Functional Analysis,
Approximation and
Computation
6 (2) (2014), 51–59

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

An iterative method for computing the pseudoinverse operator in Hilbert spaces

Gheorghe Grigore[a], Daniel Stănică[b]

[a]*Faculty of Mathematics and Computer Science, University of Bucharest, 14, Academiei str., 010014-Bucharest, Romania*
[b]*Faculty of Mathematics and Computer Science, University of Bucharest, 14, Academiei str., 010014-Bucharest, Romania*

Abstract. In this paper we construct an iterative method, based on classical Newton method, for approximating the pseudoinverse of a linear bounded operator with closed range between Hilbert spaces. The method is always convergent and, moreover, we obtain an estimation of the error. Also we consider the behavior of the method when the iterates are subject to perturbations.

1. Introduction

The role of the pseudoinverse (Moore-Penrose inverse) of a linear bounded operator with closed range between Hilbert spaces in statistics, prediction theory, control system analysis, curve fitting and numerical analysis is well recognized, as many publications show (see, for example, [3], [5]).

The most famous iterative method for computing the pseudoinverse of a matrix is described by Adi Ben-Israel in two articles (see [1], [2]). His method extends the Schulz method for the matrix inversion, which is based on classical Newton method. In the first article, the initial conditions to ensure the convergence of the method are restrictive and the algorithm requires the knowledge of a special projection operator. In the second article it is necessary to know an eigenvalue of a particular matrix. The method of Ben-Israel was extended in many papers in the finite-dimensional case (see, for example [4], [7]-[10]).

In order to calculate the solutions of the inconsistent equations in Hilbert spaces, the pseudoinverse plays an important role, because it determines the minimum least square solution.

In this paper we extend the Ben-Israel method for determining the pseudoinverse of a linear bounded operator with closed range between Hilbert spaces. We get an initial iteration which does not depend on a particular projection operator or on a particular eigenvalue. It is only defined by the adjoint and the norm of the operator. With this initial iteration the method is always convergent. Moreover, we obtain an estimation of the error. Finally we present stability problems of the method. In our proofs we do not use the functional calculus method as in [6], p. 56, where the initial iteration depends on the of the spectrum of a particular operator.

The stable perturbations of pseudoinverse operator is described in [13], Chapter 3.

2010 *Mathematics Subject Classification*. Primary 47A05; Secondary 65F20.
Keywords. Pseudoinverse operator; Moore-Penrose inverse; Iterative method; Hilbert spaces.
Received: 24 August 2014; Accepted: 9 September 2014
Communicated by Dijana Mosić
Email addresses: (Gheorghe Grigore), stanicad@fmi.unibuc.ro (Daniel Stănică)

Let X and Y be two real Hilbert spaces and let $\mathcal{L}(X,Y)$ denote the space of all bounded linear operators between X and Y. For $T \in \mathcal{L}(X,Y)$ let $R(T)$, $N(T)$ and T^* denote the range, the null space and the adjoint of T.

Let $T \in \mathcal{L}(X,Y)$ be a self-adjoint operator. T is positive (respectively, strictly positive) if $\langle Tx, x\rangle \geq 0$, $\forall\, x \in X$ (respectively, $\langle Tx, x\rangle > 0$, $\forall\, x \neq 0$). $T \geq 0$ (respectively, $T > 0$)means that T is positive (respectively, strictly positive). If $T, U \in \mathcal{L}(X,Y)$ are self-adjoint operators, we say that $T \leq U$ (respectively, $T < U$) if $T - U$ is positive (respectively, strictly positive).

It is well known that if $T, U \in \mathcal{L}(X,Y)$ are self-adjoint positive operators (respectively, strictly positive operators) and commute, then TU is a self-adjoint positive operator (respectively, strictly positive operator).

Definition 1.1. *The operator $T \in \mathcal{L}(X,Y)$ admits pseudoinverse (Moore-Penrose inverse) if there exists an operator $S \in \mathcal{L}(X,Y)$ such that $TST = T$, $STS = S$, $(TS)^* = TS$, $(ST)^* = ST$.*

It is well known that the operator $T \in \mathcal{L}(X,Y)$ admits pseudoinverse if and only if the subspace $R(T)$ is closed. Let $\mathcal{LC}(X,Y)$ denote the space of all bounded linear operators with closed range. If $T \in \mathcal{LC}(X,Y)$ then the pseudoinverse operator is unique and it is denoted by T^+.

