Functional Analysis,
Approximation and
Computation
7 (1) (2015), 15–28

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Some properties of the $M-$ essential spectra of closed linear operator on a Banach space

Aymen Ammar, Mohammed Zerai Dhahri, Aref Jeribi

Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Route de soukra Km 3.5, B. P. 1171, 3000, Sfax, Tunisia

Abstract. In this paper, we study a detailed treatment of some subsets of M-essential spectra of closed linear operators subjected to additive perturbations not necessarily belonging to any ideal of the algebra of bounded linear operators and we investigate some properties of the M-essential spectra of 2×2 matrix operator acting on a Banach space. This study led us to generalize some well known results for essential spectra of closed linear operator.

1. Introduction

Let X and Y be two infinite-dimensional Banach spaces. By an operator A from X to Y we mean a linear operator with domain $\mathcal{D}(A) \subset X$ and range $R(A) \subset Y$. We denote by $\mathcal{C}(X, Y)$ (resp. $\mathcal{L}(X, Y)$) the set of all closed, densely defined linear operators (resp. the Banach algebra of all bounded linear operators) from X into Y and we denote by $\mathcal{K}(X, Y)$ the subspace of all compact operators from X into Y. We denote by $\sigma(A)$ and $\rho(A)$ respectively the spectrum and the resolvent set of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y.

Let A and M be two operators on X such that M is nonzero and bounded and A is closed. We define the M-resolvent set by:

$$\rho_M(A) := \Big\{\lambda \in \mathbb{C} \text{ such that } \lambda M - A \text{ has a bounded inverse}\Big\}.$$

The M-spectrum of an operator A acting on a Banach space X is usually defined as

$$\sigma_M(A) := \mathbb{C} \backslash \rho_M(A).$$

Subsequently, the operator M should be taken as non invertible. For, otherwise the M-resolvent coincides with usual resolvent of the operator $M^{-1}A$, this analysis is meaningless.

2010 *Mathematics Subject Classification*. Primary 47A53, 47A55 ; Secondary 47A10.
Keywords. M-essential spectra; A-Fredholmness; 2×2 operator matrix.
Received: 18 November 2014; Accepted: 22 November 2014
Communicated by Dragan S. Djordjević
Email addresses: ammar__aymen84@yahoo.fr (Aymen Ammar), dhahri.mohammed@gmail.com (Mohammed Zerai Dhahri), Aref.Jeribi@fss.rnu.tn (Aref Jeribi)

Now, we introduce the following important operator classes: The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) = \{A \in \mathcal{C}(X,Y) \text{ such that } \alpha(A) < \infty, R(A) \text{ is closed in } Y\}.$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X,Y) = \{A \in \mathcal{C}(X,Y) \text{ such that } R(A) < \infty, R(A) \text{ is closed in } Y\}.$$

The set of Fredholm operators from X into Y is defined by

$$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

The set of bounded upper (resp. lower) semi-Fredholm operator from X into Y is defined by

$$\Phi_+^b(X,Y) = \Phi_+(X,Y) \cap \mathcal{L}(X,Y) \quad (\text{resp. } \Phi_-(X,Y) \cap \mathcal{L}(X,Y)).$$

We denote by $\Phi^b(X,Y) = \Phi(X,Y) \cap \mathcal{L}(X,Y)$ the set of bounded Fredholm operators from X into Y. If A is semi-Fredholm operator (either upper or lower) the index of A, is defined by $i(A) = \alpha(A) - \beta(A)$. It is clear that if $A \in \Phi(X,Y)$ then $i(A) < \infty$. If $A \in \Phi_+(X,Y) \setminus \Phi(X,Y)$ then $i(A) = -\infty$ and if $A \in \Phi_-(X,Y) \setminus \Phi(X,Y)$ then $i(A) = +\infty$. A complex number λ is in $\Phi_{+A,M}$, $\Phi_{-A,M}$ or $\Phi_{A,M}$ if $\lambda M - A$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ respectively.

Proposition 1.1. [2, Proposition 1.1.] *Let $A \in \mathcal{C}(X)$ and M a non null bounded linear operator on X. Then we have the following results*
(i) $\Phi_{A,M}$ is open.
(ii) $i(\lambda M - A)$ is constant on any component of $\Phi_{A,M}$.
(iii) $\alpha(\lambda M - A)$ and $\beta(\lambda M - A)$ are constant on any component of $\Phi_{A,M}$ except on a discrete set of points at which they have larger values.

There are several and in general non-equivalent definitions of the essential spectrum of a bounded linear operator on a Banach space. For a self-adjoint operator in a Hilbert space, there seems to be only one reasonable way to define the essential spectrum: The set of all points of the spectrum that are not isolated eigenvalues of finite algebraic multiplicity. Numerous mathematical and physical problems lead to operator pencils, $\lambda M - A$ (operator-valued functions of a complex argument) (see, for example, [13] and [20]). Recently, the spectral theory of operator pencils attracts an attention of many mathematicians. If X is a Banach space and $A \in \mathcal{C}(X)$, $M \in \mathcal{L}(X)$ various notions of essential $M-$ spectrum appear in application of spectral theory. In the following of this paper we introduce the M-essential spectra (see, for instance[1, 2]) and the references therein.

