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Algebraic elementary operators

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Abstract. A Banach space operator A is algebraic if there exists a non-trivial polynomial $p(.)$ such that $p(A) = 0$. Equivalently, A is algebraic if $\sigma(A)$ is a finite set consisting of poles. The sum of two commuting Banach space algebraic operators is algebraic, and the generalized derivation $\delta_{AB} = L_A - R_B$ (and, for non-nilpotent A and B, the left right multiplication operator $L_A R_B$) is algebraic if and only if A and B are algebraic. We prove: If $\mathrm{asc}(d_{AB} - \lambda) \leq 1$ for all complex λ, and if A^*, B have SVEP, then $d_{AB} - \lambda$ has closed range for every complex λ if and only if A, B are algebraic; if A, B are simply polaroid, then $d_{AB} - \lambda$ has closed range for every $\lambda \in \mathrm{iso}\,\sigma(d_{AB})$; and if A, B are normaloid, then $L_A R_B - \lambda$ has closed range at every λ in the peripheral spectrum of $L_A R_B$ if and only if $L_A R_B$ is left polar at λ.

1. Introduction

For a Banach space $\mathcal{X}$, let $B(\mathcal{X})$ denote the algebra of operators, equivalently bounded linear transformations, on $\mathcal{X}$ into itself. Given an operator $T \in B(\mathcal{X})$, the kernel $T^{-1}(0)$ of T is orthogonal to the range $T(\mathcal{X})$ of T, $T^{-1}(0) \perp T(\mathcal{X})$, in the sense of G. Birkhoff if $\|x\| \leq \|x + y\|$ for all $x \in T^{-1}(0)$ and $y \in T(\mathcal{X})$ [6, Page 25]. Clearly, $T^{-1}(0) \perp T(\mathcal{X}) \Longrightarrow T^{-1}(0) \cap \overline{T(\mathcal{X})} = \{0\} \Longrightarrow T^{-1}(0) \cap T(\mathcal{X}) = \{0\}$. (Here, as also in the sequel, $\overline{T(\mathcal{X})}$ denotes the closure of $T(\mathcal{X})$.) The range-kernel orthogonality of an operator is related to its ascent. The ascent of $T \in B(\mathcal{X})$, $\mathrm{asc}(T)$, is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$; if no such integer n exists, then $asc(T) = \infty$. It is well known [1, 6] that $\mathrm{asc}(T) \leq m < \infty$ if and only if $T^{-n}(0) \cap T^m(X) = \{0\}$ for all integers $n \geq m$, and that $T^{-1}(0) \perp T(\mathcal{X})$ implies $\mathrm{asc}(T) \leq 1$.

The range-kernel orthogonality $T^{-1}(0) \perp T(\mathcal{X})$ of Banach space operators has been studied by a number of authors over the past few decades. A classical result of Sinclair [19, Proposition 1] says that "if 0 is in the boundary of the numerical range of a $T \in B(\mathcal{X})$, then $T^{-1}(0) \perp T(\mathcal{X})$". Anderson [2], and Anderson and Foiaş [3], considered the generalized derivation $\delta_{AB} = L_A - R_B \in B(B(\mathcal{H}))$, $\delta_{AB}(X) = AX - XB$, to prove that if $A, B \in B(\mathcal{H})$ are normal (Hilbert space) operators, then $\delta_{AB}^{-1}(0) \perp \delta_{AB}(B(\mathcal{H}))$. These results have since been extended to a variety of elementary operators $\Phi_{\mathbf{AB}}(X) = A_1 X B_1 - A_2 X B_2$ for a variety of choices of tuples

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of operators $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$ (see [9, 11, 14, 15, 20] for further references). The range-kernel orthogonality of an operator $T \in B(\mathcal{X})$ does not imply that the range $T(\mathcal{X})$ is closed or that $\mathcal{X} = T^{-1}(0) \oplus \overline{T(\mathcal{X})}$; see [3, Example 3.1 and Theorem 4.1] and [19, Remark 2]. Indeed, range-kernel orthogonality neither implies nor is implied by range closure. Thus, every bounded below operator has closed range and satisfies range-kernel orthogonality, an injective compact quasi-nilpotent operator (for example, the Volterra integral operator on $L^2(0,1)$) satisfies range-kernel orthogonality but does not have closed range, and no operator T (whether it has closed range or not) with $2 \leq \operatorname{asc}(T) < \infty$ satisfies range-kernel orthogonality. The implication $T^{-1}(0) \perp T(\mathcal{X}) \Longrightarrow \operatorname{asc}(T) \leq 1$ is strictly one way; if $A_i, B_i \in B(\mathcal{H})$, $1 \leq i \leq 2$, are normal Hilbert space operators such that A_1 commutes with A_2 and B_1 commutes with B_2, then $\operatorname{asc}(\Phi_{\mathbf{AB}}) \leq 1$ [12, Theorem 3.4] but $\Phi^{-1}_{\mathbf{AB}}(0) \perp \Phi_{\mathbf{AB}}(B(\mathcal{H}))$ if and only if $(A_1 \oplus B_1^*)^{-1}(0) \cap (A_2 \oplus B_2^*)^{-1}(0) = \{0\}$ [20, Corollary 2.3].