If $y \in Y$ then $x = T^+y$ is the unique element

$$x \in A := \left\{x' \in X \mid \|Tx' - y\| = \inf_{z\in X} \|Tz - y\|\right\}$$

such that $\|x\| \leq \|x'\|$ for all $x' \in A$. x is the minimum norm solution of the previous least square problem.

2. Main result

Let X, Y be a Hilbert spaces and $T \in \mathcal{LC}(X,Y)$. We consider the sequence of operators $(X_n)_{n\in\mathbf{N}}$ defined by

$$X_0 = T^*TT^*$$

and

$$X_{n+1} = 2X_n - X_nTX_n,\ \forall\, n \in \mathbf{N}.$$

It is useful to observe, using the method of mathematical induction, that the operators TX_n and X_nT are self-adjoint operators.

Lemma 2.1. *If $T \in \mathcal{LC}(X,Y)$ and $\|T\| < \sqrt[4]{2}$, then*

$$0 \leq X_nT < 2\mathbf{I}_X,\ \ \forall\, n \in \mathbf{N}, \tag{1}$$

where $\mathbf{I}_X$ is the identity operator of the space X.

Proof. We have

$$0 \leq X_0T < 2\mathbf{I}_X \Leftrightarrow 0 \leq (T^*T)^2 < 2\mathbf{I}_X.$$

Obviously, $(T^*T)^2 \geq 0$, and

$$(T^*T)^2 < 2\mathbf{I}_X \Leftrightarrow (\sqrt{2}\mathbf{I}_X - T^*T)(\sqrt{2}\mathbf{I}_X + T^*T) > 0.$$

Since $\sqrt{2}\mathbf{I}_X + T^*T$ is strictly positive and commutes with $\sqrt{2}\mathbf{I}_X - T^*T$, it will be sufficient to demonstrate that $\sqrt{2}\mathbf{I}_X - T^*T > 0$. Indeed, if $x \neq 0$,

$$\langle T^*Tx, x\rangle \leq \|T\|^2\|x\| < \sqrt{2}\langle x, x\rangle.$$

For $n \geq 1$, from (1), $X_{n+1}T = X_nT(2\mathbf{I}_X - X_nT) \geq 0$ and then

$$2\mathbf{I}_X - X_{n+1}T = (X_nT)^2 - 2X_nT + 2\mathbf{I}_X = (X_nT - \mathbf{I}_X)^2 + \mathbf{I}_X > 0.$$

□

Lemma 2.2. *If $T \in \mathcal{LC}(X, Y)$ and $\|T\| < \sqrt[4]{2}$, then*

$$N(X_n) = N(X_0) = R(T)^{\perp}, \ \forall\, n \in \mathbf{N}.$$

Proof. We have $N(X_0) = N(T^*TT^*)$. If $x \in N(T^*TT^*)$, then $T^*x \in N(T^*T) = N(T)$. Thus, $x \in N(TT^*) = N(T^*)$, hence $N(X_0) = N(T^*TT^*) = N(T^*) = R(T)^{\perp}$.

We deduce from the recurrence relation that $N(X_n) \subset N(X_{n+1})$.

Let $y \in N(X_{n+1})$. Therefore $X_{n+1}y = 0$, i.e. $X_n y = \frac{1}{2} X_n T X_n y$. Then

$$X_n y = \frac{1}{2^k}(X_n T)^k X_n y, \ \forall\, k \geq 1.$$

We have

$$2^k \mathbf{I}_X - (X_n T)^k = (2\mathbf{I}_X - X_n T)(2^{k-1}\mathbf{I}_X + 2^{k-2} X_n T + \cdots + (X_n T)^{k-1})$$

and from Lemma 2.1 it follows that $(X_n T)^k < 2^k \mathbf{I}_X$. If we suppose that $X_n y \neq 0$, then

$$\langle X_n y, X_n y\rangle = \frac{1}{2^k}\langle (X_n T)^k X_n y, X_n y\rangle < \langle X_n y, X_n y\rangle,$$

which is a contradiction. It remains that $X_n y = 0$. Therefore $N(X_{n+1}) \subset N(X_n)$ and

$$N(X_{n+1}) = N(X_n) = N(X_0).$$

□

Let $T \in \mathcal{LC}(X, Y)$ and let P, respectively Q denote the orthogonal projection on $R(T^*)$, respectively on $R(T)$.