$$\begin{array}{rcl}
\sigma_{e1,M}(A) & := & \Big\{\lambda \in \mathbb{C} \ \text{ such that } \ \lambda M - A \notin \Phi_+(X)\Big\} := \mathbb{C}\backslash\Phi_{+A,M} \\
\sigma_{e2,M}(A) & := & \Big\{\lambda \in \mathbb{C} \ \text{ such that } \ \lambda M - A \notin \Phi_-(X)\Big\} := \mathbb{C}\backslash\Phi_{-A,M} \\
\sigma_{e3,M}(A) & := & \Big\{\lambda \in \mathbb{C} \ \text{ such that } \ \lambda M - A \notin \Phi_\pm(X)\Big\} := \mathbb{C}\backslash\Phi_{\pm A,M} \\
\sigma_{e4,M}(A) & := & \Big\{\lambda \in \mathbb{C} \ \text{ such that } \ \lambda M - A \notin \Phi(X)\Big\} := \mathbb{C}\backslash\Phi_{A,M} \\
\sigma_{e5,M}(A) & := & \mathbb{C}\backslash\rho_{e5,M}(A) \\
\sigma_{e6,M}(A) & := & \mathbb{C}\backslash\rho_{e6,M}(A) \\
\sigma_{eap,M}(A) & := & \mathbb{C}\backslash\rho_{eap,M}(A) \\
\sigma_{e\delta,M}(A) & := & \mathbb{C}\backslash\rho_{e\delta,M}(A)
\end{array}$$

where $\rho_{e5,M}(A) := \Big\{\lambda \in \mathbb{C} \ \text{ such that } \ \lambda M - A \in \Phi(X) \text{ and } i(\lambda M - A) = 0\Big\}$,

$$\rho_{e6,M}(A) := \Big\{\lambda \in \rho_{e5,M}(A) \ \text{ such that all scalars near } \lambda \text{ are in } \rho_M(A)\Big\},$$

$$\rho_{eap,M}(A) := \Big\{\lambda \in \mathbb{C} \text{ such that } \lambda M - A \in \Phi_+(X) \text{ and } i(\lambda M - A) \leq 0\Big\},$$

and

$$\rho_{e\delta,M}(A) := \Big\{\lambda \in \mathbb{C} \text{ such that } \lambda M - A \in \Phi_-(X) \text{ and } i(\lambda M - A) \geq 0\Big\}.$$

They can be ordered as

$$\sigma_{e5,M}(A) = \Big(\sigma_{eap,M}(A) \cup \sigma_{e\delta,M}(A)\Big) \subset \sigma_{e_6,M}(A).$$

$$\sigma_{e1,M}(A) \subset \sigma_{eap,M}(A) \text{ and } \sigma_{e2,M}(A) \subset \sigma_{e\delta,M}(A).$$

Note that if $M = I$, we recover the usual definition of the essential spectra of a closed linear operator A. We call $\sigma_{e1,I}(.)$ and $\sigma_{e2,I}(.)$ the Gustafson and Weidmann essential spectra [5], $\sigma_{e3,I}(.)$ is the Kato essential spectrum [12], $\sigma_{e4,I}(.)$ is the Wolf essential spectrum [5, 6, 8], and $\sigma_{e5,I}(.)$ the Schechter essential spectrum [5, 8, 9, 18, 19]. $\sigma_{eap,I}(.)$ is the essential approximate point spectrum [10, 15, 16] and $\sigma_{e\delta,I}(.)$ is the essential defect spectrum [7, 10, 16, 21].

Remark 1.2. *If M is invertible, then $\sigma_{ei,M}(A) = \sigma_{ei}(M^{-1}A)$, $i \in \Big\{1, 2, 3, 4, 5, ap, \delta\Big\}$.*

In the next, we will suppose that M is not invertible and we denote the complement of a subset $\Omega \subset \mathbb{C}$ by $C\Omega$.

Lemma 1.3. *Let $A \in \mathcal{C}(X)$, $M \in \mathcal{L}(X)$. Then,*

(i) $\sigma_{e5,M}(A) := \bigcap\limits_{K \in \mathcal{K}(X)} \sigma_M(A + K) = \bigcap\limits_{K \in \mathcal{F}_0(X)} \sigma_M(A + K) = \bigcap\limits_{K \in \mathcal{F}(X)} \sigma_M(A + K).$

(ii) $\sigma_{eap,M}(A) := \bigcap\limits_{K \in \mathcal{K}(X)} \sigma_{ap,M}(A + K) = \bigcap\limits_{K \in \mathcal{F}_0(X)} \sigma_{ap,M}(A + K) = \bigcap\limits_{K \in \mathcal{F}_+(X)} \sigma_{ap,M}(A + K).$

(ii) $\sigma_{e\delta,M}(A) := \bigcap\limits_{K \in \mathcal{K}(X)} \sigma_{\delta,M}(A + K) = \bigcap\limits_{K \in \mathcal{F}_0(X)} \sigma_{\delta,M}(A + K) = \bigcap\limits_{K \in \mathcal{F}_-(X)} \sigma_{\delta,M}(A + K).$

where

$$\sigma_{ap,S}(A) := \Big\{\lambda \in \mathbb{C} \textit{ such that } \inf_{\|x\|=1,\, x \in \mathcal{D}(A)} \|(\lambda M - A)\| = 0\Big\},$$

$$\sigma_{\delta,M}(A) := \Big\{\lambda \in \mathbb{C} \textit{ such that } \lambda M - A \textit{ is not surjective}\Big\}.$$

Proof. (i) Let $\lambda \notin O = \bigcap\limits_{K \in \mathcal{F}_0(X)} \sigma_M(A + K)$. Then, there exists $K \in \mathcal{F}_0(X)$ such that $\lambda \in \rho_M(A + K)$, then $A + K - \lambda M \in \Phi(X)$ and $i(A + K - \lambda M) = 0$. Now, the operator $A - \lambda M$ can be written in the form

$$A - \lambda M = A + K - \lambda M - K.$$

By [17, Theorem 3.1] we have $A - \lambda M \in \Phi(X)$ and $i(A - \lambda S) = 0$. Then, $\lambda \notin \sigma_{e5,M}(A)$.

