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On class p-$wA(s,t)$ operators

T. Prasad[a], K. Tanahashi[b]

[a]The Institute of Mathematical Sciences, CIT Campus, Chennai-600 113, India
[b]Department of Mathematics, Tohoku Pharmaceutical University, Sendai 981-8558, Japan

Abstract. Let T be a bounded linear operator on a complex Hilbert space H and let $T = U|T|$ be the polar decomposition of T. T is called class p-$wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$ and $(|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}$ where $0 < s,t$ and $0 < p \leq 1$. Also, T is called class p-$A(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$. We study some properties of class p-$wA(s,t)$ operators. Also, we prove tensor product $T \otimes S$ is class p-$wA(s,t)$ if and only if T and S are class p-$wA(s,t)$.

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on a complex Hilbert space H. Aluthge [1] found p-hyponormal operator T which is defined as $(T^*T)^p \geq (TT^*)^p$ where $0 < p \leq 1$. If $p = 1$, T is called hyponormal. Hence this is a generalization of hyponormal operator. This class of operator have many interesting properties, for example, Putnam's inequality, Fuglede-Putnam type theorem, Bishop's property (β), Weyl's theorem, polaroid. After this discovery, many authors are investigating new generalizations of hyponormal operator. We summarize several classes of operators. Let $T = U|T|$ be the polar decomposition of T. Then the Aluthge transformation

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$$

was introduced by Aluthge[1]. An operator T is called w-hyponormal if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$. The class of w-hyponormal operators was introduced and studied by Aluthge and Wang [2, 3]. It is well known that the class of w-hyponormal operators contains p-hyponormal operators. An operator T is called class A if $|T^2| \geq |T|^2$. Class A operators has been introduced and studied by Furuta *et.,al* [9]. An operator T is called p-w-hyponormal if $|\tilde{T}|^p \geq |T|^p \geq |\tilde{T}^*|^p$ ([13]). If $p = 1$, then p-w-hyponormal operator is w-hyponormal. If T is w-hyponormal, then $\tilde{T}$ is $\frac{1}{2}$-hyponormal. If T is p-w-hyponormal, then $\tilde{T}$ is $\frac{p}{2}$-hyponormal ([13]). An operator T is called class $A(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ ([6]) where $0 < s,t$. An operator T is called class $wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$ ([7]).

In this note we introduce and study some new classes of operators, i.e.,class p-$wA(s,t)$, class p-$A(s,t)$ and class p-A.

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Email addresses: prasadvalapil@gmail.com (T. Prasad), tanahasi@tohoku-pharm.ac.jp (K. Tanahashi)

2. Class p-$wA(s,t)$ operators

The following definition generalizes p-w-hyponormal operators.

Definition 2.1. *Let $T = U|T|$ be the polar decomposition of T and let $s, t > 0$ and $0 < p \leq 1$. T is called class p-$wA(s,t)$ if*

(1) $\quad (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$

and

(2) $\quad (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}$.

We remark that if $p = 1$, T is $wA(s,t)$. Now we define class p-A and class p-$A(s,t)$ as generalizations of class A and class $A(s,t)$.

Definition 2.2. *Let $T = U|T|$ be the polar decomposition of T and $0 < p \leq 1, 0 < s, t$.*
(i) *T is called class p-A if $|T^2|^p \geq |T|^{2p}$.*
(ii) *T is called class p-$A(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$.*

If $p = s = t = 1$, then class p-$A(s,t)$ coincides with class A operators.
The following result seems new, even for class $A(s,t)$ operators.

Theorem 2.3. *If T is p-$wA(s,t)$ and T is invertible, then T^{-1} is p-$wA(t,s)$.*

Proof. Let $T = U|T|$ the polar decomposition of T. Then

$$|T^{-1}|^2 = (T^{-1})^*T^{-1} = (T^*)^{-1}T^{-1} = (TT^*)^{-1} = |T^*|^{-2}.$$

Hence

$$|T^{-1}| = |T^*|^{-1}.$$

Also,

$$|(T^{-1})^*|^2 = (T^{-1})(T^{-1})^* = T^{-1}(T^*)^{-1} = (T^*T)^{-1} = |T|^{-2}.$$

Hence

$$|(T^{-1})^*| = |T|^{-1}.$$

Then

$$\{|(T^{-1})^*|^s|T^{-1}|^{2t}|(T^{-1})^*|^s\}^{\frac{sp}{s+t}} = (|T|^{-s}|T^*|^{-2t}|T|^{-s})^{\frac{sp}{s+t}} = (|T|^s|T^*|^{2t}|T|^s)^{\frac{-ps}{s+t}} \geq |T|^{-2sp} = |(T^{-1})^*|^{2sp}$$

and

$$\{|T^{-1}|^t|(T^{-1})^*|^{2s}|T^{-1}|^t|\}^{\frac{tp}{s+t}} = \{|T^*|^{-t}|T|^{-2s}|T^*|^{-t}\}^{\frac{tp}{s+t}} = (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{-tp}{s+t}} \leq |T^*|^{-2tp} = |T^{-1}|^{2tp}.$$