Letting iso $\sigma(A)$ (resp., iso $\sigma_a(A)$) denote the set of isolated points of the spectrum $\sigma(A)$ (resp., approximate point spectrum $\sigma_a(A)$) of $A \in B(\mathcal{X})$, we say that *A is polar at $\lambda \in$ iso $\sigma(A)$* (resp., *left polar at $\lambda \in$ iso $\sigma_a(A)$*) if λ is a pole of the resolvent of A (resp., there exists an integer $d \geq 1$ such that $\operatorname{asc}(A - \lambda) \leq d$ and $(A-\lambda)^{d+1}(\mathcal{X})$ is closed); A is *polaroid* (resp., *left polaroid*) if A is polar at every $\lambda \in$ iso $\sigma(A)$ (resp., left polar at every $\lambda \in$ iso $\sigma_a(A)$). A well known result of Anderson and Foiaş [3, Theorem 4.2] says that if $A, B \in B(\mathcal{H})$ are scalar Hilbert space operators, then $\delta_{AB} - \lambda$ has closed range for every complex λ if and only if $\sigma(A) \cup \sigma(B)$ is finite. Scalar Hilbert space operators are similar to normal operators, and normal operators are *simply polar* (i.e., they have ascent less than or equal to 1). Hence, [1, Theorem 3.83], if $A, B \in B(\mathcal{H})$ are scalar operators, then $\delta_{AB} - \lambda$ has closed range for every complex λ if and only if A, B are algebraic operators.

This paper considers algebraic elementary operators. We start by observing that an $A \in B(\mathcal{X})$ is algebraic if and only if L_A and R_A are algebraic. The algebraic property transfers from commuting $A, B \in B(\mathcal{X})$ to $A + B$, δ_{AB} is algebraic if and only if A and B are algebraic, and if A, B are non-nilpotent then $L_A R_B$ is algebraic if and only if A, B are algebraic. Let d_{AB} denote either of δ_{AB} and $L_A R_B$, where $A, B \in B(\mathcal{X})$ are non-trivial. In considering applications, we prove that: (i) If $\operatorname{asc}(d_{AB} - \lambda) \leq 1$ for all complex λ, and if A^*, B have SVEP, then $d_{AB} - \lambda$ has closed range for every complex λ if and only if A, B are algebraic; (ii) if A, B are simply polaroid, then $d_{AB} - \lambda$ has closed range for every $\lambda \in$ iso $\sigma(d_{AB})$; and (iii) if A, B are normaloid operators, then $L_A R_B - \lambda$ has closed range at every λ in the peripheral spectrum of $L_A R_B$ if and only if $L_A R_B$ is left polar at λ.

2. Results — Part A: Algebraic

Let $\mathbb{C}$ denote the set of complex numbers. An operator $A \in B(\mathcal{X})$, has *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(A - \lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. *A has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.* The single valued extension property plays an important role in local spectral theory and Fredholm theory [1, 17]. Evidently, A has SVEP at points in the resolvent set and the boundary $\partial\sigma(A)$ of $\sigma(A)$

Let $A \in B(\mathcal{X})$. The *quasinilpotent part* $H_0(A - \lambda)$ and the *analytic core* $K(A - \lambda)$ of $(A - \lambda)$ are defined by

$$H_0(A - \lambda) = \{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} \|(A - \lambda)^n x\|^{\frac{1}{n}} = 0\}$$

and

$$\begin{aligned} K(A - \lambda) = \{x \in \mathcal{X} : &\text{ there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for} \\ &\text{which } x = x_0, (A - \lambda)(x_{n+1}) = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \ldots\}. \end{aligned}$$

$H_0(A - \lambda)$ and $K(A - \lambda)$ are (generally) non-closed hyperinvariant subspaces of $(A - \lambda)$ such that $(A - \lambda)^{-q}(0) \subseteq H_0(A - \lambda)$ for all $q = 0, 1, 2, \ldots$ and $(A - \lambda)K(A - \lambda) = K(A - \lambda)$; also, if $\lambda \in$ iso $\sigma(A)$, then $H_0(A - \lambda)$ and $K(A - \lambda)$ are closed and $\mathcal{X} = H_0(A - \lambda) \oplus K(A - \lambda)$ [1].

$A \in B(\mathcal{X})$ is an *algebraic operator* if there exists a non-trivial polynomial $p(.)$ such that $p(A) = 0$. It is easily seen, [1, Theorem 3.83], that an operator $A \in B(\mathcal{X})$ is algebraic if and only if $\sigma(A)$ is a finite set consisting of the poles of the resolvent of A (i.e., if and only if $\sigma(A)$ is a finite set and A is polaroid). Since $\sigma(A) = \sigma(L_A) = \sigma(R_A)$, and since A is polaroid if and only if L_A (R_A) is polaroid [4, Theorem 11], we have:

Proposition 2.1. *Let $A \in B(\mathcal{X})$, and let $\mathcal{E}_A = L_A$ or R_A. Then $\mathcal{E}_A$ is algebraic if and only if A is algebraic.*

The *algebraic property* transfers from commuting $A, B \in B(\mathcal{X})$ to $A + B$.