Conversely, we suppose that $\lambda \notin \sigma_{e5,M}(A)$ then, $(A - \lambda M) \in \Phi(X)$ and $i(A - \lambda M) = 0$.
Let $n = \alpha(A - \lambda M) = \beta(A - \lambda M)$, $\Big\{x_1, ..., x_n\Big\}$ be bases for the $N((A - \lambda M)')$ and $\Big\{y_1', ..., y_n'\Big\}$ be basis for annihilator $R(A - \lambda M)^{\perp}$. By [17, Theorems 1.2.5, 1.2.6] there are functionals $x_1', ..., x_n'$ in X' (the adjoint space of X) and elements $y_1, ..., y_n$ such that

$$x_j'(x_k) = \delta_{jk} \text{ and } y_j'(y_k) = \delta_{jk},\ \ 1 \leq j,\ k \leq n,$$

where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$. The operator K is defined by :

$$Kx = \sum_{k=1}^{n} x_k'(x) y_k,\ x \in X.$$

Clearly K is a linear operator defined everywhere on X. It is bounded, since

$$\|Kx\| \leq \left(\sum_{k=1}^{n} \|x_k'\|\|y_k\|\right)\|x\|.$$

Moreover the range of K is contained in a finite dimensional subspace of X. Then K is a finite rank operator in X ([17, Lemma 1.3]). We prove that

$$N(A - \lambda M) \cap N(K) = \{0\} \ \text{ and } R(A - \lambda M) \cap R(K) = \{0\}. \tag{1}$$

Let $x \in N(A - \lambda M)$, then

$$x = \sum_{k=1}^{n} \alpha_k x_k,$$

therefore $x_j'(x) = \alpha_j,\ 1 \leq j \leq n$. On the other hand, if $x \in N(K)$ then $x_j'(x) = 0,\ 1 \leq j \leq n$. This proves the first relation in Eq. (1). The second inclusion is similar.
In fact, if $y \in R(K)$, then

$$y = \sum_{k=1}^{n} \alpha_k y_k,$$

and hence,

$$y_j(y) = \alpha_j,\ 1 \leq j \leq n.$$

But, if $y \in R(A - \lambda M)$, then,

$$y_j'(y) = 0,\ 1 \leq j \leq n.$$

This gives the second relation in Eq. (1). On the other hand K is a compact operator. We deduce from [17, Theorem 3.1] that $\lambda \in \Phi_{A,M}$ and $i(A - \lambda M + K) = 0$. If $x \in N(A - \lambda M + K)$ then $(A - \lambda M)x$ is in $R(A - \lambda M) \cap R(K)$ this implies that $x \in N(A - \lambda M) \cap N(K)$ hence $x = 0$. Thus $\alpha(A - \lambda M + K) = 0$. In the same way, one proves that $R(A - \lambda M + K) = X$. We get $\lambda \notin \mathcal{O}$. Also, $\sigma_{e5,M}(A) := \bigcap_{K \in \mathcal{F}_0(X)} \sigma_M(A + K)$.
Let $\mathcal{O}_1 := \bigcap_{F \in \mathcal{F}(X)} \sigma_M(A+F)$. Since, $\mathcal{F}_0(X) \subset \mathcal{F}(X)$ we infer that $\mathcal{O} \subset \sigma_{e5,M}(A)$. Conversely, let $\lambda \notin \mathcal{O}_1$ then there exist $F \in \mathcal{F}(X)$ such that $\lambda \notin \sigma_M(A + F)$. Then, $\lambda \in \rho_M(A + F)$. So, $A + F - \lambda M \in \Phi(X)$ and $i(A + F - \lambda M) = 0$. The use of [10, Lemma 2.1] makes us conclude that $A - \lambda M \in \Phi(X)$ and $i(A - \lambda M) = 0$. Then, $\lambda \notin \sigma_{e5,M}(A)$. So, $\sigma_{e5,M}(A) := \bigcap_{K \in \mathcal{F}_0(X)} \sigma_M(A + K) = \bigcap_{K \in \mathcal{F}(X)} \sigma_M(A + K)$.
Now, we use the following relations $\mathcal{F}_0(X) \subset \mathcal{K}(X) \subset \mathcal{F}(X)$, we have
$\sigma_{e5,M}(A) = \bigcap_{K \in \mathcal{F}(X)} \sigma_M(A + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma_M(A + K) \subset \bigcap_{K \in \mathcal{F}_0(X)} \sigma_M(A + K) = \sigma_{e5,M}(A)$.

Statement (ii) and (iii) can be checked similarly from the assertion (i). □

Lemma 1.4. *Let $A \in \mathcal{C}(X)$ and $M \in \mathcal{L}(X)$.*

(a) *If $\Phi_{A,M}$ is connected and $\rho_M(A) \neq \emptyset$, then*

(i) $\sigma_{e5,M}(A) = \sigma_{e4,M}(A)$.

(ii) $\sigma_{e1,M}(A) = \sigma_{eap,M}(A)$.

(iii) $\sigma_{e2,M}(A) = \sigma_{e\delta,M}(A)$.

(b) *If $C\sigma_{e5,M}(A)$ is connected and $\rho_M(A) \neq \emptyset$, then*

$$\sigma_{e5,M}(A) = \sigma_{e6,M}(A).$$

Proof. (*i*) The inclusion $\sigma_{e4,M}(A) \subset \sigma_{e5,M}(A)$ is known, it suffices to show that $\lambda \in \sigma_{e5,M}(A) \subset \sigma_{e4,M}(A)$ which is equivalent to

$$C\sigma_{e4,M}(A) \cap \{\lambda \in \mathbb{C} \text{ such that } i(A - \lambda M) \neq 0\} = \emptyset.$$

Suppose that $C\sigma_{e4,M}(A) \cap \{\lambda \in \mathbb{C}$ such that $i(A - \lambda M) \neq 0\} \neq \emptyset$ and let $\lambda_0 \in C\sigma_{e4,M}(A) \cap \{\lambda \in \mathbb{C}$ such that $i(A - \lambda M) \neq 0\}$. Since $\rho_M(A) \neq \emptyset$, then there exists $\lambda_1 \in \rho_M(A)$ and consequently $\lambda_1 M - A \in \Phi(X)$ and $i(\lambda_1 M - A) = 0$. On the other side, $\Phi_{A,M}$ is connected, it follows from Proposition 1.1 (*ii*) that $i(\lambda M - A)$ is constant on any component of $\Phi_{A,M}$. Therefore $i(\lambda_1 M - A) = i(\lambda_0 M - A) = 0$, which is a contradiction. Then $\sigma_{e5,M}(A) \subset \sigma_{e4,M}(A)$.

