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Decomposability of weighted composition operators on L^p of atomic measure space

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Abstract. In this paper, we discuss the decomposability of weighted composition operator uC_ϕ on $L^p(X)(1 \leq p < \infty)$ of a σ-finite atomic measure space $(X, \mathcal{S}, \mu)$ with the assumption that $u \in L^\infty(X)$ and $|u|$ has positive ess inf. We prove that if the analytic core of uC_ϕ is zero and uC_ϕ is not quasinilpotent, then it is not decomposable. We also show that if ϕ is either injective almost everywhere or surjective almost everywhere but not both, then uC_ϕ is not decomposable. Finally, we give a necessary condition for decomposability of uC_ϕ.

1. Introduction

In 1952 Nelson Dunford ([9], [10]) introduced the notion of the Single Valued Extension Property (abbreviated as SVEP) for bounded linear operators on Banach spaces. His idea of SVEP gave rise to evolution of the local spectral theory of bounded linear operators. The local spectral theory in its present form was shaped by Colojoara, Foias, Lange, Erdelyi, Laursen, Neumann and others (see [2], [3], [7]). It includes many properties such as Dunford's condition (C), Bishop's property (β), decomposition property (δ), decomposability etc. The SVEP is fundamental among these properties while properties (β) and (δ) are dual in nature. The decomposability of an operator is governed by both the properties (β) and (δ). For detailed study of the local spectral theory, we refer the reader to [2], [3], [7] and [11].

A composition operator is a bounded linear operator on the Banach space of functions on a set A, which is induced by a self- map ϕ on A. The theory of composition operators bridges the gap between operator theory and function theory. These operators also constitute diverse and illuminating examples in the framework of local spectral theory as mentioned by Laursen et al ([7]). The composition operators on the $L^p(X)(1 \leq p \leq \infty)$ spaces were extensively studied by Nordgren, Singh, Manhas and others. For further details about the composition operators, we refer to [1], [18]. The local spectral theory of composition operators on H^p spaces was studied by Shapiro [5] and Smith [16].

In this paper, we study the decomposability of the weighted composition operators on $L^p(X)(1 \leq p < \infty)$, where $(X, \mathcal{S}, \mu)$ is a σ-finite atomic measure space, with the assumption that the weight $u \in L^\infty(X)$ has ess

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$\inf |u| > 0$. Under these assumptions we establish the equality between the hyper-range and the analytic core of uC_ϕ and, with the help of this equality, we obtain a sufficient condition for non-decomposability of uC_ϕ. In addition, we show that if the self-map ϕ on X is either surjective almost everywhere or injective almost everywhere but not both, then uC_ϕ is not decomposable. We also give a necessary condition for the decomposability of uC_ϕ when ϕ is bijective almost everywhere.

Notations: Throughout the paper $\mathbb{N}, \mathbb{Z}$ and $\mathbb{C}$ denote the set of positive integers, the set of integers and the set of complex numbers respectively. The spaces $L^p(X)$ are considered for σ-finite atomic measure space $(X, \mathcal{S}, \mu)$ and $1 \leq p < \infty$. The symbol ϕ denotes a non-singular transformation of X into X and ϕ_n denotes the n-th iterate of ϕ. That is, $\phi_n = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}$. The notations $\sigma(T)$, $\sigma_p(T)$ and $\overline{B}$ respectively, are used to denote the spectrum, point spectrum of the operator T and closure of the set B. The symbols $f|A$ and χ_A denote the restriction of the function f on the set A and the characteristic function of the set A respectively.

2. Preliminaries

In this section we collect some definitions and basic results of the local spectral theory and composition operators. Most of these results can be found in [7], [11], [12], [18] and [19].

Let X be a complex Banach space and $\mathcal{B}(X)$ denote the Banach algebra of bounded linear operators on X.

Definition 2.1. *(1) An operator $T \in \mathcal{B}(X)$ is said to have the SVEP if for every open set $G \subseteq \mathbb{C}$, the only analytic solution $f : G \longrightarrow X$, of the equation $(\lambda - T)f(\lambda) = 0$, for all $\lambda \in G$, is the zero function on G.*

(2) An operator T is said to have Bishop's property (β) if for every open subset G of $\mathbb{C}$ and every sequence of analytic functions $f_n : G \longrightarrow X$ with the property that $(\lambda - T)f_n(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$, locally uniformly on G, then $f_n(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$, locally uniformly on G.

From the above definitions, it is easy to see that property (β) implies SVEP, and also, it is clear that an operator whose point spectrum has empty interior has SVEP. The localized versions of SVEP and property (β) were studied by [13] and [17] respectively. There is another property which lies in between (β) and SVEP, named as Dunford's property (C). The property (C) includes the idea of local spectral subspaces, which are defined as follows.

Definition 2.2. *For $x \in X$, the local resolvent of T at x, denoted by $\rho_T(x)$, is defined as the union of all open subsets G of $\mathbb{C}$ for which there is an analytic function $f : G \longrightarrow X$ satisfying $(\lambda - T)f(\lambda) = x$ for all $\lambda \in G$. The complement of $\rho_T(x)$ is called the local spectrum of T at x and is denoted by $\sigma_T(x)$. For a closed subset $F \subseteq \mathbb{C}$, the local spectral subspace of T, denoted by $X_T(F)$, is defined as $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$.*

An operator T is said to have Dunford's property (C) if $X_T(F)$ is closed for every closed subset F of $\mathbb{C}$.

Proposition 2.3. *If $T \in \mathcal{B}(X)$, then*

$$T \text{ has property } (\beta) \Rightarrow T \text{ has property } (C) \Rightarrow T \text{ has SVEP.}$$

For $T \in \mathcal{B}(X)$ and a closed set $F \subseteq \mathbb{C}$, the glocal spectral subspace $\mathcal{X}_T(F)$ is defined as the set of all $x \in X$ such that there is an analytic function $f : \mathbb{C} \setminus F \longrightarrow X$ satisfying $(\lambda - T)f(\lambda) = x$. It is well known that $\mathcal{X}_T(F) = X_T(F)$ if and only if T has SVEP.