Proposition 2.3. *Let $T \in \mathcal{LC}(X, Y)$, $T \neq 0$, $\|T\| \leq 1$. Then there exists $\gamma \in (0, 1]$ such that*

$$\gamma P \leq X_n T \leq P, \ \forall\, n \in \mathbf{N}$$

and

$$\gamma Q \leq T X_n \leq Q, \ \forall\, n \in \mathbf{N}.$$

Proof. The operator $T^*T : N(T)^{\perp} \to N(T)^{\perp}$ is invertible. Indeed, if $x, y \in N(T)^{\perp}$ and $T^*Tx = T^*Ty$, then $x - y \in N(T^*T) = N(T)$, therefore $x = y$. For $y \in N(T)^{\perp}$ there exists $x \in N(T)^{\perp}$ such that $T^*Tx = y$ because $N(T)^{\perp} = R(T^*)$ and $R(T^*T) = R(T^*)$. Then there exists $\gamma > 0$ such that

$$\|T^*Tx\| \geq \sqrt{\gamma}\|x\|, \forall\, x \in N(T)^{\perp}.$$

From

$$\sqrt{\gamma}\|x\| \leq \|T^*Tx\| \leq \|T^*T\|\|x\| = \|T\|^2\|x\| \leq \|x\|,$$

it follows that $\gamma \leq 1$. Next

$$\langle X_0 T x, x\rangle = \langle (T^*T)^2 x, x\rangle = \|T^*Tx\|^2 \geq \gamma\|x\|^2, \ \forall\, x \in N(T)^{\perp},$$

i.e.

$$\langle X_0 T x, x\rangle \geq \gamma\langle Px, x\rangle, \ \forall\, x \in N(T)^{\perp}.$$

We have $\gamma Px = X_0Tx = 0,\ \forall\, x \in N(T)$. Finally, if $x \in X$, $x = x_1 + x_2$ with $x_1 \in N(T)$ and $x_2 \in N(T)^\perp$, we have

$$\langle X_0Tx, x\rangle - \gamma\langle Px, x\rangle = \langle X_0Tx_2, x_1\rangle + \langle X_0Tx_2, x_2\rangle - \gamma\langle x_2, x_2\rangle =$$

$$= \langle X_0Tx_2, x_2\rangle - \gamma\langle x_2, x_2\rangle \geq 0.$$

It results that $\gamma P \leq X_0T$.

The inequality $X_0T \leq P$ is proved in a similar manner: if $x \in N(T)^\perp$, we have $\|T^*Tx\|^2 \leq \|T^*T\|^2\|x\|^2 \leq \|x\|^2$ and therefore

$$\langle X_0Tx, x\rangle \leq \langle Px, x\rangle,\ \forall\, x \in N(T)^\perp.$$

If $x \in N(T)$, $\langle X_0Tx, x\rangle = \langle Px, x\rangle = 0$. As above, from $R(X_0T) = R((T^*T)^2) \subset R(T^*) = N(T)^\perp$ it follow that $X_0T \leq P$.

Then we proceed by induction. We suppose that

$$\gamma P \leq X_nT \leq P.$$

Obviously, $X_nTP = X_nT$. Since X_n is a composition which has in the left side the operator T^*, we have $R(X_n) \subset R(T^*) = N(T)^\perp$ and $PX_nT = X_nT$. It results that

$$(X_nT - P)^2 = (X_nT)^2 - 2X_nT + P$$

and

$$P - X_{n+1}T = P^2 + (X_nT)^2 - 2X_nT \geq 0,$$

i.e. $X_{n+1}T \leq P$. Then

$$X_{n+1}T \geq \gamma P \Leftrightarrow (X_nT)^2 - 2X_nT + \gamma P^2 \leq 0 \Leftrightarrow$$

$$\Leftrightarrow (X_nT - (1 - \sqrt{1-\gamma})P)(X_nT + (1 + \sqrt{1-\gamma})P) \leq 0,$$

which is true because $(1 - \sqrt{1-\gamma})P \leq \gamma P \leq X_nT$.