(*ii*) It is easy to check that $\sigma_{e1,M}(A) \subset \sigma_{eap,M}(A)$. For the second inclusion we take $\lambda \in C\sigma_{e1,M}(A)$, then $\lambda \in \Big(\Phi_{A,M} \cup (\Phi_{+A,M} \setminus \Phi_{A,M})\Big)$. Hence, we will discuss the following two cases:

Case 1: If $\lambda \in \Phi_{A,M}$ then $i(A - \lambda M) = 0$. Indeed, let $\lambda_0 \in \rho_M(A)$, then $\lambda_0 \in \Phi_{A,M}$ and $i(A - \lambda_0 M) = 0$. It follows from Proposition 1.1 that $i(A - \lambda M)$ is constant on any component of $\Phi_{A,M}$, therefore $\rho_M(A) \subseteq \Phi_{A,M}$, then $i(A - \lambda M) = 0$ for all $\lambda \in \Phi_{A,M}$. This shows that $\lambda \in \rho_{eap,M}(A)$.

Case 2: If $\mu \in (\Phi_{+A,M} \setminus \Phi_{A,M})$, then $\alpha(A - \lambda M) < \infty$ and $\beta(A - \mu M) = +\infty$. So, $i(A - \lambda M) = -\infty < 0$. Thus, we obtain from the above $\sigma_{eap,M}(A) \subset \sigma_{e1,M}(A)$.

Statement (*iii*) can be checked similarly from the assertion (*ii*).

(*b*) The inclusion $\sigma_{e5,M}(A) \subset \sigma_{e6,M}(A)$ is known, it suffices to show that $\sigma_{e6,M}(A) \subset \sigma_{e5,M}(A)$. We have the set $\rho_{e5,M}(A) \neq \emptyset$, because it contains points of $\rho_{e5,M}(A)$. Because $\alpha(\lambda M - A)$ and $\beta(\lambda M - A)$ are constant on any component of $\Phi_{M,A}$ except possibly on a discrete set of points at which they have large values (see Proposition 1.1 (*iii*)) then $\rho_{e5,M}(A) \subset \rho_{e6,M}(A)$. that is equivalent to $\sigma_{e6,M}(A) \subset \sigma_{e5,M}(A)$ and so we have the equality. □

Definition 1.5. *Let $F \in \mathcal{L}(X, Y)$.*
(*i*) *F is called Fredholm perturbation if $A + F \in \Phi^b(X, Y)$ whenever $A \in \Phi^b(X, Y)$.*

(*ii*) *F is called an upper (resp. lower) semi-Fredholm perturbation if $A + F \in \Phi^b_+(X, Y)$ (resp. $A + F \in \Phi^b_-(X, Y)$) whenever $A \in \Phi^b_+(X, Y)$ (resp. $A \in \Phi^b_-(X, Y)$).*

The sets of Fredholm, upper semi Fredholm and lower semi Fredholm perturbations are denoted by $\mathcal{F}^b(X, Y)$, $\mathcal{F}^b_+(X, Y)$ and $\mathcal{F}^b_-(X, Y)$ respectively. These classes of operators were introduced and investigated in [3]. In particular, it is shown that $\mathcal{F}^b_+(X, Y)$ and $\mathcal{F}^b(X, Y)$ are closed subsets of $\mathcal{L}(X, Y)$ and if $X = Y$ then $\mathcal{F}^b_+(X)$ and $\mathcal{F}^b(X)$ are closed two-sided ideals of $\mathcal{L}(X)$. We recall the following useful result due to Gohberg, Markus and Fel'dman [3, page 69-70].

Lemma 1.6. *Let X, Y and Z be three Banach spaces.*
(*i*) *$F_1 \in \mathcal{F}^b(X, Y)$ and $A \in \mathcal{L}(Y, Z)$ then $AF_1 \in \mathcal{F}^b(X, Z)$.*
(*ii*) *$F_2 \in \mathcal{F}^b(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ then $F_1 B \in \mathcal{F}^b(Y, Z)$.*

Definition 1.7. *Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$. F is called strictly singular, if for every infinite-dimensional closed subspace $\mathcal{M}$ of X, the restriction of F to $\mathcal{M}$ is not an homeomorphism.*

Let $SS(X, Y)$ denotes the set of strictly singular operators from X into Y. If $X = Y$, the set of strictly singular operators on X will be denoted by $SS(X)$.

The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [11] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators, we refer to [4, 11]. Note that $\mathcal{SS}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space, then $\mathcal{SS}(X) = \mathcal{K}(X)$.

Definition 1.8. *Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$. F is called strictly cosingular if there exists no closed subspace N of X with codim(N) $= \infty$ such that $\pi_N F : X \longrightarrow X/N$ is surjective.*

Let $\mathcal{SC}(X)$ denote the set of strictly cosingular operators on X. This class of operators was introduced by Pelczynski [14], it forms a closed two-sided ideal of $\mathcal{L}(X)$ ([22]).