Definition 2.4. *An operator T is said to have the decomposition property (δ) if for every open cover $\{U, V\}$ of $\mathbb{C}$, $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$.*

The properties (β) and (δ) are dual in nature. That is, if one of the T and T^* has property (β), then other has property (δ).

Definition 2.5. *An operator T is said to be decomposable if for every open cover $\{U, V\}$ of $\mathbb{C}$ there exist T-invariant closed subspaces Y and Z of X such that $\sigma(T|Y) \subseteq U$, $\sigma(T|Z) \subseteq V$ and $X = Y + Z$.*

Theorem 2.6. *Let $T \in \mathcal{B}(X)$, then T is decomposable if and only if T has both the properties (β) and (δ).*

Definition 2.7. *Let $\mathcal{S}$ be the $\sigma-$algebra of all Borel subsets of the complex plane $\mathbb{C}$. A map $E : \mathcal{S} \longrightarrow \mathcal{B}(X)$ is called a spectral measure if*

(1) $E(\emptyset)=0$,

(2) $E(\mathbb{C})=I$,

(3) $E(A \cap B)=E(A)E(B)$ for all $A, B \in \mathcal{S}$ and

(4) $E(\bigcup_{n=1}^{\infty} B_n)x = \sum_{n=1}^{\infty} E(B_n)x$ for every countable family of pairwise disjoint Borel sets B_n and for all $x \in X$.

An operator $T \in \mathcal{B}(X)$ is called a spectral operator if there exists a spectral measure E on $\mathcal{S}$ which satisfies

$$E(B)T = TE(B) \text{ and } \sigma(T|E(B)(X)) \subseteq \overline{B} \text{ for all } B \in \mathcal{S}.$$

Proposition 2.8. *Every spectral operator is decomposable.*

Definition 2.9. *Let $T \in \mathcal{B}(X)$. The analytic core of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(x_n)_{n=0}^{\infty}$ in X and a constant $\delta > 0$ such that*

(a) $x = x_0$, and $Tx_{n+1} = x_n$ for every $n \geq 0$,

(b) $\|x_n\| \leq \delta^n \|x\|$ for every $n \geq 0$.

It is well-known that $K(T)$ is a subspace of X and can be easily seen that $K(T) \subseteq T^{\infty}(X)$, where $T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X)$ is called the hyper-range of T. Following result establishes a connection between analytic core and local spectral subspace of an operator. The proof of this result can be found in [8] or [15].

Theorem 2.10. *Let $T \in \mathcal{B}(X)$, then for each $\lambda \in \mathbb{C}$, $K(\lambda I - T) = X_T(\mathbb{C} \setminus \{\lambda\})$.*

Definition 2.11. *Let A be a non-empty set and $V(A)$ denote the vector space of complex functions on A. If ϕ is a self-map on A, then ϕ induces a linear transformation C_ϕ from $V(A)$ to $V(A)$, defined as*

$$C_\phi f = f \circ \phi \text{ for all } f \in V(A).$$

If $V(A)$ is a Banach space and C_ϕ is bounded, we say that C_ϕ is a composition operator.

Definition 2.12. *Let $(X, \mathcal{S}, \mu)$ be a measure space. A measurable set E is called an atom if $\mu(E) \neq 0$ and for each measurable subset F of E either $\mu(F) = 0$ or $\mu(F) = \mu(E)$. A measure space $(X, \mathcal{S}, \mu)$ is called atomic if each measurable subset of non-zero measure contains an atom.*

Let $(X, \mathcal{S}, \mu)$ be a σ-finite atomic measure space. Then $X = \bigcup_{n=1}^{\infty} A_n$, where A_n's are disjoint atoms of finite measure [19]. These atoms are unique in the sense that if $X = \bigcup_{n=1}^{\infty} B_n$, where B_n's are disjoint atoms of finite measure, then $A_n = B_n$ up to a nullset for each $n \geq 1$. A measurable transformation $\phi : X \longrightarrow X$ is called non-singular if the measure $\mu\phi^{-1}$ is absolutely continuous with respect to μ. If ϕ is non-singular, then ϕ maps atoms into atoms. A non-singular transformation ϕ of X into X is called injective almost everywhere if the inverse image of every atom under ϕ contains at most one atom. It is called surjective almost everywhere if the inverse image of every atom under ϕ contains at least one atom. If ϕ is both injective and surjective almost everywhere, then it is called bijective almost everywhere. Also each $f \in L^p(X)$ $(1 \leq p \leq \infty)$ is constant almost everywhere on each atom.

Theorem 2.13. *Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space. A necessary and sufficient condition that a non-singular transformation ϕ of X into X induces a composition operator on $L^p(X)(1 \leq p < \infty)$ is that there is a $K > 0$ such that*

$$\mu\phi^{-1}(E) \leq K\mu(E) \text{ for each } E \in \mathcal{S}.$$

Let $u \in L^\infty(X)$ and ϕ be a non-singular transformation of X into X such that C_ϕ is a composition operator on $L^p(X)(1 \leq p < \infty)$, then weighted composition operator uC_ϕ on $L^p(X)(1 \leq p < \infty)$ is defined as

$$(uC_\phi)f = uf(\phi) \text{ for all } f \in L^p(X).$$

So a "weighted composition operator" is just the product M_uC_ϕ of a multiplication operator M_u and a composition operator C_ϕ.

3. Decomposability

In this section we take $L^p(X)(1 \leq p < \infty)$ of a σ-finite atomic measure space $(X, \mathcal{S}, \mu)$, where $X = \bigcup_{n=1}^{\infty} A_n$ and A_n's are disjoint atoms of finite measure. We start our main results with the following easy observations.