□

Corollary 2.4. *If T is $A(s,t)$ and T is invertible, then T^{-1} is $A(t,s)$.*

M. Ito and T. Yamazaki [8] proved if $p = 1$, then (1) implies (2). They define that T is class $A(s,t)$ if T satisfies (1) with $p = 1$. Thus they proved that $A(s,t)$ implies $wA(s,t)$. It is clear that $wA(s,t)$ implies $A(s,t)$. Hence $wA(s,t)$ and $A(s,t)$ are equivalent. However it is not known whether class p-$A(s,t)$ implies class p-$wA(s,t)$ for $0 < p < 1$ or not.

Definition 2.5. *Let $T = U|T|$ be the polar decomposition of T and let $s, t > 0$. Then generalized Aluthge transformation is defined as follows*

$$\tilde{T_{s,t}} = |T|^sU|T|^t.$$

Also, we define

$$\tilde{T^*_{s,t}} = \left(\tilde{T_{s,t}}\right)^* = |T|^tU^*|T|^s.$$

Theorem 2.6. *Let $T = U|T|$ be the polar decomposition of T. Then T is class p-wA(s,t) if and only if $|\tilde{T_{s,t}}|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$ and $|T|^{2sp} \geq |\tilde{T}^*_{s,t}|^{\frac{2sp}{s+t}}$.*

Proof.

$$\begin{aligned}
&(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp} \\
&\Longleftrightarrow (U|T|^tU^*|T|^{2s}U|T|^tU^*)^{\frac{tp}{s+t}} \geq U|T|^{2tp}U^* \\
&\Longleftrightarrow U(|T|^tU^*|T|^{2s}U|T|^t)^{\frac{tp}{s+t}}U^* \geq U|T|^{2tp}U^* (\text{ [7, Lemma 2.1]}) \\
&\Longleftrightarrow (|T|^tU^*|T|^{2s}U|T|^t)^{\frac{tp}{s+t}} \geq |T|^{2tp} (\text{ [7, lemma 2.1]}) \\
&\Longleftrightarrow |\tilde{T_{s,t}}|^{\frac{2tp}{s+t}} \geq |T|^{2tp}.
\end{aligned}$$

Also,

$$(|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp} \Longleftrightarrow (|T|^sU|T|^{2t}U^*|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp} \Longleftrightarrow |\tilde{T}^*_{s,t}|^{\frac{2sp}{s+t}} \leq |T|^{2sp}.$$

□

Theorem 2.6 yields the following result.

Corollary 2.7. *If T is class p-wA(s,t), then $\tilde{T_{s,t}}$ is $\frac{min\{sp,tp\}}{s+t}$-hyponormal.*

3. Tensor product of class *p-wA*(s,t) operators

Let $s,t > 0$ and $0 < p \leq 1$. In [4], Duggal proved that tensor product $T \otimes S$ is p-hyponormal if and only if T and S are p-hyponormal. The passage of calss A operators T and S to their tensor product $T \otimes S$ is studied in [5] by Jeon and Duggal. In [12], K. Tanahashi and M. Chō proved that the tensor product $T \otimes S$ is of class $A(s,t)$ if and only if T and S are class $A(s,t)$ operators. Now we will prove similar result for class p-$A(s,t)$ operators by adopting the ideas in [12],[5].

Theorem 3.1. *Let $T \in B(H)$ and $S \in B(K)$ be non zero operators. Then $T \otimes S$ is class p-wA(s,t) if and only if S, T are class p-wA(s,t)*

Lemma 3.2. *[11] Let $T_1, T_2 \in B(H), S_1, S_2 \in B(K)$ be non negative operators. If $T_1 \neq 0$ and $S_1 \neq 0$, then the following conditions are equivalent*
(1) $T_1 \otimes S_1 \leq T_2 \otimes S_2$.
(2) There exists $c > 0$ such that $T_1 \leq cT_1$ and $S_1 \leq c^{-1}S_2$.

Lemma 3.3. *[12] Let $T = U_T|T|$ and $S = U_S|S|$ be the polar decompositions of $T \in S(H)$ and $S \in S(K)$, respectively. Then the following assertions hods.*
(1) $|T \otimes S| = |T| \otimes |S|$.
(2) $T \otimes S = (U_T \otimes U_S)(|T| \otimes |S|)$
(3) $(\widetilde{T \otimes S})_{s,t} = \tilde{T_{s,t}} \otimes \tilde{S_{s,t}}$ for $s,t > 0$.