Let A be a closed linear operator on a Banach space X. For $x \in \mathcal{D}(A)$ the graph norm of x is defined by

$$\|x\|_A := \|x\| + \|Ax\|.$$

It follows from the closedness of A that $\mathcal{D}(A)$ endowed with the norm $\|.\|_A$ is a Banach space. Let X_A denote $(\mathcal{D}(A), \|.\|_A)$. In this new space the operator T satisfies $\|Ax\| \leq \|x\|_A$ and consequently A is a bounded operator from X_A into X.

Definition 1.9. *Let $A \in \mathcal{C}(X)$ and let B be an arbitrary A defined linear operator on X. We say that B is A-compact (resp. A-weakly compact, A-strictly singular, A-strictly cosingular) if $\hat{B} \in \mathcal{K}(X_A, X)$ (resp. $\hat{B} \in \mathcal{W}(X_A, X)$, $\hat{B} \in \mathcal{SS}(X_A, X)$, $\hat{B} \in \mathcal{SC}(X_A, X)$).*

Let $A\mathcal{K}(X)$, $A\mathcal{W}(X)$, $A\mathcal{SS}(X)$ and $A\mathcal{SC}(X)$, denote, respectively, the sets of A-compact, A-weakly compact, A-strictly singular and A-strictly cosingular operators on X.

Definition 1.10. *Let $A \in \mathcal{C}(X)$ and let B be an A-defined linear operator on X. We say that B is A-Fredholm perturbation if $\hat{B} \in \mathcal{F}^b(X_A, X)$. B is called an upper (resp. lower) A-semi-Fredholm perturbation if $\hat{B} \in \mathcal{F}^b_+(X_A, X)$ (resp. $\hat{B} \in \mathcal{F}^b_-(X_A, X)$).*

Let $A\mathcal{F}(X)$, $A\mathcal{F}_+(X)$ and $A\mathcal{F}_-(X)$ designate the sets of A-Fredholm, upper A-semi Fredholm and lower A-semi-Fredholm perturbations, respectively.

Remark 1.11. (*i*) *If B is bounded, then B is A-bounded, B is compact (resp. weakly compact, strictly singular, strictly cosingular) implies that B is A-compact (resp. A-weakly compact, A-strictly singular, A-strictly cosingular).*
(*ii*) *Notice that the concept of A-compactness and A-Fredholmness are not connected with the operator A itself, but only with its domain.*
(*iii*) *Using the Definition 1.10 and* [3, page 69] *we have*

$$A\mathcal{K}(X) \subseteq A\mathcal{SS}(X) \subseteq A\mathcal{F}_+(X) \subseteq A\mathcal{F}(X).$$

$$A\mathcal{K}(X) \subseteq A\mathcal{CS}(X) \subseteq A\mathcal{F}_-(X) \subseteq A\mathcal{F}(X).$$

(*iv*) *Let B be an arbitrary A-Fredholm perturbaion operator, hence we can regard A and B as operators from X_A into X, they will be denoted by $\hat{A}$ and $\hat{B}$ respectively, these belong to $\mathcal{L}(X_A, X)$. Furthermore, we have the obvious relations*

$$\begin{cases} \alpha(\hat{A}) = \alpha(A), \ \ \beta(\hat{B}) = \beta(B), \quad R(\hat{A}) = R(A), \\ \alpha(\hat{A} + \hat{B}) = \alpha(A + B), \\ \beta(\hat{A} + \hat{B}) = \beta(A + B) \text{ and } R(\hat{A} + \hat{B}) = R(A + B). \end{cases} \tag{2}$$

The first purpose of this work is inspired by [1, 2] where the author studied the various types of M-essential spectra of linear bounded operators on a Banach space X. We begin by study a detailed treatment of some subsets of M-essential spectra of closed linear operators subjected to additive perturbations not necessarily belonging to any ideal of the algebra of bounded linear operators and we investigate some properties of the M-essential spectra of 2×2 matrix operator acting on a Banach space. We organize our paper in the following way: In Section 2, we give the characterization of different M-essential spectra of closed linear operator and in Section 3, we study the stability the $M-$ essential spectra of the matrix operator.

2. Stability of M-essential spectra of closed linear operator

The purpose of this this Section, we also the following useful stability result for the M-essential spectra of a closed, densely defined linear operator on a Banach space X. we begin with the following useful result.

Theorem 2.1. *Let $A \in \mathcal{C}(X)$, $M \in \mathcal{L}(X)$ and let B be an operator on X.*
(i) If $A - \lambda M \in \Phi(X)$ and $B \in A\mathcal{F}(X)$ then $A + B - \lambda M \in \Phi(X)$ and $i(A + B - \lambda M) = i(A - \lambda M)$.
(ii) If $A - \lambda M \in \Phi_+(X)$ and $B \in A\mathcal{F}_+(X)$ then $A + B - \lambda M \in \Phi_+(X)$
(iii) If $A - \lambda M \in \Phi_-(X)$ and $B \in A\mathcal{F}_-(X)$ then $A + B - \lambda M \in \Phi_-(X)$.
(iv) $A - \lambda M \in \Phi_\pm(X)$ and $B \in A\mathcal{F}_+(X) \cap A\mathcal{F}_-(X)$ then $A + B - \lambda M \in \Phi_\pm(X)$.