Proposition 2.2. *If $A, B \in B(\mathcal{X})$ are algebraic operators such that $[A, B] = AB - BA = 0$, then $A + B$ is algebraic.*

A proof of the proposition (in a certain sense, a more direct proof) may be obtained as a consequence of the easily proved fact that if A and B are commuting algebraic elements of an algebra, then each polynomial $p(A, B)$ is also algebraic: In keeping with the spirit of this paper, in the following we draw upon *local spectral theory* to prove the proposition.

Proof. If $A \in B(\mathcal{X})$ is algebraic, then there is an integer $n \geq 1$ such that $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ (for some scalars λ_i, $1 \leq i \leq n$), $\mathcal{X} = \bigoplus_{i=1}^n H_0(A - \lambda_i)$, and to each i there corresponds an integer $p_i \geq 1$ such that $H_0(A - \lambda_i) = (A - \lambda_i)^{-p_i}(0)$. Let $A_i = A|_{H_0(A-\lambda_i)}$; then $A = \bigoplus_{i=1}^n A_i$, $A_i - \lambda_j$ is nilpotent for all $1 \leq i = j \leq n$, and $A_i - \lambda_j$ is invertible for all $1 \leq i \neq j \leq n$. Furthermore, if we let $B_i = B|_{H_0(A-\lambda_i)}$ for all $1 \leq i \leq n$, then $B = \bigoplus_{i=1}^n B_i$ and (since $[A, B] = 0$) $[A_i, B_i] = 0$ for all $1 \leq i \leq n$. Trivially, B algebraic implies $\sigma(B_i)$ is a finite set for all i. Consider $A_i + B_i - \lambda = (A_i - \lambda_i) + (B_i - \lambda + \lambda_i)$, where $\lambda \in \sigma(B_i)$ $(= iso\sigma(B_i))$. If $\lambda - \lambda_i \notin \sigma(A_i - \lambda_i + B_i) = \sigma(B_i)$, then $A_i + B_i - \lambda$ is invertible, and hence

$$H_0(A_i + B_i - \lambda) = \{0\} = (A_i + B_i - \lambda)^{-r_i}(0)$$

for every positive integer r_i. If, on the other hand, $\lambda - \lambda_i \in \sigma(A_i - \lambda_i + B_i) = \sigma(B_i)$, then $H_0(B_i + \lambda_i - \lambda) = (B_i + \lambda_i - \lambda)^{-r_i}(0)$ for some integer $r_i \geq 1$. Observe that

$$\begin{aligned} \|(B_i + \lambda_i - \lambda)^t x\|^{\frac{1}{t}} &= \|\{(A_i + B_i - \lambda) - (A_i - \lambda_i)\}^t\|^{\frac{1}{t}} \\ &= \|\sum_{j=0}^{t} (-1)^j \begin{pmatrix} t \\ j \end{pmatrix} (A_i + B_i - \lambda)^{t-j}(A_i - \lambda_i)^j x\|^{\frac{1}{t}} \\ &\leq \|\sum_{j=0}^{t} \{\begin{pmatrix} t \\ j \end{pmatrix} \|(A_i - \lambda_i)\|^j\}^{\frac{1}{t}} \|(A_i + B_i - \lambda)^{t-j} x\|^{\frac{1}{t}} \end{aligned}$$

for all $x \in \mathcal{X}$ implies

$$H_0(B_i + \lambda_i - \lambda) \subseteq H_0(A_i + B_i - \lambda).$$

By symmetry

$$H_0(A_i + B_i - \lambda) \subseteq H_0(A_i + B_i - \lambda - A_i + \lambda_i) \subseteq H_0(B_i + \lambda_i - \lambda),$$

and hence

$$H_0(A_i + B_i - \lambda) = H_0(B_i + \lambda_i - \lambda) = (B_i + \lambda_i - \lambda)^{-r_i}(0).$$

Now let $r_i p_i = m_i$. Then, for all $x \in (B_i + \lambda_i - \lambda)^{-m_i}(0)$,

$$(A_i + B_i - \lambda)^{m_i} x = \sum_{j=p_i+1}^{m_i} \{\begin{pmatrix} m_i \\ j \end{pmatrix} (B_i + \lambda_i - \lambda)^{m_i - j}(A_i - \lambda_i)^{j-p_i}\}(A_i - \lambda_i)^{p_i} x = 0$$

implies

$$H_0(A_i + B_i - \lambda) = (B_i + \lambda_i - \lambda)^{-m_i}(0) \subseteq (A_i + B_i - \lambda)^{-m_i}(0) \subseteq H_0(A_i + B_i - \lambda).$$

