in terms of the weighted Drazin inverse of A and the generalized Schur complement $D - BWA_{d,W}WC$. The results extend the earlier works about the Drazin inverse.

1. Introduction

The Drazin inverse and the weighted Drazin inverse are very useful because of their various applications which can be found in [1,2,4,9].

Let $\mathbb{C}^{n\times n}$ denote the set of $n \times n$ complex matrices. For $A \in \mathbb{C}^{m\times m}$, the smallest nonnegative integer k such that $rank(A^{k+1}) = rank(A^k)$ is called the index of A, and is denoted by $k = ind(A)$.

Let $A \in \mathbb{C}^{m\times m}$ with $ind(A) = k$, and $X \in \mathbb{C}^{m\times m}$ be a matrix such that

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA, \quad (1)$$

then X is called the Drazin inverse of A, and is denoted $X = A^d$. In particular, when $ind(A) = 1$, the matrix X which is satisfying (1) is called the group inverse of A, and is denoted by $X = A^\#$.

Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ with $ind(AW) = k$, and $X \in \mathbb{C}^{m\times n}$ be a matrix such that

$$(AW)^{k+1}XW = (AW)^k, \quad XWAWX = X, \quad AWX = XWA, \quad (2)$$

then X is called the W-weighted Drazin inverse of A, and is denoted by $X = A_{d,W}$. In particular when A is an square matrix and $W = I$, where I is the identity matrix with proper size, (2) coincides with (1), and $A_{d,W} = A^d$.

Wei [11] studied the expressions of the Drazin inverse of a modified square matrix $A - CB$. Chen and Xu [3] discussed some representations for the weighted Drazin inverse of a modified rectangular matrix $A - CB$ under some conditions. These results can be applied to update finite Markov chains.

In [5] Cvetković-Ilić, Ljubisavljević present expressions for the Drazin inverse of generalized Schur complement $A - CD^dB$ in terms of Drazin inverse of A and the generalized Schur complement $D - BA^dC$.

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Dopazo, Martínez-Serrano [6], Mosić [8] and Shakoor, Yang, Ali [10] give representations for the Drazin inverses of a modified matrix $A - CD^dB$ under new conditions to generalize some results in the literature.

Recently Zhang and Du give representations for the Drazin inverse of the generalized Schur complement $A - CD^dB$ in terms of the Drazin inverses of A and the generalized Schur complement $D - BA^dC$ under less and weaker conditions, which generalized results of [5,6,8,10,11]. These results extends the formula of Sherman-Morrison-Woodbury type

$$(A - CD^{-1}B)^{-1} = A^{-1} + A^{-1}C(D - BA^{-1}C)^{-1}BA^{-1}$$

where the matrices A, D and the Schur complement $D - BA^{-1}C$ are invertible.

Throughout this papper, let $A, B, C, D \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$, and $(AW)^\pi = I - AWA_{d,W}W$. The generalized Schur complements will be denoted by $S = A - CWD_{d,W}WB$ and $Z = D - BWA_{d,W}WC$.

In this paper we present conditions under which the weighted Drazin inverse of a modified matrix $A - CWD_{d,W}WB$ can be expressed in terms of the weighted Drazin inverse of A and the generalized Schur complement $D - BWA_{d,W}WC$. The results extend the earlier works about the Drazin inverse.

2. Weighted Drazin inverse of a modified matrix

Some conclusions in [12] are obtained directly from the results.

Let $A, B, C, D \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. Throughout this section we use the following notations:

$$S = A - CWD_{d,W}WB, \qquad Z = D - BWA_{d,W}WC, \qquad (3)$$
$$K = A_{d,W}WC, \qquad H = BWA_{d,W} \qquad (4)$$
and
$$(AW)^\pi = I - AWA_{d,W}W, \quad (DW)^\pi = I - DWD_{d,W}W. \qquad (5)$$

Lemma 2.1. If $(AW)^\pi CWD_{d,W}WB = 0$, then

$$(SW)^d = (S_AW)^d + \sum_{i=0}^{k-1}((S_AW)^d)^{i+2}SW(AW)^i(AW)^\pi \qquad (6)$$

where $S_A = SWAWA_{d,W}$ and $k = ind(AW)$.