Proof. Assume that $A - \lambda M \in \Phi(X)$. Then, using (2) we infer that $\hat{A} - \lambda\hat{M} \in \Phi^b(X_A, X)$. Hence, it follows from [18, Theorem 1.4 p 108] that there exist $A_0 \in \mathcal{L}(X, X_A)$ and $K \in \mathcal{K}(X)$ such that

$$(\hat{A} - \lambda\hat{M})A_0 = I - K, \quad \text{on } X \tag{3}$$

Thus,

$$(\hat{A} + \hat{B} - \lambda\hat{M})A_0 = I - K + \hat{B}A_0, \quad \text{on } X \tag{4}$$

Next, using Eq. (3) we get $(\hat{A} - \lambda\hat{M})A_0 \in \Phi^b(X)$ and $i\Big[(\hat{A} - \lambda\hat{M})A_0\Big] = 0$. So, using of [18, Theorem 3.4 p 117] and [18, Theorem 2.3 p 111] we implies that $A_0 \in \Phi^b(X_A, X)$ and

$$i(\hat{A} - \lambda\hat{M}) = -i(A_0).$$

Since, $B \in A\mathcal{F}(X)$ and $A_0 \in \mathcal{L}(X)$. Applying Lemma 3.2 we have $\hat{B}A_0 \in \mathcal{F}^b(X)$, so $K - \hat{B}A_0 \in \mathcal{F}^b(X)$. Using Eq. (4) we get $(\hat{A} + \hat{B} - \lambda\hat{M})A_0 \in \Phi^b(X)$ and $i((\hat{A} + \hat{B} - \lambda\hat{M})A_0) = 0$. As, $A_0 \in \Phi^b(X, X_A)$, and according of the [18, Theorem 3.4 p 117] we have $(\hat{A} + \hat{B} - \lambda\hat{M}) \in \Phi^b(X_A, X)$ and

$$i(\hat{A} + \hat{B} - \lambda\hat{M}) = -i(A_0). \tag{5}$$

Now, by Eqs. (2), (3) and (5) we find that $i(A + B - \lambda M) = i(A - \lambda M)$ which completes the proof of (*i*).
The assertion (*ii*), the first part of (*iii*) and (*iv*) are immediate. To prove the second part of (*iii*) we proceed as follows. Let $A - \lambda M \in \Phi_-(X)$. [12, Theorem 5.13 p. 234] we infer that $(A - \lambda M)^* = A^* - \lambda M^* \in \Phi_+(X)$. Since $B^* \in A\mathcal{F}_+(X^*)$ then implied that $(A + B - \lambda M)^* = A^* + B^* - \lambda M^* \in \Phi_+(X^*)$. According of the [12, Theorem 5.13 p. 234] we get $A + B - \lambda M \in \Phi_-(X)$. □

Corollary 2.2. *Let $A \in \mathcal{C}(X)$, $M \in \mathcal{L}(X)$ and let B be an operator on X.*
(i) If $A - \lambda M \in \Phi_+(X)$ and $B \in A\mathcal{SS}(X)$ then $A + B - \lambda M \in \Phi_+(X)$.
(ii) If $A - \lambda M \in \Phi_-(X)$ and $B \in A\mathcal{CS}(X)$ then $A + B - \lambda M \in \Phi_-(X)$.

Theorem 2.3. *Let $A \in \mathcal{C}(X)$, B be an operator on X and $M \in \mathcal{L}(X)$. The following statements are satisfied.*
(i) If $B \in A\mathcal{F}_+(X)$ then

$$\sigma_{e1,M}(A + B) = \sigma_{e1,M}(A)$$

If in addition we suppose that the sets $\Phi_{A,M}$ and $\Phi_{A+B,M}$ are connected and the sets $\rho_M(A)$ and $\rho_M(A + B)$ are not empty, then

$$\sigma_{eap,M}(A + B) = \sigma_{eap,M}(A).$$

(ii) If $B \in A\mathcal{F}_-(X)$ then

$$\sigma_{e2,M}(A + B) = \sigma_{e2,M}(A).$$

If in addition we suppose that the sets $\Phi_{A,M}$ and $\Phi_{A+B,M}$ are connected and the sets $\rho_M(A)$ and $\rho_M(A + B)$ are not empty, then

$$\sigma_{e\delta,M}(A + B) = \sigma_{e\delta,M}(A).$$

(iii) If $B \in A\mathcal{F}_+(X) \cap A\mathcal{F}_-(X)$ then

$$\sigma_{e3,M}(A + B) = \sigma_{e3,M}(A)$$

(*iv*) *If* $B \in \mathcal{AF}(X)$ *then*

$$\sigma_{ei,M}(A+B) = \sigma_{ei,M}(A), \;\; i = 4, \; 5.$$

Moreover, if $C_{\sigma_{e5,M}(A)}$ *is connected. If neither* $\rho_M(A)$ *nor* $\rho_M(A+B)$ *is empty, then*

$$\sigma_{e6,M}(A+B) = \sigma_{e6,M}(A).$$

Proof. (*i*) Let $\lambda \notin \sigma_{e1,M}(A)$ then $\lambda \in \Phi_{+A,M}$. Since $B \in \mathcal{AF}_+(X)$, applying Theorem 2.1 (*ii*) we infer that $\lambda M - A - B \in \Phi_+(X)$. Thus, $\lambda \notin \sigma_{e1,M}(A+B)$. Conversely, let $\lambda \notin \sigma_{e1,M}(A+B)$, then $\lambda M - A - B \in \Phi_+(X)$, using Theorem 2.1 (*ii*) and since $-B \in \mathcal{AF}_+(X)$ we get $\lambda \in \Phi_{+A,M}$. So, $\lambda \notin \sigma_{e1,M}(A)$. We infer that

$$\sigma_{e1,M}(A+B) = \sigma_{e1,M}(A).$$

Now, we have $\Phi_{A,M}$ and $\Phi_{A+B,M}$ are connected and the sets $\rho_M(A)$ and $\rho_M(A+B)$ are not empty, then by Lemma 1.4 we have

$$\sigma_{eap,M}(A) = \sigma_{e1,M}(A) \text{ and } \sigma_{eap,M}(A+B) = \sigma_{e1,M}(A+B).$$

We deduce that

$$\sigma_{e\delta,M}(A+B) = \sigma_{e\delta,M}(A).$$

A similar proof as (*ii*) and (*iii*).