Proposition 3.1. *Let $u \in L^\infty(X)$ and uC_ϕ be a weighted composition operator on $L^p(X)$. If $u|A_n \to 0$, then uC_ϕ is decomposable.*

Proof. It can be easily seen that if $u|A_n \to 0$, then uC_ϕ is compact. Now the proof follows from the fact that every compact operator is decomposable. □

Proposition 3.2. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. Then range of uC_ϕ is given by*

$$(uC_\phi)(L^p) = \left\{ f \in L^p : \frac{f}{u}\Big|\phi^{-1}(A_n) \text{ is constant almost everywhere for each } n \geq 1 \right\}.$$

Proof. The proof is similar to that of [14, Theorem 2.1.1]. □

Proposition 3.3. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. Then range of uC_ϕ is closed.*

Proof. The proof is similar to that of [14, Theorem 2.1.2]. □

Proposition 3.4. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. Then the analytic core $K(uC_\phi) = (uC_\phi)^\infty(L^p(X))$, the hyper-range of uC_ϕ.*

Proof. The inclusion $K(uC_\phi) \subseteq (uC_\phi)^\infty(L^p(X))$ follows from the definition of $K(uC_\phi)$. Also from the preceding Proposition 3.2, we have

$$(uC_\phi)^k(L^p(X)) = \left\{ f \in L^p(X) : \frac{f}{u.(u \circ \phi) \ldots (u \circ \phi_{k-1})}\Big|\phi_k^{-1}(A_n) \text{ is constant for each } n \geq 1 \right\}.$$

Hence

$$\begin{aligned} (uC_\phi)^\infty(L^p(X)) &= \bigcap_{k=1}^{\infty} (uC_\phi)^k(L^p(X)) \\ &= \Big\{ f \in L^p(X) : \frac{f}{u.(u \circ \phi) \ldots (u \circ \phi_{k-1})}\Big|\phi_k^{-1}(A_n) \text{ is constant} \\ &\qquad \text{for each } n \geq 1 \text{ and for each } k \geq 1 \Big\}. \end{aligned}$$

Suppose that $f \in (uC_\phi)^\infty(L^p(X))$. Then for each $n \in \mathbb{N}$, define

$$f^{(n)}|A_m = \frac{f}{u.(u\circ\phi)\dots(u\circ\phi_{n-1})}\Big|\phi_n^{-1}(A_m) \text{ for each } m \geq 1.$$

(This $f^{(n)}$ has nothing to do with the n-th derivative.)
Since $\frac{f}{u.(u\circ\phi)\dots(u\circ\phi_{n-1})}|\phi_n^{-1}(A_m)$ is constant for each $n \geq 1$ and for each $k \geq 1$, therefore each $f^{(n)}|A_m$ is well-defined and $f^{(n)} \in L^p(X)$ for each $n \geq 1$. Now for $n \geq 0$,

$$\begin{aligned}(uC_\phi)f^{(n+1)} &= (uC_\phi)\sum_{m=1}^{\infty}(f^{(n+1)}|A_m)\chi_{A_m}\\ &= \sum_{m=1}^{\infty}\left(u|A_m\frac{f}{u.(u\circ\phi)\dots(u\circ\phi_n)}\Big|\phi_{n+1}^{-1}(\phi(A_m))\right)\chi_{A_m}.\end{aligned}$$

Note that $\phi_n^{-1}(A_m) \subseteq \phi_{n+1}^{-1}(\phi(A_m))$, therefore if $A_k \subseteq \phi_n^{-1}(A_m)$, then $\phi_n(A_k) = A_m$. Since $u|A_m\frac{f}{u.(u\circ\phi)\dots(u\circ\phi_n)}|\phi_{n+1}^{-1}(\phi(A_m))$ is constant, so we get

$$\begin{aligned}u|A_m\frac{f}{u.(u\circ\phi)\dots(u\circ\phi_n)}\Big|\phi_{n+1}^{-1}(\phi(A_m)) &= \frac{u|A_m.f|A_k}{u|A_k.u|\phi(A_k)\dots u|\phi_n(A_k)}\\ &= \frac{u|A_m.f|A_k}{u|A_k.u|\phi(A_k)\dots u|A_m}\\ &= \frac{f|A_k}{u|A_k.u|\phi(A_k)\dots u|\phi_{n-1}(A_k)}\\ &= \frac{f}{u.(u\circ\phi)\dots(u\circ\phi_{n-1})}\Big|\phi_n^{-1}(A_m)\\ &= f^{(n)}|A_m.\end{aligned}$$

Thus $(uC_\phi)f^{(n+1)} = f^{(n)}$ for each $n \geq 0$, where $f^{(0)} = f$. Further, for any $n \geq 0$,

$$\begin{aligned}\|f^{(n)}\|^p &= \sum_{k=1}^{\infty}\left\|f^{(n)}|A_k\right\|^p\\ &= \sum_{k=1}^{\infty}\left\|\frac{f}{u.(u\circ\phi)\dots(u\circ\phi_{n-1})}\Big|\phi_n^{-1}(A_k)\right\|^p\\ &\leq \frac{1}{\alpha^{np}}\sum_{k=1}^{\infty}\left\|f|\phi_n^{-1}(A_k)\right\|^p,\ \alpha = ess\ inf\ |u|\\ &= \frac{1}{\alpha^{np}}\sum_{k=1}^{\infty}\left(\sum_{A_m\subseteq\phi_n^{-1}(A_k)}\Big|f|A_m\Big|^p\mu(A_m)\right)\\ &= \frac{\|f\|^p}{\alpha^{np}}.\end{aligned}$$

That is, $\|f^{(n)}\| \leq \frac{1}{\alpha^n}\|f\|$. Now taking $\delta = \frac{1}{\alpha}$, the sequence $(f^{(n)})_{n=0}^{\infty}$ satisfies all the conditions of $K(uC_\phi)$. Consequently, $(uC_\phi)^\infty(L^p(X)) \subseteq K(uC_\phi)$. Hence, $K(uC_\phi) = (uC_\phi)^\infty(L^p(X))$. □

Corollary 3.5. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. Then the analytic core of uC_ϕ on $L^p(X)$ is closed.*

Proof. The proof is evident from the above proposition and the foregoing Proposition 3.3. □

Corollary 3.6. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. If ϕ is injective almost everywhere, then $K(uC_\phi) = L^p(X)$.*

Proof. If ϕ is injective almost everywhere and ess inf $|u| > 0$, then uC_ϕ is surjective. Therefore from above Proposition 3.4, $K(uC_\phi) = L^p(X)$. □

The proof of next lemma can be found in [20]. We state it for the sake of completeness.

