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Invariant subspace problem for ExB-Operators

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Abstract. In this paper, we show the following: Let A, B^* be ExB-operators on a complex Hilbert space $\mathcal{H}$. If there exists a non-zero compact operator K such that $AK = aKB + bK$ $(0 \neq a, b \in \mathbb{C})$, then A and B have a non-trivial invariant subspace.

1. Introduction

In this paper we denote infinite dimensional Hilbert space and Banach space by $\mathcal{H}$ and $\mathcal{X}$, respectively. Let $B(\mathcal{H})$ and $B(\mathcal{X})$ be the sets of all bounded linear operators on $\mathcal{H}$ and $\mathcal{X}$, respectively. Lauric in [3] showed the following nice result: Let A, B^* be hyponormal operators on a complex Hilbert space $\mathcal{H}$. If there exists a non-zero Hilbert-Schmidt operator K such that $AK = aKB + bK$ $(0 \neq a, b \in \mathbb{C})$, then A and B have a non-trivial invariant subspace. First we show that, for hyponormal operators A, B^*, there exists a non-zero operator $K \in C_p$ for some p $(1 \leq p < \infty)$ such that $AK = aKB + bK$ $(0 \neq a, b \in \mathbb{C})$, then A and B have a non-trivial invariant subspace, where C_p is a Schatten p-class. Next we introduce ExB-operators and show that, for ExB-operators A, B^*, if there exists a non-zero compact operator K such that $AK = aKB + bK$ $(0 \neq a, b \in \mathbb{C})$, then A and B have a non-trivial invariant subspace.

2. Hyponormal operators

In the case of Hilbert space operator $T \in B(\mathcal{H})$, T is said to be *hyponormal* if $T^*T - TT^* \geq 0$ (that is, $((T^*T - TT^*)x, x) \geq 0$ for every $x \in \mathcal{H}$), where T^* is the adjoint operator of T. The numerical range of T is denoted by $W(T)$, i.e.,

$$W(T) = \{(Tx, x) \ : \ \|x\| = 1\}.$$

In the case of Banach space operator $T \in B(\mathcal{X})$, the numerical range $V(T)$ of T is defined by

$$V(T) = \{f(Tx) \ : \ (x, f) \in \Pi(\mathcal{X})\},$$

where $\Pi(\mathcal{X}) = \{(x, f) \in \mathcal{X} \times \mathcal{X}^* \ : \ \|f\| = f(x) = \|x\| = 1\}$ and $\mathcal{X}^*$ is the dual space of $\mathcal{X}$.

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Definition 1. For an operator $T \in B(\mathcal{X})$, T is called *hermitian* if $V(T) \subset \mathbb{R}$.
T is said to be *hyponormal* if there exist hermitian operators H, K such that $T = H + iK$ and

$$V(\overline{T}T - T\overline{T}) \subset \mathbb{R}_+ = [0, \infty),$$

where $\overline{T} = H - iK$. Hence if $T \in B(\mathcal{H})$, then $\overline{T} = T^*$. See detail [1] and [2].

In the case of Banach space operator, there exist an hermitian operator H and an operator T such that H^2 is not hermitian and $T \neq A + iB$ for all hermitian operators A, B, respectively. See detail [5].

First we show the following theorem.

Theorem 1. *Let $A, B^* \in B(\mathcal{H})$ be hyponormal operators. If there exists a non-zero operator $K \in \mathcal{C}_p$ for some p $(1 \le p < \infty)$ such that $AK = aKB + bK$ $(0 \neq a, b \in \mathbb{C})$, then A and B have a non-trivial invariant subspace.*

For a proof of Theorem 1, the following results are essential. Mattila proved it for strictly c-convex spaces but it holds for strictly convex spaces.

Proposition 1 (Theorem 2.4, Mattila [4]).
Let a Banach space $\mathcal{X}$ be strictly convex and $T = H + iK \in B(\mathcal{X})$ be hyponormal. If $Tx = 0$ $(x \in \mathcal{X})$, then $Hx = Kx = 0$.

Proposition 2 (Corollary 1.3, Shaw [7]).
Let $A, B \in B(\mathcal{H})$ and Δ be $\Delta(T) = AT - TB$ $(T \in \mathcal{C}_p)$. Then Δ is an operator on a Banach space $\mathcal{C}_p$ and it holds

$$V(\Delta) \ \subset \ \overline{W(A)} - \overline{W(B)},$$

where $\overline{E}$ is the clousure of E.