(*iv*) For $i = 5$. Let $\lambda \notin \sigma_{e5,M}(A)$ then $\lambda \in \Phi_{A,M}$ and $i(\lambda M - A) = 0$. Since $B \in \mathcal{AF}(X)$, applying Theorem 2.1 (*i*) we infer that $\lambda \in \Phi_{A+B,M}$ and $i(\lambda M - A - B) = 0$, and therefore $\lambda \notin \sigma_{e5,M}(A+B)$. Thus $\sigma_{e5,M}(A+B) \subseteq \sigma_{e5,M}(A)$. Similarly, If $\lambda \notin \sigma_{e5,M}(A+B)$ then using Theorem 2.1 (*i*) and arguing as above we derive the opposite inclusion $\sigma_{e5,M}(A) \subseteq \sigma_{e5,M}(A+B)$. Now, we get $C_{\sigma_{e5,M}(A+B)} = C_{\sigma_{e5,M}(A)}$, which is connected by hypothesis. Thus by, Lemma 1.4 we have

$$\sigma_{e5,M}(A) = \sigma_{e6,M}(A) \text{ and } \sigma_{e5,M}(A+B) = \sigma_{e6,M}(A+B).$$

We deduce that $\sigma_{e6,M}(A+B) = \sigma_{e6,M}(A)$. □

Theorem 2.4. *Let* $A \in \mathcal{C}(X)$ *and let* $\mathcal{I}_i(X), \; i \in \{1, \; 2, \; 3\}$ *be any be any subset of operators satisfying*
(*i*) $\mathcal{K}(X) \subseteq \mathcal{I}_1(X) \subseteq \mathcal{AF}(X)$. *Then,*

$$\sigma_{e5,M}(A) = \bigcap_{B \in \mathcal{I}_1(X)} \sigma_M(A+B).$$

(*ii*) $\mathcal{K}(X) \subseteq \mathcal{I}_2(X) \subseteq \mathcal{AF}_+(X)$. *Then,*

$$\sigma_{eap,M}(A) = \bigcap_{B \in \mathcal{I}_2(X)} \sigma_{ap,M}(A+B).$$

(*iii*) $\mathcal{K}(X) \subseteq \mathcal{I}_3(X) \subseteq \mathcal{AF}_-(X)$. *Then,*

$$\sigma_{e\delta,M}(A) = \bigcap_{B \in \mathcal{I}_3(X)} \sigma_{\delta,M}(A+B).$$

Proof. (*i*) Let $\mathcal{O} = \bigcap_{B \in \mathcal{I}_1(X)} \sigma_M(A+B)$. According of the Remark 1.11, we have $\mathcal{K}(X) \subseteq \mathcal{AK}(X) \subseteq \mathcal{AF}(X)$. So, $\mathcal{O} \subseteq \sigma_{e5,M}(A)$. So, we have only to prove that $\sigma_{e5,M}(A) \subseteq \mathcal{O}$. Let $\lambda_0 \notin \mathcal{O}$, then there exists $B \in \mathcal{I}(X)$ such that $\lambda_0 \in \rho_M(A+B)$. Let $x \in X$ and put $y = (\lambda_0 M - A - B)^{-1}x$. It follows from the estimate

$$\begin{aligned}
\|y\|_{A+B} &= \|y\| + \|(\hat{A}+\hat{B})y\| = \|y\| + \|x - \lambda_0 \hat{M} y\| \\
&= \|(\lambda_0 \hat{M} - \hat{A} - \hat{B})^{-1}x\| + \|x - \lambda_0 \hat{M}(\lambda_0 \hat{M} - \hat{A} - \hat{B})^{-1}x\| \\
&\leq \Big(1 + (1 + |\lambda_0| \|\hat{M}\|)\|(\lambda_0 M - \hat{A} - \hat{B})^{-1}\|\Big)\|x\|.
\end{aligned}$$

Thus, $(\lambda_0\hat{M} - \hat{A} - \hat{B})^{-1} \in \mathcal{L}(X, X_{A+B})$. Since $B \in \mathcal{I}(X) \subseteq A\mathcal{F}(X)$, applying Lemma 1.6 we conclude that $(\lambda_0\hat{M} - \hat{A} - \hat{B})^{-1}\hat{B} \in \mathcal{F}^b(X_A, X_{A+B})$. Let $\Im$ denote the imbedding operator which maps every $x \in X_A$ onto the same element $x \in X_{A+B}$. Clearly we have $N(\Im) = 0$ and $R(\Im) = X_{A+B}$. So,

$$\begin{array}{rcl} \|\Im(x)\| & = & \|x\|_{A+B} \leq \|x\| + \|Ax\|_X + \|Bx\|_X \\ & \leq & \left(1 + \|B\|_{\mathcal{L}(X,X_{A+B})}\right)\|x\|_{X_A}, \ \ \forall x \in X_A. \end{array}$$

Thus, $\Im \in \Phi^b(X_A, X_{A+B})$ and $i(\Im) = 0$. Next, since $(\lambda_0\hat{M} - \hat{A} - \hat{B})^{-1}\hat{B} \in \mathcal{F}^b(X_A, X_{A+B})$ and using Theorem 2.1 (i) we get

$$\Im + (\lambda_0\hat{M} - \hat{A} - \hat{B})^{-1}\hat{B} \in \Phi^b(X_A, X_{A+B}) \text{ and } i(\Im + (\lambda_0\hat{M} - \hat{A} - \hat{B})^{-1}\hat{B}) = 0. \tag{6}$$

On the other hand, since $\lambda_0 \in \rho_M(A + B)$ it follows from Eq. (2) that

$$(\lambda_0\hat{M} - \hat{A} - \hat{B}) \in \Phi^b(X_A, X_{A+B}) \text{ and } i(\lambda_0\hat{M} - \hat{A} - \hat{B}) = 0. \tag{7}$$