Thus, there exists an integer $m_i \geq 1$ such that

$$H_0(A_i + B_i - \lambda) = (A_i + B_i - \lambda)^{-m_i}(0)$$

for every $\lambda \in \text{iso}\,\sigma(B_i)$. Let $m = \max_{1\leq i\leq n} m_i$, and let $\lambda \in \sigma(A+B) = \text{iso}\,\sigma(A+B)$. Then

$$H_0(A+B-\lambda) = \bigoplus_{i=1}^{n} H_0(A_i+B_i-\lambda) = \bigoplus_{i=1}^{n} (A_i+B_i-\lambda)^{-m_i}(0) = (A+B-\lambda)^{-m}(0)$$

at every $\lambda \in \sigma(A+B)$. Since

$$\begin{aligned} & \mathcal{X} = H_0(A+B-\lambda) \oplus K(A+B-\lambda) = (A+B-\lambda)^{-m}(0) \oplus K(A+B-\lambda) \\ \Longrightarrow\quad & \mathcal{X} = (A+B-\lambda)^{-m}(0) \oplus (A+B-\lambda)^{m}\mathcal{X} \end{aligned}$$

for every $\lambda \in \sigma(A+B)$, $A+B$ is polaroid. This, since $\sigma(A+B) \subseteq \sigma(A)+\sigma(B)$ is finite, implies $A+B$ is algebraic. □

The descent of $A \in B(\mathcal{X})$, dsc(A), is the least non-negative integer n such that $A^n(\mathcal{X}) = A^{n+1}(\mathcal{X})$; if no such integer exists, then dsc$(A) = \infty$. Evidently, A is polar at λ if and only if asc$(A-\lambda) = $ dsc$(A-\lambda) < \infty$, and a necessary and sufficient condition for an operator A with dsc$(A-\lambda)$ to be polar at λ is that A has SVEP at λ [1, Theorem 3.81]. The following corollary is immediate from Proposition 2.2 and [1, Theorem 3.83].

Corollary 2.3. *If $A, B \in B(\mathcal{X})$ are commuting algebraic operators, then the following statements are mutually equivalent:*
(i) There exists a non-trivial polynomial $p(.)$ such that $p(A+B) = 0$.
(ii) $dsc(A+B-\lambda) < \infty$ for all complex λ.
(iii) $dsc(A+B-\lambda) < \infty$ for every λ in the topological boundary $\partial\sigma(A+B)$ of $\sigma(A+B)$.
(iv) $A+B-\lambda$ is polar (at 0) for every complex λ.

The converse of Proposition 2.2 is false: For a general non-algebraic operator $A \in B(\mathcal{X})$, $A - A = 0$ is always algebraic. Propositions 2.1 and 2.2 have a number of consequences. Recall from [11, Lemma 3.8] that if A^n is polaroid for some integer $n \geq 1$ (and $A \in B(\mathcal{X})$), then A is polaroid. Since $\sigma(A^n) = \sigma(A)^n$, we have:

Corollary 2.4. *$A \in B(\mathcal{X})$ is algebraic if and only if A^n is algebraic for all natural numbers n.*

Combining this corollary with Proposition 2.2 we have:

Corollary 2.5. *If $A, B \in B(\mathcal{X})$ are commuting algebraic operators, then AB is algebraic.*

Proof. If $AB = BA$, then $AB = \frac{1}{4}\{(A+B)^2 - (A-B)^2\}$. □

The converse of Corollary 2.5 is false: If $A \in B(\mathcal{X})$ is a nilpotent and $B \in B(\mathcal{X})$ is an operator which commutes with A, then AB being nilpotent is algebraic irrespective of whether B is or is not. It is immediate from Proposition 2.2 and Corollary 2.5 that $A, B \in B(\mathcal{X})$ algebraic implies δ_{AB}, L_AR_B, and $\triangle_{AB} = L_AR_B - \lambda$ algebraic for all complex λ. The following proposition shows that the converse holds in the case of δ_{AB}.

Proposition 2.6. *Let $A, B \in B(\mathcal{X})$.*
(a) *δ_{AB} is algebraic if and only if A and B are algebraic.*
(b) *L_AR_B algebraic does not imply A and B algebraic. However, if L_AR_B is algebraic, then at least one of A and B is algebraic.*
(c) *Furthermore, if neither of A and B is nilpotent, then L_AR_B is algebraic if and only if A and B are algebraic.*

Proof. **(a)** Assume that δ_{AB} is algebraic, i.e., assume that there exists a polynomial $p(.)$ such that $p(\delta_{AB}) = \sum_{i=0}^{n} \alpha_i \delta_{AB}^{n-i} = 0$. Then there exist scalars a_i, $1 \leq i \leq n$, not all zero such that

$$A^nX + a_1A^{n-1}XB + \cdots + a_{n-1}AXB^{n-1} + a_nXB^n = 0$$

for all $X \in B(\mathcal{X})$. Considering only those powers B^i (including $B^0 = I$) of B for which $a_i \neq 0$, it is seen that the linear independence of this set implies that $A^i = 0$ for every power of A which appears in the identity above (see [16, Theorem 1]). Hence B^n is a linear combination of elements from a maximal linearly independent subset of the set $\{I, B, B^2, \cdots, B^{n-1}\}$. Thus B is algebraic, and hence R_B is algebraic. Since $L_A = \delta_{AB} + R_B$, A is also algebraic.