Lemma 3.7. *Let X be a complex Banach space and $T \in \mathcal{B}(X)$. Suppose that $\sigma(T)$ is not a singleton and $\bigcap_{\substack{x\in X\\ x\neq 0}} \sigma_T(x) \neq \emptyset$, then T has SVEP but does not have decomposition property (δ) and hence, T is not decomposable.*

Corollary 3.8. *Let X be a complex Banach space and $T \in \mathcal{B}(X)$. Suppose that $\sigma(T)$ is not a singleton and $K(T) = \{0\}$. Then T has SVEP but does not have decomposition property (δ).*

Proof. The proof follows from Theorem 2.10 and Lemma 3.7 above. □

Proposition 3.9. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. If uC_ϕ is not quasinilpotent and $K(uC_\phi) = \{0\}$, then uC_ϕ has SVEP but does not have decomposition property (δ).*

Proof. If $K(uC_\phi) = \{0\}$, then from Theorem 2.10 above, we get $L^p_{uC_\phi}(X)(\mathbb{C}\setminus\{0\}) = \{0\}$. That is, $0 \in \bigcap_{\substack{x\in L^p(X)\\ x\neq 0}} \sigma_{uC_\phi}(x)$. Also, since uC_ϕ is not quasinilpotent, therefore $\sigma(uC_\phi)$ is not a singleton. Hence, from Lemma 3.7, it follows that uC_ϕ has SVEP but does not have decomposition property (δ). □

Proposition 3.10. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. If ϕ is injective but is not surjective almost everywhere, then uC_ϕ does not have SVEP.*

Proof. Let $n_0 \in \mathbb{N}$ be such that $\phi^{-1}(A_{n_0}) = \emptyset$. For each positive integer k, put $A_{n_k} = \phi_k(A_{n_0})$. Since ϕ is injective almost everywhere, all A_{n_k}'s are disjoint. Suppose that $\alpha = ess\ inf\ |u|$. Let $G = \{\lambda \in \mathbb{C} : |\lambda| < \alpha\}$. Now define a map $f : G \longrightarrow L^p(X)$ by $f(\lambda) = f_\lambda$, where

$$f_\lambda|A_n = \begin{cases} 1, & \text{if } n = n_0 \\ \frac{\lambda^k}{u(u\circ\phi)\ldots(u\circ\phi_{k-1})}\Big|A_{n_0}, & \text{if } n = n_k, k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then for each $\lambda \in G$,

$$\begin{aligned} (\lambda - uC_\phi)f_\lambda &= (\lambda - uC_\phi)\sum_{k=0}^{\infty}(f_\lambda|A_{n_k})\chi_{A_{n_k}} \\ &= \lambda\chi_{A_{n_0}} + \sum_{k=1}^{\infty}\left(\frac{\lambda^{k+1}}{u(u\circ\phi)\ldots(u\circ\phi_{k-1})}\Big|A_{n_0}\right)\chi_{A_{n_k}} \\ &\quad -\lambda\chi_{A_{n_0}} - \sum_{k=1}^{\infty}(u|A_{n_k})(f_\lambda|A_{n_{k+1}})\chi_{A_{n_k}} \\ &= \sum_{k=1}^{\infty}\left(\frac{\lambda^{k+1}}{u(u\circ\phi)\ldots(u\circ\phi_{k-1})}\Big|A_{n_0}\right)\chi_{A_{n_k}} \\ &\quad - \sum_{k=1}^{\infty}\left((u|A_{n_k})\frac{\lambda^{k+1}}{u(u\circ\phi)\ldots(u\circ\phi_k)}\Big|A_{n_0}\right)\chi_{A_{n_k}} \\ &= 0. \end{aligned}$$

Clearly, f is a non-zero analytic function. Hence uC_ϕ does not have SVEP. □

The following corollary easily follows from above proposition.

Corollary 3.11. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. If ϕ is injective but is not surjective almost everywhere, then $\{\lambda \in \mathbb{C} : |\lambda| \leq \alpha\} \subseteq \sigma_p(uC_\phi)$, where $\alpha = ess\ inf\ |u|$.*

Proposition 3.12. *Let $u \in L^\infty(X)$ and uC_ϕ be a weighted composition operator on $L^p(X)$. If there is a positive integer N such that $\phi_N(A_n) = A_n$ up to a null set for all $n \geq 1$, then uC_ϕ is decomposable.*

Proof. Note that if $\phi_N(A_n) = A_n$ up to a null set for all $n \geq 1$, then $(uC_\phi)^N$ is a multiplication operator induced by the function $v = u(u \circ \phi)(u \circ \phi_2) \dots (u \circ \phi_{N-1})$. As observed by Rho and Yoo ([4], Example1), the multiplication operator M_v, induced by the function v, is spectral. In fact, the spectral measure E is given by

$$E(B) = M_{\chi_B \circ v} \text{ for all Borel sets } B \text{ of } \mathbb{C}.$$

Hence $(uC_\phi)^N = M_v$ is decomposable. If U is any open disk containing $\sigma(uC_\phi)$, then $f : U \longrightarrow \mathbb{C}$, defined as

$$f(\lambda) = \lambda^N \text{ for all } \lambda \in U,$$

is non-constant analytic function. Hence by [7, Theorem 3.3.9], uC_ϕ is decomposable. □

If ϕ is bijective and $(A_{n_k})_{k=-\infty}^{\infty}$ is a cycle of infinite length with $\phi(A_{n_k}) = A_{n_{k+1}}$, we put $a_p = \liminf_{k\to\infty} \mu(A_{n_k})^{\frac{1}{kp}}$ and $b_p = \liminf_{k\to\infty} \mu(A_{n_{-k}})^{\frac{1}{kp}}$. In view of Theroem 2.13, it is easy to see that $a_p > 0$ and $b_p < \infty$. Henceforth, we assume that $a_p < \infty$ and $b_p > 0$.

Proposition 3.13. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. Suppose that ϕ is bijective almost everywhere. If each atom lies in a cycle of finite length, then uC_ϕ has SVEP. If there is an atom A_{n_0} which lies in a cycle of infinite length, say $(A_{n_k})_{k=-\infty}^{\infty}$ with $\phi(A_{n_k}) = A_{n_{k+1}}$, then uC_ϕ has SVEP if and only if*

$$\frac{1}{a_p} \liminf_{k\to\infty} \left| u(u \circ \phi) \dots (u \circ \phi_{k-1}) | A_{n_0} \right|^{\frac{1}{k}} \leq b_p \limsup_{k\to\infty} \left| (u \circ \phi^{-1})(u \circ \phi_2^{-1}) \dots (u \circ \phi_k^{-1}) | A_{n_0} \right|^{\frac{1}{k}}$$

for each such A_{n_0}.