Proof. Since $(AW)^\pi CWD_{d,W}WB = 0$, we have $(AW)^\pi(A - S) = 0$ or alternatively $(AW)^\pi A = (AW)^\pi S$. Now we can obtain that

$$\begin{aligned}
(AW)^\pi S_A &= (AW)^\pi SWAWA_{d,W}\\
&= (AW)^\pi AWAWA_{d,W}\\
&= (I - AWA_{d,W}W)AWAWA_{d,W}\\
&= AWAWA_{d,W} - AWA_{d,W}WAWAWA_{d,W}\\
&= AWAWA_{d,W} - AWAWA_{d,W}WAWA_{d,W}\\
&= AWAWA_{d,W} - AWAWA_{d,W}\\
&= 0.
\end{aligned}$$

Since $((AW)^\pi)^2 = (AW)^\pi$ and $(AW)^\pi AW = AW(AW)^\pi$, then

$$\begin{aligned}
(SW(AW)^\pi)^i &= SW(AW)^\pi SW(AW)^\pi ... SW(AW)^\pi\\
&= SW(AW)^\pi AW(AW)^\pi ... AW(AW)^\pi\\
&= SWAW(AW)^\pi ... AW(AW)^\pi\\
&= ...\\
&= SW(AW)^{i-1}(AW)^\pi,
\end{aligned}$$

for any positive integer i.
Since

$$\begin{aligned}(AW)^k(AW)^\pi &= (AW)^k(I - AWA_{d,W}W)\\ &= (AW)^k - (AW)^{k+1}A_{d,W}W\\ &= (AW)^k - (AW)^k\\ &= 0,\end{aligned}$$

we have $SW(AW)^\pi$ is nilpotent, $k \le ind(SW(AW)^\pi) \le k+1$ and so

$$(SW(AW)^\pi)^d = 0 \text{ and } (SW(AW)^\pi)^\pi = I.$$

Let $ind(SW(AW)^\pi) = s$. By [7, Theorem 2.1] for $P = SW(AW)^\pi$ and $Q = S_AW$ we have

$$\begin{aligned}PQ &= SW(AW)^\pi S_AW = 0,\\ P+Q &= SW(AW)^\pi + S_AW = SW(I - AWA_{d,W}W) + SWAWA_{d,W}W = SW,\\ (SW)^d &= (SW(AW)^\pi + S_AW)^d\\ &= \sum_{i=0}^{s-1}((S_AW)^d)^{i+1}(SW(AW)^\pi)^i(SW(AW)^\pi)^\pi\\ &= (S_AW)^d + \sum_{i=1}^{s}((S_AW)^d)^{i+1}SW(AW)^{i-1}(AW)^\pi\\ &= (S_AW)^d + \sum_{i=0}^{s-1}((S_AW)^d)^{i+2}SW(AW)^i(AW)^\pi.\end{aligned}$$

Since $s-1 \le k \le s$ and $(AW)^i(AW)^\pi = 0$ for any $i \ge k$, we get

$$(SW)^d = (S_AW)^d + \sum_{i=0}^{k-1}((S_AW)^d)^{i+2}SW(AW)^i(AW)^\pi.$$

□

Let $(AW)^e = AWA_{d,W}W$ and $M = A_{d,W} + KWZ_{d,W}WH$. It is not difficult to prove

$$MW = (AW)^eMW = MW(AW)^e$$

and

$$S_AW = S_AW(AW)^e, \text{ where } S_A = SWAWA_{d,W}.$$

If $(AW)^\pi CWD_{d,W}WB = 0$ from Lemma 2.1 is satisfied then we have

$$\begin{aligned}&(AW)^\pi S_A = 0,\\ &(I - AWA_{d,W}W)SWAWA_{d,W} = 0,\\ &SWAWA_{d,W} = AWA_{d,W}WSWAWA_{d,W},\end{aligned}$$

or

$$S_AW = (AW)^eS_AW.$$

Now we give the following result.

Lemma 2.2. Let $S_A = SWAWA_{d,W}$ and $M = A_{d,W} + KWZ_{d,W}WH$, then the following statements are equivalent:

$$KW(DW)^\pi Z_{d,W}WHW = KWD_{d,W}W(ZW)^\pi HW; \quad (7)$$
$$AWA_{d,W}WS_AWMW = AWA_{d,W}W; \quad (8)$$
$$MWAWA_{d,W}WS_AW = AWA_{d,W}W; \quad (9)$$
$$KW(ZW)^\pi D_{d,W}WHW = KWZ_{d,W}W(DW)^\pi HW. \quad (10)$$

Furthermore, $(AWA_{d,W}WS_AW)^{\#} = MW$.