Proof of Theorem 1. Since $\mathcal{C}_p \subset \mathcal{C}_q$ for $1 \le p < q < \infty$, we may assume $p > 1$, that is, a compact operator K belongs $\mathcal{C}_p$ $(p > 1)$. Then it's known that $\mathcal{C}_p$ is uniformly convex and hence strictly convex (See detail [6]). First we assume that A, B^* are hyponormal operators and $AK = KB$. Define $\Delta(T) = AT - TB$. Let $A = A_1 + iA_2, B = B_1 + iB_2$ be the Cartesian decompositions of A, B, respectively. Let

$$\Delta_1(T) = A_1T - TB_1, \ \ \Delta_2(T) = A_2T - TB_2.$$

By Proposition 2, $V(\Delta_i) \subset \overline{W(A_i)} - \overline{W(B_i)}$ $(i = 1, 2)$. Hence operators Δ_1 and Δ_2 are hermitian and $\Delta = \Delta_1 + i\Delta_2$. Since $(\overline{\Delta}\Delta - \Delta\overline{\Delta})(T) = (A^*A - AA^*)T - T(B^*B - BB^*)$, by Proposition 2 it holds

$$V(\overline{\Delta}\Delta - \Delta\overline{\Delta}) \subset \overline{W(A^*A - AA^*)} - \overline{W(B^*B - BB^*)}.$$

Since A, B^* are hyponormal, the operator Δ is hyponormal on $\mathcal{C}_p$. By the assumption it holds $\Delta(K) = 0$. Since $\mathcal{C}_p$ is strictly convex, by Proposition 1 we have

$$\Delta_1(K) = A_1K - KB_1 = 0 \ \text{ and } \ \Delta_2(K) = A_2K - KB_2 = 0.$$

Hence it holds $A_1K = KB_1$ and $A_2K = KB_2$ and hence $K^*A_1 = B_1K^*$ and $K^*A_2 = B_2K^*$. Therefore, $K^*A = BK^*$ and

$$KK^*A = KBK^* = AKK^*.$$

Hence, A commutes with non-zero compact hermitian operator KK^*. Since there exists a non-zero eigenvalue α of KK^* such that $|\alpha| = ||KK^*||$, $\ker(KK^* - \alpha)$ is a non-trivial invariant subspace for A.

Since $K^*KB = K^*AK = BK^*K$, B commutes with the compact hermitian operator K^*K and similarly B has a non-trivial invariant subspace.

Next in the case of $AK = aKB + bK$, let $B' = aB + bI$. Put $\Delta'(T) = AT - TB'$. Since B'^* is hyponormal, Δ' is hyponormal on $\mathcal{C}_p$ and $\Delta'(K) = AK - KB' = 0$. Hence by the same way A and B have a non-trivial invariant subspace. □

3. ExB-operators

Definition 2. Let $T \in B(\mathcal{X})$ have the Cartesian decomposition $T = H + iK$. An operator $T \in B(\mathcal{X})$ is said to be *ExB-operator* if there exists a positive number M such that

$$\|e^{z\overline{T}} \cdot e^{-\overline{z}T}\| \leq M \quad \text{for all } z \in \mathbb{C}.$$

In [5] Mattila defined a $*$-hyponormal operator T as follows: $T \in B(\mathcal{X})$ is said to be *$*$-hyponormal* if

$$\|e^{z\overline{T}} \cdot e^{-\overline{z}T}\| \leq 1 \quad \text{for all } z \in \mathbb{C}.$$

In [5] Mattila showed that if T is $*$-hyponormal, then T is hyponormal. The following implications

$$\text{normal} \Longrightarrow \text{subnormal} \Longrightarrow *\text{-hyponormal} \Longrightarrow \text{hyponormal}$$

hold. See detail [5]. If $A \in B(\mathcal{H})$, then

$$e^{z(aA+bI)^*} \cdot e^{-\overline{z}(aA+bI)} = e^{z\bar{b}-\overline{z\bar{b}}} \cdot e^{z\bar{a}A^*} \cdot e^{-\overline{z\bar{a}}A}.$$

Since $|e^{z\bar{b}-\overline{z\bar{b}}}| = 1$, it holds that if $A \in B(\mathcal{H})$ is an ExB-operator, then so is $aA + bI$ for every $a, b \in \mathbb{C}$ and it holds

$$\|e^{z\overline{T}}x\| \leq M \cdot \|e^{\overline{z}T}x\| \quad \text{for all } z \in \mathbb{C} \text{ and } x \in \mathcal{H}.$$

In this section we show the following theorem.