Proof. Suppose that each atom lies in a cycle of finite length. For each $k \geq 1$, put

$$B_k = \{\lambda \in \mathbb{C} : \lambda^k = (u|A_{n_1})(u|A_{n_2}) \dots (u|A_{n_k}) \text{ for } n_1, n_2, \dots, n_k \in \mathbb{N}\}.$$

Then B_k is countable for each $k \geq 1$. Let $\lambda \in \sigma_p(uC_\phi)$. Then

$$(\lambda - uC_\phi)f = 0 \text{ for some non-zero } f \in L^p(X). \tag{1}$$

Hence there is an atom A_{n_1} such that $f|A_{n_1} \neq 0$. Suppose that A_{n_1} lies in a cycle $(A_{n_1}, A_{n_2}, \dots, A_{n_k})$ of length k. Then from (1) we have,

$$\begin{aligned}
\lambda f|A_{n_1} &= (u|A_{n_1})(f|A_{n_2}) \\
\lambda f|A_{n_2} &= (u|A_{n_2})(f|A_{n_3}) \\
&\dots \\
\lambda f|A_{n_{k-1}} &= (u|A_{n_{k-1}})(f|A_{n_k}) \\
\lambda f|A_{n_k} &= (u|A_{n_k})(f|A_{n_1}).
\end{aligned}$$

Thus, $(\lambda^k - (u|A_{n_1})(u|A_{n_2}) \dots (u|A_{n_k}))f|A_{n_1} = 0$. Since $f|A_{n_1} \neq 0$, therefore $\lambda^k = (u|A_{n_1})(u|A_{n_2}) \dots (u|A_{n_k})$. That is, $\lambda \in B_k$. Hence $\sigma_p(uC_\phi) \subseteq \bigcup_{k=1}^{\infty} B_k$. Thus, in this case, the eigenvalues of uC_ϕ are countable and consequently uC_ϕ has SVEP.

Suppose that there is an atom A_{n_0} which lies in a cycle of infinite length, say $(A_{n_k})_{k=-\infty}^{\infty}$ with $\phi(A_{n_k}) = A_{n_{k+1}}$. Let C denote the collection of all such atoms. Let $\lambda \in \sigma_p(uC_\phi)$. Then there is a non-zero $f \in L^p(X)$ such that

$$(\lambda - uC_\phi)f = 0. \tag{2}$$

Since $f \neq 0$, there is an atom A_m such that $f|A_m \neq 0$. If A_m lies in a cycle of finite length, then, as above, $\lambda \in \bigcup_{k=1}^{\infty} B_k$. If A_m lies in a cycle of infinite length, then without loss of generality we may assume that $A_m = A_{n_0}$. Then from (2) we have

$$\begin{array}{rcl} & \cdots & \\ \lambda f|A_{n_{-1}} & = & (u|A_{n_{-1}})(f|A_{n_0}) \\ \lambda f|A_{n_0} & = & (u|A_{n_0})(f|A_{n_1}) \\ \lambda f|A_{n_1} & = & (u|A_{n_1})(f|A_{n_2}) \\ & \cdots & \end{array}$$

Solving above we get

$$f|A_{n_k} = \frac{\lambda^k f}{u(u\circ\phi)\ldots(u\circ\phi_{k-1})}\Big|A_{n_0} \text{ for each } k \geq 1 \tag{3}$$

and

$$f|A_{n_{-k}} = \frac{(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})f}{\lambda^k}\Big|A_{n_0} \text{ for each } k \geq 1. \tag{4}$$

Also, we must have $\sum_{k=0}^{\infty} \left|f|A_{n_k}\right|^p \mu(A_{n_k}) < \infty$, which gives that $\sum_{k=0}^{\infty} \left|\frac{\lambda^k f}{u(u\circ\phi)\ldots(u\circ\phi_{k-1})}\Big|A_{n_0}\right|^p \mu(A_{n_k}) < \infty$. The radius of convergence R_1 of the series

$$\sum_{k=0}^{\infty} \left(\frac{\lambda^k f}{u(u\circ\phi)\ldots(u\circ\phi_{k-1})}\Big|A_{n_0}\right)^p \mu(A_{n_k})$$

is given by,

$$\begin{array}{rcl} \frac{1}{R_1} & = & \limsup_{k\to\infty} \left|\frac{f}{u(u\circ\phi)\ldots(u\circ\phi_{k-1})}\Big|A_{n_0}\right|^{\frac{1}{k}} \mu(A_{n_k})^{\frac{1}{kp}} \\ & \geq & \limsup_{k\to\infty} \left|\frac{1}{u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}}\right|^{\frac{1}{k}} \liminf_{k\to\infty} \mu(A_{n_k})^{\frac{1}{kp}} \\ & = & \frac{a_p}{\liminf_{k\to\infty} \left|u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}\right|^{\frac{1}{k}}}. \end{array}$$

Hence $|\lambda| \leq \frac{1}{a_p} \liminf_{k\to\infty} \left|u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}\right|^{\frac{1}{k}}$. Again, since $\sum_{k=1}^{\infty} \left|f|A_{n_{-k}}\right|^p \mu(A_{n_{-k}}) < \infty$ or $\sum_{k=1}^{\infty} \left|\frac{(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})}{\lambda^k}\Big|A_{n_0}\right|^p \mu(A_{n_{-k}}) < \infty$, therefore, the radius of convergence R_2 of the series

$$\sum_{k=1}^{\infty} \left(\frac{(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})}{\lambda^k}\Big|A_{n_0}\right)^p \mu(A_{n_{-k}})$$

is given by, $R_2 \leq \frac{1}{b_p}\frac{1}{\limsup\limits_{k\to\infty}\left|(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})|A_{n_0}\right|^{\frac{1}{k}}}$. Hence $|\lambda| \geq b_p\limsup\limits_{k\to\infty}\left|(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})|A_{n_0}\right|^{\frac{1}{k}}$. Thus

$$\begin{aligned}\sigma_p(uC_\phi) \quad \subseteq \quad & \bigcup_{A_{n_0}\in C}\Big\{\lambda : b_p\limsup_{k\to\infty}\left|(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})|A_{n_0}\right|^{\frac{1}{k}} \leq |\lambda| \leq \\ & \frac{1}{a_p}\liminf_{k\to\infty}\left|u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}\right|^{\frac{1}{k}}\Big\} \cup (\bigcup_{k=1}^{\infty}B_k).\end{aligned}$$