(b) The example of the operator $A = 0$ and B is a quasinilpotent proves that $L_A R_B$ algebraic does not imply A and B algebraic. The hypothesis $L_A R_B$ algebraic implies the existence of scalars a_i, $1 \le i \le n$, not all 0 such that

$$A^n X B^n + a_1 A^{n-1} X B^{n-1} + \cdots + a_{n-1} A X B + a_n X = 0$$

for all $X \in B(\mathcal{X})$. Denote by $\{a_{n_1}, a_{n_2}, \ldots, a_{n_m}, 1\}$ the set of coefficients a_{n-i}, $0 \le i \le n-1$, which are non-zero, and arrange the corresponding sets of ascending powers of B and A by $S_B = \{B_1, B_2, \cdots, B_m, B^n\}$ and $S_A = \{A_1, A_2, \cdots, A_m, A^n\}$. If the set S_B is linearly independent, then $A^n = 0$, and if S_B is not linearly independent then B^n is a linear combination of powers B^i, $i < n$, of B [16, Theorem 1]. Thus either A or B is algebraic.

(c) Assume now that neither of A and B is nilpotent. Then the preceding argument implies that B is algebraic. If $\{B_1, B_2, \cdots, B_k\}$ is a maximal linearly independent subset of S_B, then there exist scalars α_{kj}, not all zero, such that $A_t = \sum_{j=k+1}^{m} \alpha_{kj} A_j$ for all $1 \le t \le k$ [16, Theorem 1]. Hence A is also algebraic. □

If $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbf{B} = (B_1, B_2, \cdots, B_n)$ are n-tuples of mutually commuting operators in $B(\mathcal{X})$, then $[L_{A_i} R_{B_i}, L_{A_j} R_{B_j}] = 0$ for all $1 \le i, j \le n$. Since A_i and B_i algebraic implies $L_{A_i} R_{B_i}$ algebraic, we have:

Corollary 2.7. *If $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbf{B} = (B_1, B_2, \cdots, B_n)$ are n-tuples of mutually commuting algebraic operators in $B(\mathcal{X})$, then the operator $\mathcal{E}_{\mathbf{AB}} - \lambda$, $(\mathcal{E}_{\mathbf{AB}} - \lambda)(X) = \sum_{i=1}^{n} A_i X B_i - \lambda X$, is algebraic for all complex λ.*

Remark 2.8. (i) Given two complex infinite-dimensional Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X}\overline{\otimes}\mathcal{Y}$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$; let, for $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$, $A \otimes B \in B(\mathcal{X}\overline{\otimes}\mathcal{Y})$ denote the tensor product operator defined by A and B. If A and B are non-nilpotent operators, then $A \otimes B$ is an algebraic operator if and only if A and B are algebraic operators: this may be proved directly or deduced from Proposition 2.2(b) using an argument of Eschmeier [13, Pages 50 and 51] relating tensor products to the operator of left-right multiplication in the operator ideal $B(B(\mathcal{Y}, \mathcal{X}))$. (Here, in using [13], one observes that B is algebraic if and only if B^* is algebraic.) It is evident from Proposition 2.2 that if A_i and B_i are algebraic for all $1 \le i \le n$ and $[A_i, A_j] = 0 = [B_i, B_j]$ for all $1 \le i, j \le n$, then $\sum_{i=1}^{n} A_i \otimes B_i$ is an algebraic operator.

(ii) An operator $A \in B(\mathcal{X})$ is *meromorphic* if its non-zero spectral points are poles of the resolvent [17, Page 225]. Clearly, a meromorphic operator possesses at most countably many spectral points $\{\lambda_i\}$ (and 0 as its only accumulation point) which we may arrange by decreasing modulus by $|\lambda_1| \ge |\lambda_2| \ge \cdots$. Recall that the polaroid property transfers from A and B to L_A, R_A, $L_A R_B$ and $L_A - R_B$ [4, 5, 10]. Evidently, A meromorphic implies L_A and R_A meromorphic. Let A and $B \in B(\mathcal{X})$ be meromorphic operators, and let $0 \ne \lambda \in \sigma(L_A R_B) = \sigma(A)\sigma(B)$. Then $\lambda = \mu\nu$ for some $0 \ne \mu \in \sigma(A)$ and $0 \ne \nu \in \sigma(B)$, and it follows that $L_A R_B$ is polar at λ. Conclusion: If A and $B \in B(\mathcal{X})$ are meromorphic, then $L_A R_B$ is meromorphic. This fails for the operator $L_A - R_B$, for the reason that $\sigma(L_A - R_B) = \sigma(A) - \sigma(B)$ (and hence every $\mu \in \sigma(A)$ and every $-\nu \in \sigma(B)$ is a point of accumulation. Note however that $L_A - R_B$ is polaroid.