Proof. Firstly, we have

$$\begin{aligned} AWA_{d,W}WS_AW &= AWA_{d,W}WSWAWA_{d,W}W \\ &= AWA_{d,W}W(A - CWD_{d,W}WB)WAWA_{d,W}W \\ &= AWA_{d,W}WAWAWA_{d,W}W - AWA_{d,W}WCWD_{d,W}WBWAWA_{d,W}W \\ &= AWAWA_{d,W}WAWA_{d,W}W - AWKWD_{d,W}WBWA_{d,W}WAW \\ &= AWAWA_{d,W}W - AWKWD_{d,W}WBWA_{d,W}WAW \\ &= AWA_{d,W}WAW - AWKWD_{d,W}WBWA_{d,W}WAW \end{aligned}$$

and

$$\begin{aligned} AWA_{d,W}WS_AWMW &= (AWA_{d,W}WAW - AWKWD_{d,W}WBWA_{d,W}WAW)(A_{d,W}W + KWZ_{d,W}WHW) \\ &= AWA_{d,W}WAWA_{d,W}W + AWA_{d,W}WAWKWZ_{d,W}WHW \\ &\quad -AWKWD_{d,W}WBWA_{d,W}WAWA_{d,W}W \\ &\quad -AWKWD_{d,W}WBWA_{d,W}WAWKWZ_{d,W}WHW \\ &= AWA_{d,W}W + AWKWZ_{d,W}WHW - AWKWD_{d,W}WBWA_{d,W}W \\ &\quad -AWKWD_{d,W}WBWKWZ_{d,W}WHW \\ &= AWA_{d,W}W + AWKWZ_{d,W}WHW - AWKWD_{d,W}WHW \\ &\quad -AWKWD_{d,W}W(D - Z)WZ_{d,W}WHW \\ &= AWA_{d,W}W + AWKW[Z_{d,W}W - D_{d,W}W - D_{d,W}W(D - Z)WZ_{d,W}W]HW \\ &= AWA_{d,W}W + AWKW[(DW)^{\pi}Z_{d,W}W - D_{d,W}W(ZW)^{\pi}]HW. \end{aligned}$$

From this it follows (7) is equivalent to (8). Similarly, (9) is equivalent to (10). Let us prove that (8) implies (9). Let $(AW)^e = AWA_{d,W}W$. Now $(AW)^eS_AWMW = (AW)^e$ i.e., $(AW)^eS_AW(AW)^eMW = (AW)^e$, by [12, Lemma 2.3] we have

$$\begin{aligned} &(AW)^eMW(AW)^e(AW)^eS_AW(AW)^e = (AW)^e \text{ or} \\ &MW(AW)^eS_AW = (AW)^e. \end{aligned}$$

Similarly (9) implies (8). Thus, the statements (8) and (9) are equivalent.

If any of the four conditions is satisfied, then

$$\begin{aligned} &MW(AW)^eS_AW = (AW)^eS_AWMW, \\ &MW(AW)^eS_AWMW = MW(AW)^e = MW \end{aligned}$$

and

$$\begin{aligned} ((AW)^eS_AW)^2MW &= (AW)^eS_AW(AW)^eS_AWMW \\ &= (AW)^eS_AW(AW)^e \\ &= (AW)^eS_AW. \end{aligned}$$

Hence, $((AW)^eS_AW)^{\#} = MW$. □

Theorem 2.1. If $(AW)^{\pi}CWD_{d,W}WB = 0$ and $KW(DW)^{\pi}Z_{d,W}WHW = KWD_{d,W}W(ZW)^{\pi}HW$, then

$$(SW)^d = (A_{d,W} + KWZ_{d,W}WH)W + \sum_{i=0}^{k-1}((A_{d,W} + KWZ_{d,W}WH)W)^{i+2}SW(AW)^i(AW)^{\pi} \quad (11)$$

and

$$S_{d,W} = ((SW)^d)^2 S;$$

or alternatively

$$\begin{aligned}(SW)^d = \;& (A_{d,W} + A_{d,W}WCWZ_{d,W}WBWA_{d,W})W \\ & - \sum_{i=0}^{k-1}((A_{d,W} + A_{d,W}WCWZ_{d,W}WBWA_{d,W})W)^{i+1}A_{d,W}WCWZ_{d,W}WBW(AW)^i(AW)^\pi \\ & + \sum_{i=0}^{k-1}((A_{d,W} + A_{d,W}WCWZ_{d,W}WBWA_{d,W})W)^{i+1}A_{d,W}WCW(Z_{d,W}W(DW)^\pi - (ZW)^\pi D_{d,W}W)BW(AW)^i, \quad (12)\end{aligned}$$

where $k = ind(AW)$.