The inequalities $\gamma Q \leq TX_n \leq Q$ are proved in a similar manner, following the next steps: $TX_0 \geq \gamma Q$, $TX_0 \leq Q$. Any X_n being as AT^* and $N(X_n) = R(T)^\perp$ (from Lemma 2.2), we have $TX_nQ = TX_n$ and obviously $QTX_n = TX_n$.Whence $(TX_n - Q)^2 = (TX_n)^2 - 2TX_n + Q$ and then $Q - TX_{n+1} = Q + (TX_n)^2 - 2TX_n \geq 0$. Next

$$TX_{n+1} \geq \gamma Q \Leftrightarrow (TX_n - (1 - \sqrt{1-\gamma})Q)(TX_n + (1 - \sqrt{1-\gamma})Q) \leq 0,$$

which is true because $(1 - \sqrt{1-\gamma})Q \leq \gamma Q \leq X_nT$. □

Proposition 2.4. *If $T \in \mathcal{LC}(X, Y)$ and $\|T\| \leq 1$, then the sequence $(X_n)_{n\in\mathbf{N}}$ is pointwise convergent.*

Proof. We have

$$X_{n+1}T - X_nT = X_nT - (X_nT)^2 = X_nT(P - X_nT) = (P - X_nT)X_nT.$$

From the Proposition 2.3 it results that the sequence of self-adjoint operators $(X_nT)_{n\in\mathbf{N}}$ is increasing (i.e. $X_nT \leq X_{n+1}T,\ \forall\, n \in \mathbf{N}$) and $X_nT \leq P,\ \forall\, n \in \mathbf{N}$. In this case we know that the sequence has an upper bound which is the pointwise limit of the sequence.

Let $Ux = \lim_{n\to\infty} X_nTx,\ \forall\, x \in X$. We prove that the sequence $(X_n)_{n\in\mathbf{N}}$ is also pointwise convergent.

If $y \in R(T)$, i.e. $y = Tx$ with $x \in X$, then $X_ny = X_nTx$, hence $\lim_{n\to\infty} X_ny = \lim_{n\to\infty} X_nTx = Ux$. The limit does not depend on the representation $y = Tx$. Indeed, if $y = Tz$ then $U_nTy = U_nTz,\ \forall\, n \in \mathbf{N}$ and, consequently, the sequences $(U_nTy)_{n\in\mathbf{N}}$, $(U_nTz)_{n\in\mathbf{N}}$ have the same upper bound.

If $y \in R(T)^\perp$, according to the Lemma 2.2, we have $X_ny = 0$, so $\lim_{n\to\infty} X_ny = 0$. Finally, for $y \in Y$, $y = y_1 + y_2$, $y_1 \in R(T)$, $y_2 \in R(T)^\perp$ we have $X_ny = X_ny_1$, hence $\lim_{n\to\infty} X_ny$ exists and is equal to the limit $\lim_{n\to\infty} X_ny_1$. □

Corollary 2.5. *If* $Sy = \lim_{n\to\infty} X_n y$, $\forall\, y \in Y$, *then* $STS = S$.

Proof. From Banach−Steinhaus theorem, the sequence $(\|X_n\|)_{n\in\mathbf{N}}$ is bounded and S is linear and bounded. For $n \in \mathbf{N}$ and $y \in Y$, we have

$$\|X_nTX_ny - STSy\| \le \|X_nTX_ny - X_nTSy\| + \|X_nTSy - STSy\| \le$$

$$\le \|X_n\|\|T\|\|X_ny - Sy\| + \|X_nTSy - STSy\|.$$

It results that $\lim_{n\to\infty} X_nTX_ny = STSy$. From $X_{n+1} = 2X_n - X_nTX_n$ we deduce that $Sy = 2Sy - STSy$, hence $STSy = Sy$. □

Proposition 2.6. *If* $T \in \mathcal{LC}(X, Y)$ *and* $\|T\| \le 1$, *then the sequence* $(X_nT)_{n\in\mathbf{N}}$ *converges to* P *and we have*

$$ST = P,$$

$$\|ST - X_nT\| \le (1-\gamma)^{2^n},\ \forall\, n \in \mathbf{N}.$$

Proof. If $\|T\| = 1$ and $\gamma = 1$, then $X_nT = P$, $\forall\, n \in \mathbf{N}$ and the proof is obvious.