Now if $\frac{1}{a_p}\liminf\limits_{k\to\infty}\left|u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}\right|^{\frac{1}{k}} \leq b_p\limsup\limits_{k\to\infty}\left|(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})|A_{n_0}\right|^{\frac{1}{k}}$ for each $A_{n_0}\in C$, then $\sigma_p(uC_\phi)$ has empty interior and hence uC_ϕ has SVEP.

Conversely, suppose that there is an atom A_{n_0} lying in a cycle $(A_{n_k})_{k=-\infty}^{\infty}$ of infinite length for which

$$b_p\limsup_{k\to\infty}\left|(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})|A_{n_0}\right|^{\frac{1}{k}} < \frac{1}{a_p}\liminf_{k\to\infty}\left|u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}\right|^{\frac{1}{k}}.$$

Put $G = \Big\{\lambda : b_p\limsup\limits_{k\to\infty}\left|(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})|A_{n_0}\right|^{\frac{1}{k}} < |\lambda| < \frac{1}{a_p}\liminf\limits_{k\to\infty}\left|u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}\right|^{\frac{1}{k}}\Big\}$ and define $f : G \longrightarrow L^p(X)$ by $f(\lambda) = f_\lambda$ for each $\lambda\in G$, where

$$f_\lambda|A_n = \begin{cases} 1, & \text{if } n = n_0 \\ \frac{\lambda^k}{u(u\circ\phi)\ldots(u\circ\phi_{k-1})}\Big|A_{n_0}, & \text{if } n = n_k, k\geq 1, \\ \frac{(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})}{\lambda^k}\Big|A_{n_0}, & \text{if } n = n_{-k}, k\geq 1 \\ 0, & \text{otherwise.}\end{cases}$$

Then it is easy to see that $f(\lambda)$ is a non-zero analytic function which satisfies $(\lambda - uC_\phi)f(\lambda) = 0$ for each $\lambda\in G$. Hence uC_ϕ does not have SVEP. □

Lemma 3.14. *Let $u\in L^\infty(X)$ and uC_ϕ be a weighted composition operator on $L^p(X)(1<p<\infty)$. Then $(uC_\phi)^*$: $L^q(X)\longrightarrow L^q(X)$ $(\frac{1}{p}+\frac{1}{q}=1)$ is given by*

$$(uC_\phi)^*\sum_{n=1}^{\infty}(f|A_n)\chi_{A_n} = \sum_{n=1}^{\infty}\left(\sum_{A_k\in\phi^{-1}(A_n)}(u|A_k)(f|A_k)\right)\chi_{A_n}$$

for each $f\in L^q(X)$.

Proof. Let $f\in L^q(X)$. Then for each $g\in L^p(X)$, we have

$$\begin{aligned}\left((uC_\phi)^*\sum_{n=1}^{\infty}(f|A_n)\chi_{A_n}\right)\sum_{n=1}^{\infty}(g|A_n)\chi_{A_n} \quad &= \quad \sum_{n=1}^{\infty}(f|A_n)\chi_{A_n}(uC_\phi)\sum_{n=1}^{\infty}(g|A_n)\chi_{A_n} \\ &= \quad \sum_{n=1}^{\infty}(f|A_n)(u|A_n)(g|\phi(A_n)) \\ &= \quad \left(\sum_{n=1}^{\infty}\Big(\sum_{A_k\in\phi^{-1}(A_n)}(u|A_k)(f|A_k)\Big)\chi_{A_n}\right) \\ &\qquad \left(\sum_{n=1}^{\infty}(g|A_n)\chi_{A_n}\right).\end{aligned}$$

Therefore, $(uC_\phi)^* \sum_{n=1}^{\infty}(f|A_n)\chi_{A_n} = \sum_{n=1}^{\infty}\left(\sum_{A_k\in\phi^{-1}(A_n)}(u|A_k)(f|A_k)\right)\chi_{A_n}$. □

Proposition 3.15. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)(1 < p < \infty)$. Suppose that ϕ is bijective almost everywhere. If each atom lies in a cycle of finite length, then $(uC_\phi)^*$ has SVEP. If there is an atom A_{n_0} which lies in a cycle of infinite length, say $(A_{n_k})_{k=-\infty}^{\infty}$ with $\phi(A_{n_k}) = A_{n_{k+1}}$, then $(uC_\phi)^*$ has SVEP if and only if*

$$\frac{1}{b_q}\liminf_{k\to\infty}\left|(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})|A_{n_0}\right|^{\frac{1}{k}} \le a_q \limsup_{k\to\infty}\left|u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}\right|^{\frac{1}{k}}$$

for each such A_{n_0}.