Part B: Range Closure

An operator $A \in B(\mathcal{X})$ is *left polar* at a point $\lambda \in \text{iso}\, \sigma_a(A)$ if there exists a positive integer d such that $\text{asc}(A - \lambda) \le d$ and $(A - \lambda)^{d+1}(\mathcal{X})$ is closed; A is *left polaroid* if it is left polar at every $\lambda \in \text{iso}\, \sigma_a(T)$. Trivially, a Banach space operator T, in particular the operator d_{AB} or the operator $\mathcal{E}_{\mathbf{AB}}$ above, with ascent less than or equal to one has closed range if and only if it left polar (at 0). Furthermore, if $\text{asc}(T - \lambda) \le 1$ and T^* has SVEP (everywhere), then $T - \lambda$ has closed range for all complex λ if and only if T is an algebraic operator. To prove this, start by observing that T algebraic implies T polaroid, and hence if $\text{asc}(T - \lambda) \le 1$ then $T - \lambda$ has closed range for all λ. Conversely, the hypothesis T^* has SVEP implies $\sigma(T) = \sigma_a(T)$, and hence $T - \lambda$ has closed range implies $T-\lambda$ is polar for every complex λ. But then we must have that ($\sigma(T)$ has no points of accumulation, consequently) $\sigma(T)$ is a finite set. Since already T is polaroid, T is algebraic. This argument extends to the operators δ_{AB} and L_AR_B.

Proposition 2.9. *Let $A, B \in B(\mathcal{X})$ be two non-trivial operators, and let d_{AB} denote either of δ_{AB} and L_AR_B. If $asc(d_{AB}-\lambda) \leq 1$ for all complex λ, and if either* (i) *A^* and B have SVEP or* (ii) *d^*_{AB} has SVEP, then $d_{AB}-\lambda$ has closed range for all complex λ if and only if A and B are algebraic operators.*

Proof. If A and B are algebraic operators in $B(\mathcal{X})$, then so is d_{AB}. Hence, since $\text{asc}(d_{AB}-\lambda) \leq 1$ for all complex λ, $d_{AB}-\lambda$ has closed range for all complex λ. Conversely, $\text{asc}(d_{AB}-\lambda) \leq 1$ and $d_{AB}-\lambda$ has closed range for all complex λ imply d_{AB} is left polar at every complex λ. (Here, by a misuse of language we consider points λ in the resolvent set as left poles of order 0.) Now let A^* and B have SVEP. Then $\sigma(A) = \sigma_a(A)$, $\sigma(B) = \sigma_s(B)$ (= to the surjectivity spectrum of B), $\sigma_a(\delta_{AB}) = \sigma_a(A) - \sigma_s(B) = \sigma(A) - \sigma(B) = \sigma(\delta_{AB})$ and $\sigma_a(L_AR_B) = \sigma_a(A).\sigma_s(B) = \sigma(A).\sigma(B) = \sigma(L_AR_B)$. Observe also that if d^*_{AB} has SVEP, then $\sigma_a(d_{AB}) = \sigma(d_{AB})$. Hence, if either of the hypotheses (i) and (ii) is satisfied, then d_{AB} is polar at every complex λ (implies $\lambda \in \text{iso}\,\sigma(d_{AB})$ for every complex λ). Consequently, we must have that $\sigma(d_{AB})$ is a finite set and the operator d_{AB} is algebraic. This, by Proposition 2.6 (a), implies that A and B are algebraic in the case in which $d_{AB} = \delta_{AB}$. Consider now L_AR_B. Since A, B non-trivial and either of A, B nilpotent implies L_AR_B nilpotent with $\text{asc}(L_AR_B) > 1$, Proposition 2.6(c) applies and we conclude that L_AR_B algebraic implies A and B algebraic. □

The "only if part" of Proposition 2.9 fails if one relaxes the requirement that "$d_{AB}-\lambda$ has closed range for all complex λ". Thus, if A, B are two unitary (hence non–algebraic) Hilbert space operators, then $L_AR_B-\lambda$ has closed range for all $\lambda \notin \sigma(A).\sigma(B^*)$. Proposition 2.9 generalizes [3, Theorem 4.2] (and other similar results). Observe that $A, B \in B(\mathcal{H})$ normal implies δ_{AB} normal, and hence $\text{asc}(\delta_{AB}-\lambda) = \text{asc}(\delta_{(A-\lambda)B}) \leq 1$ for all complex λ and δ^*_{AB} has SVEP. $A, B \in B(\mathcal{H})$ normal does not in general imply L_AR_B normal [11, Example 2.1]; however, Proposition 2.9 applies to L_AR_B for normal $A, B \in B(\mathcal{H})$ (for the reason that A, B^* have SVEP and $\text{asc}(L_AR_B-\lambda) \leq 1$ for all complex λ — see the proof of [7, Theorem 4.1]). An alternative argument generalizing [3, Theorem 4.2], see the following proposition, is consequent from the observation that normal operators T are *simply polaroid* (i.e., $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) \leq 1$ at every $\lambda \in \text{iso}\,\sigma(T)$).