Theorem 2. *Let $A, B^* \in B(\mathcal{H})$ be ExB-operators. If there exists a non-zero compact operator K such that $AK = aKB + bK$ $(0 \neq a, b \in \mathbb{C})$, then A and B have a non-trivial invariant subspace.*

Let A, B be in $B(\mathcal{H})$ and $A = A_1 + iA_2$, $B = B_1 + iB_2$ be the Cartesian decompositions. Let $\Delta(T) = AT - TB$ $(T \in B(\mathcal{H}))$. Then $\Delta(T) = \Delta_1(T) + i\Delta_2(T)$ $(T \in B(\mathcal{H}))$, where $\Delta_1(T) = A_1T - TB_1$ and $\Delta_2(T) = A_2T - TB_2$. By Proposition 2, operators Δ_1 and Δ_2 are hermitian on $B(\mathcal{H})$, and $\overline{\Delta} = \Delta_1 - i\Delta_2$.

For a proof we prepare lemmas.

Lemma 1. *If $A, B^* \in B(\mathcal{H})$ be ExB-operators, then Δ is an ExB-operator on $B(\mathcal{H})$.*

Proof. For $A, B \in B(\mathcal{H})$, define operators L_A and R_B on $B(\mathcal{H})$ by

$$L_A(T) = AT, \quad R_B(T) = TB \quad (T \in B(\mathcal{H})).$$

Then $L_AR_B = R_BL_A$, $\Delta = L_A - R_B$ and $\overline{\Delta} = L_{A^*} - R_{B^*}$. Hence $\Delta(T) = (L_A - R_B)(T)$ and $\overline{\Delta}(T) = (L_{A^*} - R_{B^*})(T)$. Since L_A commutes with R_B, it holds

$$(1) \quad e^{\Delta} = e^{L_A - R_B} = e^{L_A} \cdot e^{-R_B} \quad \text{and} \quad e^{\overline{\Delta}} = e^{L_{A^*}} \cdot e^{-R_{B^*}}.$$

Since $A, B^* \in B(\mathcal{H})$ be ExB-operators, for some constants $M, N > 0$ let

$$(2) \quad \|e^{zA^*} \cdot e^{-\bar{z}A}\| \leq M \quad \text{and} \quad \|e^{\bar{z}B} \cdot e^{-zB^*}\| \leq N \quad \text{for all } z \in \mathbb{C}.$$

Hence by (1) it holds

$$e^{z\overline{\Delta}} \cdot e^{-\bar{z}\Delta}(T) = e^{zA^*} \cdot e^{-\bar{z}A} T e^{\bar{z}B} \cdot e^{-zB^*}.$$

Therefore by (2) since

$$\|e^{z\overline{\Delta}} \cdot e^{-\bar{z}\Delta}(T)\| = \|e^{zA^*} \cdot e^{-\bar{z}A} T e^{\bar{z}B} \cdot e^{-zB^*}\| \leq M \cdot N \cdot \|T\| \quad \text{for all } z \in \mathbb{C},$$

we have $\|e^{z\overline{\Delta}} \cdot e^{-\bar{z}\Delta}\| \leq M \cdot N$ $(\forall z \in \mathbb{C})$ and Δ is an ExB-operator on $B(\mathcal{H})$. □

Lemma 2. *If $T = H + iK \in B(\mathcal{X})$ be an ExB-operator and $Tx = 0$, then $\overline{T}x = Hx - iKx = 0$.*

Proof. For any $f \in \mathcal{X}^*$, let $g(z) = f(e^{z\overline{T}}x)$. Since $Tx = 0$, it holds $x = e^{-\bar{z}T}x$. Hence

$$|g(z)| = |f(e^{z\overline{T}} \cdot e^{-\bar{z}T}x)| \leq M \cdot \|f\| \cdot \|x\| \quad \text{for all } z \in \mathbb{C}.$$

Hence g is bounded and clearly is analytic. By Liouville's Theorem we have $g(z) = g(0)$ and $f(e^{z\overline{T}}x - x) = 0$. Since f is arbitrary, we have $e^{z\overline{T}}x = x$ and it easily follows $\overline{T}x = 0 = Hx - iKx$. □

Proof of Theorem 2. Since $aB^* + bI$ is an ExB-operator, we assume $AK - KB = 0$. Let $\Delta(T) = AT - TB$ and $A = A_1 + iA_2, B = B_1 + iB_2$ be the Cartesian decompositions of A, B, respectively. Then by Lemma 1 the operator Δ an ExB-operator on $B(\mathcal{H})$ and satisfies $\Delta(K) = \Delta_1(K) + i\Delta_2(K) = 0$. By Lemma 2 it holds $\Delta_1(K) = 0$ and $\Delta_2(K) = 0$. Hence we have $A^*K = KB^*$ and $K^*A = BK^*$. Since it holds $AK = KB$ and $K^*A = BK^*$, A and B have an invariant subspace by the same way of the proof of Theorem 1. □

Remark. Under the assumption of Theorems 1 or 2, since $\ker(KK^* - \alpha)$ is a finite dimensional invariant subspace of A, A has an eigen-value. Hence so is B.

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