Proof. From Lemma 3.14 above, $(uC_\phi)^* : L^q(X) \longrightarrow L^q(X)$ takes the form

$$(uC_\phi)^* \sum_{n=1}^{\infty}(f|A_n)\chi_{A_n} = \sum_{n=1}^{\infty}(u|\phi^{-1}(A_n))(f|\phi^{-1}(A_n))\chi_{A_n}$$

for each $f \in L^q(X)$. Now rest of the proof is similar to that of Proposition 3.13. □

Theorem 3.16. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)(1 < p < \infty)$. Suppose that ϕ is bijective almost everywhere. If uC_ϕ is decomposable, then for each cycle $(A_{n_k})_{k=-\infty}^{\infty}$ of infinite length, we have*

$$\frac{1}{a_p}\liminf_{k\to\infty}\left|u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}\right|^{\frac{1}{k}} \le b_p \limsup_{k\to\infty}\left|(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})|A_{n_0}\right|^{\frac{1}{k}}$$

and

$$\frac{1}{b_q}\liminf_{k\to\infty}\left|(u\circ\phi^{-1})(u\circ\phi_2^{-1})\ldots(u\circ\phi_k^{-1})|A_{n_0}\right|^{\frac{1}{k}} \le a_q \limsup_{k\to\infty}\left|u(u\circ\phi)\ldots(u\circ\phi_{k-1})|A_{n_0}\right|^{\frac{1}{k}}.$$

Proof. Combining Proposition 3.13 and Proposition 3.15 and using the fact that an operator T is decomposable if and only if both T and T^* have Bishop's property (β) , we get the proof. □

The conditions in Theorem 3.16 are not sufficient. For example, let $L^p(X) = l^p(\mathbb{N})$, the sequence spaces, and $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ be defined as

$$\begin{aligned}\phi(1) &= 2\\ \phi(2n) &= 2(n+1) \text{ for each } n \ge 1\\ \phi(2n+1) &= 2n-1 \text{ for each } n \ge 1.\end{aligned}$$

Taking $n_0 = 1$, we get $n_k = 2k$ and $n_{-k} = 2k+1$ for each $k \ge 1$. Now $u = (u_{n_k})_{k=-\infty}^{\infty} \in l^\infty(\mathbb{N})$ is defined as

$$\begin{aligned}u_{n_{-1}}u_{n_{-2}}\ldots u_{n_{-k}} &= \frac{1}{2^m}, \text{ if } m \le k < 2m,\ m \in \mathbb{N} \text{ and}\\ u_{n_k} &= \eta \text{ for each } k \ge 0, \text{where } \eta \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right).\end{aligned}$$

Then $a_p = a_q = b_p = b_q = 1$, $\limsup_{k\to\infty}|u_{n_{-1}}u_{n_{-2}}\ldots u_{n_{-k}}|^{\frac{1}{k}} = \frac{1}{\sqrt{2}}$ and $\liminf_{k\to\infty}|u_{n_{-1}}u_{n_{-2}}\ldots u_{n_{-k}}|^{\frac{1}{k}} = \frac{1}{2}$. Hence both the conditions of Theorem 3.16 are satisfied. Since, for $\chi_{\{n_0\}} \in l^p(\mathbb{N})$, $\|(uC_\phi)^k\chi_{\{n_0\}}\|^{\frac{1}{k}} = |u_{n_{-1}}u_{n_{-2}}\ldots u_{n_{-k}}|^{\frac{1}{k}}$ does not converge, therefore from [21, Proposition 1.5], uC_ϕ does not have property (β) and hence it is not decomposable.

Let $(X, \mathcal{S}, \mu)$ be a σ-finite atomic measure space and $\phi : X \longrightarrow X$ be surjective almost everywhere. For each atom A_n, put

$$A_{n_k} = \phi_k(A_n) \text{ for each } k \geq 0,$$

where $\phi_0(A_n) = A_{n_0} = A_n$. For $k < 0$, let $A_{n_{-k}}$ be an atom in $\phi_k^{-1}(A_n)$. Now define the set E as

$$E = \{A_n : A_{n_k}\text{'s are disjoint for all } k \in \mathbb{Z}\}.$$

Now we have the following result.

Theorem 3.17. *Let $u \in L^\infty(X)$ with ess inf $|u| > 0$ and uC_ϕ be a weighted composition operator on $L^p(X)$. Suppose that ϕ is surjective almost everywhere but is not injective almost everywhere. Then uC_ϕ does not have decomposition property (δ). Further, if*

$$\frac{1}{a_p} \liminf_{k\to\infty} \left|(u|A_{n_0})(u|A_{n_1}) \ldots (u|A_{n_{k-1}})\right|^{\frac{1}{k}} < b_p \limsup_{k\to\infty} \left|(u|A_{n_{-1}})(u|A_{n_{-2}}) \ldots (u|A_{n_{-k}})\right|^{\frac{1}{k}} \tag{5}$$

for each $A_n \in E$, then uC_ϕ has SVEP.

Proof. Since ϕ is surjective almost everywhere, $ess\ inf\ |u| > 0$ and range of uC_ϕ is closed, therefore $(uC_\phi)^*$ is surjective. But $(uC_\phi)^*$ can not have SVEP. For, if it has SVEP, it would imply that $(uC_\phi)^*$ is invertible [6]. This contradicts the fact that uC_ϕ is not invertible. Thus uC_ϕ does not have decomposition property (δ).

Further, assume that uC_ϕ does not have SVEP. Then there exist an open set G and a non-zero analytic function $f : G \longrightarrow L^p(X)$, defined as

$$f(\lambda) = f_\lambda \text{ for each } \lambda \in G$$

which satisfies $(\lambda - uC_\phi)f(\lambda) = 0$ for each $\lambda \in G$. Without loss of generality, we may assume that f is never zero on G. Choose $\lambda_0 \in G$ such that $\lambda_0 \notin \bigcup_{k=1}^{\infty} B_k \cup \{0\}$, where B_k's are as defined in Proposition 3.13. Since $f(\lambda_0) \neq 0$, therefore there is an atom A_{n_0} such that $f_{\lambda_0}|A_{n_0} \neq 0$. We claim that $A_{n_0} \in E$. For, if $A_{n_0} \notin E$, then there are distinct integers i and j such that $A_{n_i} = A_{n_j}$. Let $i < j$. Then $i + k = j$ for some $k > 0$. Since $(\lambda_0 - uC_\phi)f_{\lambda_0} = 0$, therefore we have

$$\begin{aligned}
\lambda_0 f_{\lambda_0}|A_{n_i} &= (u|A_{n_i})(f_{\lambda_0}|A_{n_{i+1}}) \\
&\ldots \\
\lambda_0 f_{\lambda_0}|A_{n_{i+k-1}} &= (u|A_{n_{i+k-1}})(f_{\lambda_0}|A_{n_{i+k}}) \\
&= (u|A_{n_{i+k-1}})(f_{\lambda_0}|A_{n_j}) \\
&= (u|A_{n_{i+k-1}})(f_{\lambda_0}|A_{n_i})
\end{aligned}$$

which gives $\left(\lambda_0^k - (u|A_{n_i})(u|A_{n_{i+1}}) \ldots (u|A_{n_{i+k-1}})\right) f_{\lambda_0}|A_{n_i} = 0$. Since $\lambda_0 \notin \bigcup_{k=1}^{\infty} B_k$, so $f_{\lambda_0}|A_{n_i} = 0$. Again,

$$\begin{aligned}
\lambda_0 f_{\lambda_0}|A_{n_0} &= (u|A_{n_0})(f_{\lambda_0}|A_{n_1}) \\
&\ldots \\
\lambda_0 f_{\lambda_0}|A_{n_{i-1}} &= (u|A_{n_{i-1}})(f_{\lambda_0}|A_{n_i}) = 0.
\end{aligned}$$