Proposition 2.10. *If A and $B \in B(\mathcal{X})$ are non-trivial simply polaroid operators, then $d_{AB}-\lambda$ has closed range for every $\lambda \in iso\,\sigma(d_{AB})$.*

Proof. In view of the fact that the polaroid property transfers from A, B to δ_{AB} and L_AR_B, we have only to prove that $\text{asc}(d_{AB}-\lambda) \leq 1$ for all $\lambda \in \text{iso}\,\sigma(d_{AB})$. Let $\lambda \in \text{iso}\,\sigma(d_{AB})$. We start by considering the case in which $\lambda \neq 0$. (Thus, if $\lambda = \mu - \nu \in \text{iso}\,\sigma(\delta_{AB})$ then (only) one of μ and ν may equal 0, and if $\lambda = \mu\nu \in \text{iso}\,\sigma(L_AR_B)$ then neither of μ and ν equals zero.) Then for every $\mu \in \text{iso}\,\sigma(A)$ and $\nu \in \text{iso}\,\sigma(B)$ such that $\lambda = \mu - \nu$ if $d_{AB} = \delta_{AB}$ and $\lambda = \mu\nu$ if $d_{AB} = L_AR_B$, $\mathcal{X} = \mathcal{X}_{11} \oplus \mathcal{X}_{12} = \mathcal{X}_{21} \oplus \mathcal{X}_{22}$, $A = A|_{\mathcal{X}_{11}} \oplus A|_{\mathcal{X}_{12}} = A_1 \oplus A_2$, $B = B|_{\mathcal{X}_{21}} \oplus B|_{\mathcal{X}_{22}} = B_1 \oplus B_2$, $A_1 - \mu$ is 1-nilpotent, $A_2 - \mu$ is invertible, $B_1 - \nu$ is 1-nilpotent and $B_2 - \nu$ is invertible. Let $X : \mathcal{X}_{21} \oplus \mathcal{X}_{22} \longrightarrow \mathcal{X}_{11} \oplus \mathcal{X}_{12}$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then

$$(\delta_{AB}-\lambda)^2(X) = 0 \Longleftrightarrow \begin{pmatrix} 0 & (\mu R^2_{B_2-\nu})(X_{12}) \\ \nu(L^2_{A_2-\mu})(X_{21}) & (\delta_{A_2B_2}-\lambda)^2(X_{22}) \end{pmatrix} = 0$$

$$\Longleftrightarrow \quad X_{12} = X_{21} = X_{22} = 0 \Longrightarrow (\delta_{AB}-\lambda)(X) = 0.$$

A similar argument shows that $(L_AR_B-\lambda)^2(X) = 0$ if and only if $(L_AR_B-\lambda)(X) = 0$. We consider next the case $\lambda = 0$. If $d_{AB} = \delta_{AB}$, then either $\mu = \nu = 0$ or $\mu = \nu \neq 0$ for every $\mu \in \text{iso}\,\sigma(A)$ and $\nu \in \text{iso}\,\sigma(B)$ such that $\lambda = \mu - \nu$. Defining A_i, B_i, $\mathcal{X}_{1i}$ and $\mathcal{X}_{2i}$, $1 \leq i \leq 2$, as above it is then seen that ($A_1 = 0 = B_1$ and) $\delta^2_{AB}(X) = 0$ implies $X_{22} = 0$ in the case in which $\mu = \nu = 0$ and $X_{12} = X_{21} = X_{22} = 0$ in the case in which $\mu = \nu \neq 0$. In either case $\delta_{AB}(X) = 0$. Finally, if $d_{AB} = L_AR_B$ and $0 \in \text{iso}\,\sigma(L_AR_B)$, then either $0 \in \text{iso}\,\sigma(A)$ and $0 \notin \sigma(B)$, or, $0 \notin \sigma(A)$ and $0 \in \text{iso}\,\sigma(B)$, or, $0 \in \text{iso}\,\sigma(A)$ and $0 \in \text{iso}\,\sigma(B)$. (Note that by hypothesis A, B are non-trivial and polaroid; hence neither of $\sigma(A)$ and $\sigma(B) = \{0\}$.) Trivially, if either of A or B is invertible, then $\text{asc}(L_AR_B) \leq 1$.

If, instead, $0 \in \{\text{iso}\,\sigma(A) \cap \text{iso}\,\sigma(B)\}$, then upon defining A_i, B_i, $\mathcal{X}_{1i}$ and $\mathcal{X}_{2i}$, $1 \leq i \leq 2$, as above it is seen that $A_1 = 0 = B_1$ and $(L_A R_B)^2(X) = 0$ implies $X_{22} = 0$. Hence $(L_A R_B)(X) = 0$. □

The hypotheses of Proposition2.10 are satisfied by a wide variety of classes of operators. We mention here one such class, the class of paranormal Banach space operators [17, Page 229].