Next, we suppose that $\|T\| < 1$ or $\|T\| = 1$ and $\gamma < 1$. From

$$P - X_{n+1}T = P^2 - 2X_nT + (X_nT)^2 = (P - X_nT)^2,$$

we deduce that

$$P - X_nT = (P - X_0T)^{2^n}.$$

We have $0 \le P - X_0T \le (1-\gamma)P$ and therefore $\|P - X_0T\| \le 1-\gamma$. It results that

$$\|P - X_nT\| \le (1-\gamma)^{2^n}. \tag{2}$$

Let $x \in Y$. Then $\|Px - X_nT\| \le \|P - X_nT\|\|x\| \le (1-\gamma)^{2^n}\|x\|$, which shows that $Px = \lim_{n\to\infty} X_nTx = ST$. From (2) we deduce that $\|ST - X_nT\| \le (1-\gamma)^{2^n}$. □

Proposition 2.7. *If* $T \in \mathcal{LC}(X, Y)$ *and* $\|T\| \le 1$, *then the sequence* $(TX_n)_{n\in\mathbf{N}}$ *converges to* Q *and we have*

$$TS = Q,$$

$$\|Q - TX_n\| \le (1-\gamma)^{2^n},\ \forall\, n \in \mathbf{N}.$$

Proof. The proof of this proposition is similar to the proof of the previous proposition. □

Remark 2.8. From the Corollary 2.5 and the Propositons 2.6, 2.7 and the fact that T^+ is the unique operator $S \in \mathcal{L}(Y, X)$ such that $STS = S$, $ST = P$ and $TS = Q$ it results that if $T \in \mathcal{LC}(X, Y)$ and $\|T\| \le 1$, then $\lim_{n\to\infty} X_ny = Sy = T^+y$, $\forall\, y \in Y$. Therefore, the sequence $(X_n)_{n\in\mathbf{N}}$ converges pointwise to T^+.

Remark 2.9. If $\|T\| = 1$ and $\gamma = 1$, then $X_0 = T^+$ and $X_n = X_0$, $\forall\, n \in \mathbf{N}$. Indeed, in this case $X_0T = P$, $TX_0 = Q$ and $X_0TX_0 = PT^*TT^* = T^*TT^* = X_0$, hence $X_0 = T^+$. For a given n, if $X_n = T^+$, then $X_{n+1} = 2T^+ - T^+TT^+ = T^+$. Thus, $X_n = T^+$, $\forall\, n \in \mathbf{N}$.

Next we present the main result of this paper.

Theorem 2.10. *Let* $T \in \mathcal{LC}(X, Y)$ *with* $\|T\| \le 1$. *Consider* $X_0 = T^*TT^*$ *and* $X_{n+1} = 2X_n - X_nTX_n$, $\forall\, n \in \mathbf{N}$. *Then the sequence* $(X_n)_{n\in\mathbf{N}}$ *converges to* T^+.

Moreover, there exist $\alpha > 0$ *and* $\gamma \in (0, 1]$ *such that*

$$\|T^+ - X_n\| \le \alpha(1-\gamma)^{2^n},\ \forall\, n \in \mathbf{N}.$$

Proof. Since $T : N(T)^{\perp} \to R(T)$ is invertible, then there exists $\beta > 0$ such that

$$\|Tx\| \geq \beta\|x\|, \ \forall\, x \in N(T)^{\perp}.$$

If $y \in R(T)$, then

$$\|Sy - X_n y\| = \|STx - X_n Tx\| \leq (1-\gamma)^{2^n}\|x\| \leq$$

$$\leq \frac{1}{\beta}(1-\gamma)^{2^n}\|y\| = \alpha(1-\gamma)^{2^n}\|y\|,$$

where $\alpha = 1/\beta$.

If $y = y_1 + y_2$, with $y_1 \in R(T)$ and $y_2 \in R(T)^{\perp}$, then

$$\|Sy - X_n y\| = \|S(y_1 + y_2) - X_n(y_1 + y_2)\| = \|Sy_1 - X_n y_1\| \leq$$

$$\leq \alpha(1-\gamma)^{2^n}\|y_1\| \leq \alpha(1-\gamma)^{2^n}\|y\|.$$

In conclusion, $\lim_{n\to\infty} X_n = S = T^+$ and

$$\|T^+ - X_n\| \leq \alpha(1-\gamma)^{2^n}, \ \forall\, n \in \mathbf{N}.$$

□

Remark 2.11. In finite dimensional case (i.e. $\dim X < \infty$, $\dim Y < \infty$) γ is $\min\{\lambda^2 \mid \lambda \in \sigma_p(T^*T),\ \lambda \neq 0\}$, where $\sigma_p(T^*T)$ is the point spectrum of T^*T. In this case, if $\|T\| = 1$ and $\gamma = 1$, it results that $T^*T = \mathbf{I}_m$, where m is the dimension of space X, and $T^+ = X_0 = T^*$.