Since $\lambda_0 \neq 0$, therefore by backward substitution, we get $f_{\lambda_0}|A_{n_0} = 0$, which is a contradiction. Hence $A_{n_0} \in E$. Now using $(\lambda_0 - uC_\phi)f_{\lambda_0} = 0$, we get

$$f_{\lambda_0}|A_{n_k} = \frac{\lambda_0^k f_{\lambda_0}|A_{n_0}}{(u|A_{n_0})(u|A_{n_1}) \ldots (u|A_{n_{k-1}})}$$

and

$$f_{\lambda_0}|A_{n_{-k}} = \frac{(u|A_{n_{-1}})(u|A_{n_{-2}})\dots(u|A_{n_{-k}}f_{\lambda_0}|A_{n_0})}{\lambda_0^k}$$

for each $k \geq 1$. Now following the steps of Proposition 3.13, we get

$$b_p \limsup_{k\to\infty} |(u|A_{n_{-1}})(u|A_{n_{-2}})\dots(u|A_{n_{-k}})|^{\frac{1}{k}} \leq |\lambda_0| \leq \frac{1}{a_p} \liminf_{k\to\infty} |(u|A_{n_0})(u|A_{n_1})\dots(u|A_{n_{k-1}})|^{\frac{1}{k}}.$$

That is,

$$b_p \limsup_{k\to\infty} |(u|A_{n_{-1}})(u|A_{n_{-2}})\dots(u|A_{n_{-k}})|^{\frac{1}{k}} \leq \frac{1}{a_p} \liminf_{k\to\infty} |(u|A_{n_0})(u|A_{n_1})\dots(u|A_{n_{k-1}})|^{\frac{1}{k}}.$$

Thus condition (5) implies that uC_ϕ has SVEP. □

4. Examples

Now we give examples which support preceding results.

Example 4.1. *Let $X = \mathbb{N}$, $\mathcal{S}$ = the power set of $\mathbb{N}$ and μ = counting measure. Let $u_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$ and define $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ as*

$$\phi(1) = \phi(2) = \phi(3) = 2$$

and

$$\phi(n) = n - 2 \text{ for all } n \geq 4.$$

Then it is easy to see that $\lim_{k\to\infty} \phi_k^{-1}(2) = \mathbb{N}$ and consequently, $(uC_\phi)^\infty(L^p(X)) = K(uC_\phi) = \{0\}$. Hence from Proposition 3.9, uC_ϕ does not have decomposition property (δ).

Example 4.2. *Let $X = \mathbb{N}$, $\mathcal{S}$ = the power set of $\mathbb{N}$ and μ = counting measure. Let $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ be defined as*

$$\phi(1) = 2,$$

$$\phi(2n) = 2n + 2 \text{ and } \phi(2n+1) = 2n - 1 \text{ for all } n \in \mathbb{N}.$$

Let $n_0 = 1$ and $(u_n)_{n=1}^\infty$ is given as

$$u_n = \begin{cases} 1 + \frac{1}{n}, & \text{if } n \text{ is odd,} \\ 2 + \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

An easy calculation shows that $\limsup_{k\to\infty} |u_{n_{-1}} u_{n_{-2}} \dots u_{n_{-k}}|^{\frac{1}{k}} = 1$, whereas $\liminf_{k\to\infty} |u_{n_0} u_{n_1} \dots u_{n_{k-1}}|^{\frac{1}{k}} = 2$. Therefore, by Theorem 3.16, uC_ϕ is not decomposable.

In case when ϕ is neither injective nor surjective almost everywhere, uC_ϕ may or may not be decomposable. The following examples illustrate our statement.

Example 4.3. *Let $X = \mathbb{N}$, $\mathcal{S}$ = the power set of $\mathbb{N}$ and μ = counting measure. Let*

$$u_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ -1, & \text{if } n \text{ is even} \end{cases}$$

and $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ *is defined as*

$$\phi(n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

Then it is not difficult to see that uC_ϕ *is a projection. That is,* $(uC_\phi)^2 = uC_\phi$. *Hence, from [7, Proposition 1.4.5],* uC_ϕ *is decomposable.*

Example 4.4. *Let* $X = \mathbb{N}$, $\mathcal{S}$ *= the power set of* $\mathbb{N}$ *and* μ *= counting measure. Suppose that* $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ *is defined as*

$$\phi(2n-1) = \phi(2n) = 2n + 3 \text{ for all } n \in \mathbb{N}$$

and $u_n = 1$ *for all* $n \in \mathbb{N}$. *Let D denote the open unit disc in the complex plane. Now define a map* $f : D \longrightarrow L^p(X)$ *as*

$$f(\lambda) = \Big(x_1(\lambda), x_2(\lambda), \dots\Big) \text{ for all } \lambda \in D,$$

where

$$x_1(\lambda) = x_2(\lambda) = x_3(\lambda) = x_4(\lambda) = 1$$

and

$$x_{4n+1}(\lambda) = x_{4n+2}(\lambda) = x_{4n+3}(\lambda) = x_{4(n+1)}(\lambda) = \lambda^n \text{ for all } n \in \mathbb{N}.$$

Then f *is a non-constant analytic function which satisfies* $(\lambda - uC_\phi)f(\lambda) = 0$ *for all* $\lambda \in D$. *Thus* uC_ϕ *does not have SVEP and hence* uC_ϕ *is not decomposable.*

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