For an operator $T \in B(\mathcal{X})$ with spectral radius $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$, the *peripheral spectrum* $\sigma_\pi(T)$ *of* T is the set $\sigma_\pi(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$. As we saw earlier on, if $A, B \in B(\mathcal{X})$ are meromorphic operators, then the operator $L_A R_B$ is meromorphic. Since A, B normaloid ($T \in B(\mathcal{X})$ is normaloid if $r(T) = \|T\|$) implies $L_A R_B$ normaloid, if A, B are normaloid then $\lambda \in \sigma_\pi(L_A R_B)$ if and only if there exist $\mu \in \sigma_\pi(A)$ and $\nu \in \sigma_\pi(B)$ such that $\lambda = \mu\nu$. Recall from [17, Proposition 54.4] that if $L_A R_B$ is a normaloid meromorphic operator, then $\text{asc}(L_A R_B - \lambda) \leq 1$ for all $\lambda \in \sigma_\pi(L_A R_B)$. Such an operator $L_A R_B$ being polaroid, we conclude: *If* $A, B \in B(\mathcal{X})$ *are normaloid meromorphic operators, then* $L_A R_B - \lambda$ *has closed range for every* $\lambda \in \sigma_\pi(L_A R_B)$. The following proposition is a generalization of this result.

Proposition 2.11. *If* $A, B \in B(\mathcal{X})$ *are normaloid operators, then the following assertions are mutually equivalent for all* $\lambda \in \sigma_\pi(L_A R_B)$*:*
(i) $L_A R_B - \lambda$ *has closed range.*
(ii) $L_A R_B - \lambda$ *is left polar at* 0*.*
(iii) $L_A R_B - \lambda$ *is polar at* 0*.*

Proof. The proof of the proposition depends upon the known fact, [8, Proposition 2.4], that $\text{asc}(L_A R_B - \lambda) \leq 1$ for all $\lambda \in \sigma_\pi(L_A R_B)$: we include a proof here for completeness.

If A, B are normaloid, then $L_A R_B$ is normaloid, $r(L_A R_B) = r(A)r(B) = \|A\|\|B\|$, and

$$\sigma_\pi(L_A R_B) = \{\lambda \in \mathbf{C} : \lambda = \mu\nu, \mu \in \sigma_\pi(A), \nu \in \sigma_\pi(B)\}.$$

If we define the contractions A_1 and B_1 by $A_1 = A/\|A\|$ and $B_1 = B/\|B\|$, then $L_{A_1} R_{B_1}$ is a contraction and

$$\sigma_\pi(L_{A_1} R_{B_1}) = \{\lambda \in \mathbf{C} : \lambda = \mu\nu, \mu \in \sigma_\pi(A_1), \nu \in \sigma_\pi(B_1), |\mu| = |\nu| = 1\}.$$

Choose a $\lambda_0 = \mu_0\nu_0 \in \sigma_\pi(L_{A_1} R_{B_1})$; let $A_{10} = A_1/\mu_0$ and $B_{10} = B_1/\nu_0$. Then

$$\begin{aligned} &\|\frac{\lambda_0}{n}\sum_{i=0}^{n-1}(L_{A_{10}}R_{B_{10}})^i(L_{A_{10}}R_{B_{10}} - 1)(Z)\| = \|\frac{\lambda_0}{n}(L^n_{A_{10}}R^n_{B_{10}} - 1)(Z)\| \\ &= \frac{1}{n}\|(L_{A^n_{10}}R_{B^n_{10}} - 1)(Z)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

for all $Z \in B(\mathcal{X})$. Set $\lambda_0\|A\|\|B\| = \lambda \in \sigma_\pi(L_A R_B)$. Then $X \in B(\mathcal{X})$ satisfies $(L_{A_{10}} R_{B_{10}})(X) = 0$ if and only if $(L_A R_B)(X) = 0$. An easy calculation shows that $X \in (L_{A_{10}} R_{B_{10}} - 1)^{-1}(0)$ implies $X = \frac{1}{n}\sum_{i=0}^{n-1}(L_{A_{10}} R_{B_{10}})^i(X)$. Hence if $X \in (L_A R_B - 1)^{-1}(0)$ and $Y = Z/\|A\|\|B\|$, then for all $Z \in B(\mathcal{X})$,

$$\begin{aligned} &\|X + \frac{\lambda_0}{n}\sum_{i=0}^{n-1}(L_{A_{10}}R_{B_{10}})^i(L_{A_{10}}R_{B_{10}} - 1)(Z)\| \\ = \;&\|\frac{1}{n}\sum_{i=0}^{n-1}(L_{A_{10}}R_{B_{10}})^i(X + \lambda_0(L_{A_{10}}R_{B_{10}} - 1)(Z))\| \\ \leq \;&\|X + \lambda_0(L_{A_{10}}R_{B_{10}} - 1)(Z)\| = \|X + (L_{A_1}R_{B_1} - \lambda_0)(Z)\| \\ = \;&\|X + (L_A R_B - \lambda)(Y)\| \end{aligned}$$

for all $Y \in B(\mathcal{X})$ and $\lambda \in \sigma_\pi(L_A R_B)$.

The two way implication (i)⟺(ii) is evident. Observe that if $L_A R_B$ is normaloid and $\lambda \in \sigma_\pi(L_A R_B)$, then λ is in the boundary of $\sigma(L_A R_B)$. Hence $(L_A R_B - \lambda)^*$ has SVEP (at 0), and so $L_A R_B$ is left polar at λ if and only if it is polar at λ. Hence (ii)⟺(iii). □

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