Proof. (Theorem 3.1.) Let $T \in S(H)$ and $S \in S(K)$ be non zero class p-$wA(s,t)$ operators. Then $|\tilde{T_{s,t}}|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$ and $|\tilde{S_{s,t}}|^{\frac{2tp}{s+t}} \geq |S|^{2tp}$ by Theorem 2.6. By applying Lemma 3.3, we obtain

$$\begin{aligned}
|(\widetilde{T \otimes S})_{s,t}|^{\frac{2tp}{s+t}} &= |\tilde{T_{s,t}} \otimes \tilde{S_{s,t}}|^{\frac{2tp}{s+t}} = |\tilde{T_{s,t}}|^{\frac{2tp}{s+t}} \otimes |\tilde{S_{s,t}}|^{\frac{2tp}{s+t}} \\
&\geq |T|^{2tp} \otimes |S|^{2tp} = |T \otimes S|^{2tp}.
\end{aligned}$$

Similarly, we have

$$|T \otimes S|^{2sp} \geq |(\widetilde{T \otimes S})^*_{s,t}|^{\frac{2sp}{s+t}}.$$

Hence $T \otimes S$ is class p-$wA(s,t)$.

Conversely, suppose that $T \otimes S$ is class p-$wA(s,t)$. Then

$$\begin{aligned}|(\widetilde{T \otimes S})_{s,t}|^{\frac{2tp}{s+t}} &= |\tilde{T}_{s,t}|^{\frac{2tp}{s+t}} \otimes |\tilde{S}_{s,t}|^{\frac{2tp}{s+t}} \\ &\geq |T|^{2tp} \otimes |S|^{2tp} = |T \otimes S|^{2tp}\end{aligned}$$

and

$$|T \otimes S|^{2sp} \geq |(\widetilde{T \otimes S})^*_{s,t}|^{\frac{2sp}{s+t}}.$$

Hence there exists $c > 0$ such that

$$c|\tilde{T}_{s,t}|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$$

and

$$c^{-1}|\tilde{S}_{s,t}|^{\frac{2tp}{s+t}} \geq |S|^{2tp}$$

by Lemma 3.2. Let x be a unit vector. Then

$$\begin{aligned}\||T|^{tp}x\|^2 &= \langle |T|^{2tp}x, x\rangle \leq \langle c|\tilde{T}_{s,t}|^{\frac{2tp}{s+t}}x, x\rangle \\ &\leq c\||\tilde{T}_{s,t}|^{\frac{tp}{s+t}}\|^2 = c\|\tilde{T}_{s,t}\|^{\frac{2tp}{s+t}} \\ &= c\||T|^s U_T |T|^t\|^{\frac{2tp}{s+t}} \\ &\leq c\left(\||T|^s\| \cdot 1 \cdot \||T|^t\|\right)^{\frac{2tp}{s+t}} = c\||T|^{tp}\|^2.\end{aligned}$$

Hence $1 \leq c$. Similarly,

$$\begin{aligned}\||S|^{tp}x\|^2 &= \langle |S|^{2tp}x, x\rangle \leq \langle c^{-1}|\tilde{S}_{s,t}|^{\frac{2tp}{s+t}}x, x\rangle \\ &\leq c^{-1}\||\tilde{S}_{s,t}|^{\frac{tp}{s+t}}\|^2 = c^{-1}\|\tilde{S}_{s,t}\|^{\frac{2tp}{s+t}} \\ &= c^{-1}\||S|^s U_S |S|^t\|^{\frac{2tp}{s+t}} \\ &\leq c^{-1}\left(\||S|^s\| \cdot 1 \cdot \||S|^t\|\right)^{\frac{2tp}{s+t}} = c^{-1}\||S|^{tp}\|^2.\end{aligned}$$

Hence $1 \leq c^{-1}$. Hence $c = 1$ and $|\tilde{T}_{s,t}|^{\frac{2tp}{s+t}} \geq |T|^{2tp}, |\tilde{S}_{s,t}|^{\frac{2tp}{s+t}} \geq |S|^{2tp}$. Similarly we have $|T|^{2sp} \geq |\tilde{T}^*_{s,t}|^{\frac{2sp}{s+t}}$ and $|S|^{2sp} \geq |\tilde{S}^*_{s,t}|^{\frac{2sp}{s+t}}$. Thus T and S are class p-$wA(s,t)$. □

Corollary 3.4. *Let $T \in B(H)$ and $S \in B(K)$ be non zero operators. Then $T \otimes S$ is class p-$A(s,t)$ if and only if S, T are class p-$A(s,t)$.*

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