Suppose now that $T \in \mathcal{L}(X, Y)$, $T \neq 0$. The operator $\frac{1}{\|T\|}T$ has the norm equal to 1. Then we can use this operator in Theorem 2.10. It results that

$$X_0 = \frac{1}{\|T\|^3}T^*TT^*$$

and

$$X_{n+1} = 2X_n - \frac{1}{\|T\|}X_n T X_n, \forall\, n \in \mathbf{N}.$$

With the notation $Y_n = \frac{1}{\|T\|}X_n$, we have

$$Y_0 = \frac{1}{\|T\|^4}T^*TT^* \tag{3}$$

and

$$Y_{n+1} = 2Y_n - Y_n T Y_n, \ \forall\, n \in \mathbf{N}. \tag{4}$$

From Theorem 2.10 it results that $\lim_{n\to\infty} X_n = \left(\frac{1}{\|T\|}T\right)^+ = \|T\|T^+$ and consequently

$$\lim_{n\to\infty} Y_n = T^+.$$

To evaluate the error of the approximation we also use Theorem 2.10. Then we deduce that there exist $\alpha > 0$ and $\gamma \in (0, 1]$ such that

$$\|T^+ - Y_n\| \leq \frac{\alpha}{\|T\|}(1-\gamma)^{2^n}, \ \forall\, n \in \mathbf{N},$$

where $\alpha = 1/\beta$, β verifies the inequality

$$\|Tx\| \geq \beta \|T\| \|x\|, \ \forall\, x \in N(T)^{\perp} = R(T^*),$$

and γ verifies the inequality

$$\|T^*Tx\| \geq \sqrt{\gamma}\|T\|^2\|x\|, \forall\, x \in N(T)^{\perp} = R(T^*).$$

In conclusion, we get the following theorem:

Theorem 2.12. *Let $T \in \mathcal{LC}(X, Y)$ with $\|T\| \neq 0$. Consider $Y_0 = \frac{1}{\|T\|^4} T^*TT^*$ and $Y_{n+1} = 2Y_n - Y_nTY_n$, $\forall\, n \in \mathbf{N}$. Then the sequence $(Y_n)_{n\in\mathbf{N}}$ converges to T^+.*

Moreover, there exist $\alpha > 0$ and $\gamma \in (0, 1]$ such that

$$\|T^+ - Y_n\| \leq \frac{\alpha}{\|T\|}(1-\gamma)^{2^n}, \ \forall\, n \in \mathbf{N}.$$

Therefore, the sequence defined by (3) and (4) always converges to the pseudoinverse operator with quadratic speed of convergence. Finally, we present some examples of this method.

Example 2.13. Let $T : l^2(\mathbf{R}) \to l^2(\mathbf{R})$ with $T(x) = (x_2, x_3, x_4, \ldots)$, $\forall x = (x_k)_{k\geq 1} \in l^2(\mathbf{R})$ (the left shift operator).

We have $\|T\| = 1$ and $T^*T = \mathbf{I}_{l^2(\mathbf{R})}$, where $T^*(x) = (0, x_1, x_2, x_3, \ldots)$, $\forall x = (x_k)_{k\geq 1} \in l^2(\mathbf{R})$. Then $X_0 = T^+$. But $X_0 = T^*TT^* = T^*$. Thus, $T^+ = T^*$.

Example 2.14. Let $T : L^2([0, 1]) \to L^2([0, 1])$, $Tx(s) = s \int_0^1 tx(t)dt$, $\forall\, x \in L^2([0, 1])$, $\forall\, s \in [0, 1]$.

We have $T \in \mathcal{L}(L^2([0, 1]), L^2([0, 1]))$, $R(T)$ is closed, $T = T^*$ and $\|T\| = 1/3$. Then $X_n = \alpha_n T$, $\forall\, n \in \mathbf{N}$, with $\alpha_0 = 1/9$ and $\alpha_{n+1} = 2\alpha_n - \frac{\alpha_n^2}{9}$, $\forall\, n \in \mathbf{N}$. The sequence $(\alpha_n)_{n\in\mathbf{N}}$ is convergent and $\lim_{n\to\infty} \alpha_n = 9$. Then $\lim_{n\to\infty} X_n = 9T$. Therefore, from Theorem 2.10, we deduce that $T^+ = 9T$.