Proof. Since $(AW)^\pi CWD_{d,W}WB = 0$, then $S_AW = (AW)^e S_AW$. Using Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}S_{A;d,W} &= M = A_{d,W} + KWZ_{d,W}WH, \\ (SW)^d &= MW + \sum_{i=0}^{k-1}(MW)^{i+2}SW(AW)^i(AW)^\pi.\end{aligned}$$

Substituting M we get (11).

Since

$$\begin{aligned}(A_{d,W} + KWZ_{d,W}WH)WSW(AW)^\pi &= (A_{d,W}W + KWZ_{d,W}WHW)(AW - CWD_{d,W}WBW)(AW)^\pi \\ &= ((AW)^e - KWD_{d,W}WBW + KWZ_{d,W}WBW(AW)^e \\ &\quad -KWZ_{d,W}W(D - Z)WD_{d,W}WBW)(AW)^\pi \\ &= -KWD_{d,W}WBW(AW)^\pi - KWZ_{d,W}W(D - Z)WD_{d,W}WBW(AW)^\pi \\ &= KW(Z_{d,W}W(DW)^\pi - (ZW)^\pi D_{d,W}W)BW(AW)^\pi - KWZ_{d,W}WBW(AW)^\pi \\ &= KW(Z_{d,W}W(DW)^\pi - (ZW)^\pi D_{d,W}W)BW - KWZ_{d,W}WBW(AW)^\pi,\end{aligned}$$

we have (12).

□

By Theorem 2.1, when A, B, C and D are square and $W = I$, we can get directly some results in [12].

Corollary 2.1. Let $A, B, C, D \in \mathbb{C}^{m\times m}$ and $W = I$ in (3),(4),(5). Suppose $A^\pi CD^dB = 0$ and $KD^\pi Z^dH = KD^dZ^\pi H$ then

$$S^d = A^d + KZ^dH + \sum_{i=0}^{k-1}(A^d + KZ^dH)^{i+2}SA^iA^\pi.$$

Corollary 2.2. If $(AW)^\pi CWD_{d,W}WB = 0$, $CW(DW)^\pi Z_{d,W}WB = 0$ and $CWD_{d,W}W(ZW)^\pi B = 0$, then

$$\begin{aligned}&(SW)^d = (A_{d,W} + A_{d,W}WCWZ_{d,W}WBWA_{d,W})W \\ &- \sum_{i=0}^{k-1}((A_{d,W} + A_{d,W}WCWZ_{d,W}WBWA_{d,W})W)^{i+1}A_{d,W}WCWZ_{d,W}WBW(AW)^i(AW)^\pi \\ &+ \sum_{i=0}^{k-1}((A_{d,W} + A_{d,W}WCWZ_{d,W}WBWA_{d,W})W)^{i+1}A_{d,W}WCW(Z_{d,W}W(DW)^\pi - (ZW)^\pi D_{d,W}W)BW(AW)^i,\end{aligned}$$

where $k = ind(AW)$.

Corollary 2.3. If $(AW)^{\pi}CWD_{d,W}WB = 0$ and $(DW)^{\pi} = (ZW)^{\pi}$, then

$$(SW)^d = (A_{d,W} + A_{d,W}WCWZ_{d,W}WBWA_{d,W})W - \sum_{i=0}^{k-1}((A_{d,W} + A_{d,W}WCWZ_{d,W}WBWA_{d,W})W)^{i+1}A_{d,W}WCWZ_{d,W}WBW(AW)^i(AW)^{\pi},$$

where $k = ind(AW)$.

The following theorem can be proved similary to Theorem 2.1.

Theorem 2.2. If $CWD_{d,W}WBW(AW)^{\pi} = 0$ and $KW(ZW)^{\pi}D_{d,W}WHW = KWZ_{d,W}W(DW)^{\pi}HW$ then

$$(SW)^d = (A_{d,W} + KWZ_{d,W}WH)W + \sum_{i=0}^{k-1}(AW)^i(AW)^{\pi}SW((A_{d,W} + KWZ_{d,W}WH)W)^{i+2}.$$

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