Example 2.15. Let $T : L^2([0, \pi]) \to L^2([0, \pi])$,

$$Tx(s) = x(s) - \frac{2}{\pi}\int_0^{\pi} \left(\sin s \sin \xi + \frac{1}{2}\cos s \cos \xi\right) x(\xi) d\xi,$$

$\forall\, x \in L^2([0, \pi])$, $\forall\, s \in [0, \pi]$.

We have $T \in \mathcal{L}(L^2([0, 1]), L^2([0, 1]))$, $R(T)$ is closed, $T = T^*$, $\|T\| = 1$ and T is not invertible (see [11], [12]). Then

$$Y_n(s) = x(s) - \frac{2}{\pi}\int_0^{\pi} (\sin s \sin \xi + \alpha_n \cos s \cos \xi)\, x(\xi) d\xi,$$

where $\alpha_0 = \frac{7}{8}$ and $\alpha_{n+1} = 2\alpha_n - \frac{1 + 2\alpha_n - \alpha_n^2}{2} = \frac{\alpha_n^2 + 2\alpha_n - 1}{2}$. The sequence $(\alpha_n)_{n\in\mathbf{N}}$ is convergent and $\lim_{n\to\infty} \alpha_n = -1$. Consequently, from Theorem 2.12, we deduce that

$$T^+x(s) = x(s) - \frac{2}{\pi}\int_0^{\pi} (\sin s \sin \xi - \cos s \cos \xi)\, x(\xi) d\xi.$$

The minimum norm solution x^* of the least square problem determined by the inconsistent equation $Tx(s) = b(s)$ with $b(s) = s$, $\forall\, s \in [0, \pi]$, i.e.

$$x(s) - \frac{2}{\pi}\int_0^{\pi} \left(\sin s \sin \xi + \frac{1}{2}\cos s \cos \xi\right) x(\xi) d\xi = s,$$

is

$$x^*(s) = T^+b(s) = s - 2\sin s - \frac{4}{\pi}\cos s.$$

Example 2.16. Let $T = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 6 & -3 \\ 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. We have rank$(T) = 3$ and from Theorem 2.12, using MATLAB, we obtain

$$Y_0 = \frac{1}{\alpha} \begin{pmatrix} 44 & 25 & 127 & 41 & 23 \\ 36 & 31 & 89 & 31 & 17 \\ 120 & 50 & 374 & 122 & 62 \\ -40 & 6 & -158 & -50 & -22 \end{pmatrix},$$

where $\alpha = \left(\frac{74}{3} + \frac{4\sqrt{703}}{3} \cos\left(\frac{1}{3} \arccos \frac{20197}{793\sqrt{793}}\right)\right)^2$ and

$$Y_{15} = \begin{pmatrix} 5/56 & 3/112 & 11/112 & -55/112 & 7/16 \\ 1/56 & 23/112 & -9/112 & 45/112 & -5/16 \\ 1/28 & 1/28 & 5/56 & 3/56 & 0 \\ 1/14 & 11/56 & -1/14 & -1/7 & 1/8 \end{pmatrix},$$

which is T^+, the matrix pseudoinverse of T.

Example 2.17. Let $A = (a_{ij})_{i,j=\overline{1,100}}$, where $a_{ii} = 2$, $\forall\, i = \overline{1,100}$, $a_{i+1\,i} = -1$ $\forall\, i = \overline{1,100}$, $a_{i\,i+1} = -1$, $\forall\, i = \overline{1,100}$ and $a_{ij} = 0$, otherwise. From Theorem 2.12, we obtain Y_{54} is the pseudoinverse (inverse) of A, with an acuracy of ten decimals. We observe that the pseudoinverse is obtained in a number of steps smaller than the dimension of the matrix A.

We now consider the behavior of the method when the iterates are subject to perturbations. One case, in finite-dimensional spaces, when the method is unstable is described in [9]. Next we consider a more general situation. We suppose that the operator $\widetilde{Y}_k$, in kth iteration differs from the original operator Y_k by the error operator Δ_k. Then

$$\widetilde{Y}_{k+1} = 2\widetilde{Y}_k - \widetilde{Y}_k T \widetilde{Y}_k \Leftrightarrow$$

$$\Delta_{k+1} = 2\Delta_k - (Y_k T \Delta_k + \Delta_k T Y_k) - \Delta_k T \Delta_k =$$
$$= 2\Delta_k - (Y_k T \Delta_k + \Delta_k T Y_k) + O(\|\Delta_k\|^2).$$

It is easy to observe, using the method of mathematical induction, that $TY_k = Y_k T$. If we ignore the expression $O(\|\Delta_k\|^2)$ and if we suppose that $\Delta_k(TY_k) = (TY_k)\Delta_k$, it results that

$$\Delta_{k+1} = 2\Delta_k(\mathbf{I}_X - TY_k).$$

Since

$$\mathbf{I}_X - TY_k = \mathbf{I}_X - 2TY_{k-1} + (TY_{k-1})^2 = (\mathbf{I}_X - TY_{k-1})^2,$$

we deduce that

$$\Delta_{k+1} = 2\Delta_k(\mathbf{I}_X - TY_0)^{2^n} = 2\Delta_k \left(\mathbf{I}_X - \frac{1}{\|T\|^4}(T^*T)^2\right)^{2^n}.$$

Then

$$\|\Delta_{k+1}\| \le 2\|\Delta_k\| \left\|\mathbf{I}_X - \frac{1}{\|T\|^4}(T^*T)^2\right\|^{2^n} =$$

$$= 2\|\Delta_k\| \left(1 - \frac{1}{\|T\|^4} \inf_{\|x\|=1} \langle (T^*T)^2 x, x\rangle \right)^{2^n} =$$

But

$$\inf_{\|x\|=1} \langle (T^*T)^2 x, x\rangle = \begin{cases} 0 & \text{if } N(T) \neq \{0\} \\ \gamma & \text{if } N(T) = \{0\} \end{cases} .$$

Therefore, if $N(T) \neq 0$, then $\|\Delta_{k+1}\| = 2\|\Delta_k\|$ and if $N(T) = 0$ then $\|\Delta_{k+1}\| = 2\|\Delta_k\| \left(1 - \frac{\gamma}{\|T\|^4}\right)^{2^n} > \|\Delta_k\| \Leftrightarrow \gamma < \left(1 + \frac{1}{n}\right)\|T\|^4 \leq 2\|T\|^4$. In conclusion, in both cases the method is eventual unstable.

Acknowledgment. We are grateful to the referees for helpful comments concerning the paper.

References

[1] A. Ben-Israel, An iterative method for computing the generalized inverse of an arbitrary matrix. *Math. of Comput.* 19 (1965), 452-455.
[2] A. Ben-Israel A note on an iterative method for generalized inversion of matrices. *Math. of Comput.* 20 (1966), 439-440.
[3] A. Ben-Israel and T. Greville, *Generalized Inverses: Theory and Applications. Second Edition.* Springer-Verlag, New-York, 2003.
[4] D.A. Bini, G. Codevico, M. Van Barel, Solving Toeplitz least squares problems by means of Newton's iteration. *Numer. Algorithms.* 33 (2003), 93-103.
[5] S. Campbell and C. Meyer, *Generalized Inverses of Linear Transformations.* SIAM, Philadelphia, 2008.
[6] C.W. Groetsch, Generalized Inverses of Linear Operators: Representation and Approximation, Marcel Dekker, New York, 1977.
[7] V. Pan and R. Schreiber, An improved Newton iteration for the generalized inverse of a matrix, with applications. *SIAM J. Sci. and Stat. Comput.* 12 (2008), 1109-1130.
[8] V Pan, M. Van Barel, X. Wang, Gianni Codevico, Iterative inversion of structured matrices. *Theoret. Comput. Sci.* 315 (2004), 581-592.
[9] M. Petkovic, P. Stanimirovic, Iterative method for computing the Moore-Penrose inverse based on Penrose equations. *Journal of Comput. and Appl. Math.* 235 (2011), 1604-1613.
[10] F. Soleymani, P. S. Stanimirovic, Malik Zaka Ullah, An accelerated iterative method for computing weighted Moore-Penrose inverse. *Appl. Math. and Comput.* 222 (2013), 365-371.
[11] I. Stakgold, *Boundary value problem of Mathematical Physics. Vol I.* Macmillan, New-York, 1969.
[12] H. Wolkowicz and S. Zlobec, Calculating the best approximate solution of an operator equation. *Math. of Comput.* 32 (1978), 1183-1213.
[13] Y. Xue, *Stable Perturbations of Operators and Related Topics.* World Scientific, Singapore, 2012.