Writing $\lambda_0\hat{M} - \hat{A}$ in the from

$$\lambda_0\hat{M} - \hat{A} = (\lambda_0\hat{M} - \hat{A} - \hat{B})(\Im + (\lambda_0\hat{M} - \hat{A} - \hat{B})^{-1}\hat{B}.$$

Using the Eqs. (6) and (7) we get

$$\lambda_0\hat{M} - \hat{A} \in \Phi^b(X_A, X) \text{ and } i(\lambda_0\hat{M} - \hat{A}) = 0.$$

Now using (2) we infer that

$$\lambda_0 M - A \in \Phi^b(X_A, X) \text{ and } i(\lambda_0 M - A) = 0.$$

We deduce that, $\sigma_{e5,M}(A) \subseteq \mathcal{O}$. A similar proof as (ii) and (iii). □

3. The M-essential spectra of 2 × 2 matrix operator

The purpose of this section is to discuss the $M-$ essential spectra of the matrix operator $\mathcal{L}$, closure of $\mathcal{L}_0$, we begin with the following useful result

Definition 3.1. [2] (i) *Let $A \in \mathcal{C}(X)$ and λ_0 be isolated point of $\sigma_M(A)$. For an admissible contour Γ_{λ_0},*

$$P_{\lambda_0,M} = -\frac{M}{2\pi i}\oint_{\Gamma_{\lambda_0}} (A - \lambda M)^{-1}\, d\lambda,$$

is called the M-Riesz integral for A, M and λ_0 with range and Kernel denote by $\mathcal{R}_{\lambda,M}$ and $K_{\lambda,M}$.
(ii) *The M-discrete spectrum of A denoted $\sigma_{d_M}(A)$, and for $\lambda \in \rho_{b,M}(A) = \sigma_{d_M}(A) \cup \rho_M(A)$. we denote by $R_{b,M}(A, \lambda) = \big(A - \lambda M) \mid \mathcal{K}_{\lambda,M})^{-1}(I - P_{\lambda,M}) + P_{\lambda,M}$.*

Proposition 3.2. *Let $A \in \mathcal{C}(X)$, $M \in \mathcal{L}(X)$. Then for any μ, $\lambda \in \rho_{b,M}(A)$ we have*

$$R_{b,M}(A, \lambda) - R_{b,M}(A, \mu) = (\lambda - \mu)R_{b,M}(A, \lambda)MR_{b,M}(A, \mu) + \mathcal{M}(\lambda, \mu), \tag{8}$$

where $\mathcal{M}(\lambda, \mu)$ is a finite rank operator with the following expression

$$\mathcal{M}(\lambda, \mu) = R_{b,M}(A, \lambda)\Big[(A - (\lambda M + 1))P_{\lambda,M} - (A - (\mu M + 1))P_{\mu,M}\Big]R_{b,M}(A, \mu) \tag{9}$$

is a finite rank operator with $rank(\mathcal{M}(\lambda, \mu)) = rank(P_{\lambda,M}) + rank(P_{\mu,M})$ in case $\lambda \neq \mu$.

Proof. We have

$$R_{b,M}(A,\lambda) - R_{b,M}(A,\mu) = R_{b,M}(A,\lambda)[A_{\mu,M} - A_{\lambda,M}]R_{b,M}(A,\mu).$$

So,

$$\begin{array}{rcl} A_{\mu,M} - A_{\lambda,M} & = & [(A-\mu M)(I-P_{\mu,M}) + P_{\mu,M}] - [(A-\lambda M)(I-P_{\lambda,M}) + P_{\lambda,M}] \\ & = & [(A-(\lambda M+1))P_{\lambda,M} - (A-(\mu M+1))P_{\mu,M}] + (\lambda-\mu)M. \end{array}$$

Therefore $R_{b,M}(A,\lambda) - R_{b,M}(A,\mu) = (\lambda-\mu)R_{b,M}(A,\lambda)MR_{b,M}(A,\mu) + \mathcal{M}(\lambda,\mu)$. □

Proposition 3.3. *Let X and Y be two complex Banach spaces. $A \in \mathcal{C}(X)$, $M \in \mathcal{L}(X)$ and $B : Y \longrightarrow X$, $C : X \longrightarrow Y$ be two linear operators. Then, we have:*

(*i*) *$R_{b,M}(A,\mu)B$ is closable for some $\mu \in \rho_{b,M}(A)$ if and only if it is closable for all $\mu \in \rho_{b,M}(A)$.*

(*ii*) *C is A-bounded if and only if $CR_{b,M}(A,\mu)$ is bounded for some (hence for every) $\mu \in \rho_{b,M}(A)$.*

(*iii*) *If B and C satisfy the conditions (i) and (ii), respectively, and B is densely defined, then $C\mathcal{M}_{A,M}(\lambda,\mu)$, $\overline{\mathcal{M}_{A,M}(\lambda,\mu)B}$, and $\overline{C\mathcal{M}_{A,M}(\lambda,\mu)B}$ are operators of finite rank for any $\mu,\ \lambda \in \rho_{b,M}(A)$.*

Proof. From the resolvent identity we have, for any $\mu,\ \lambda \in \rho_{b,M}(A)$,

$$R_{b,M}(A,\lambda)B = R_{b,M}(A,\mu)B + (\lambda-\mu)R_{b,M}(A,\lambda)M(R_{b,M}(A,\mu)B) + \mathcal{M}(\lambda,\mu)B,$$

$$CR_{b,M}(A,\lambda) = CR_{b,M}(A,\mu) + (\lambda-\mu)(CR_{b,M}(A,\lambda))MR_{b,S}(A,\mu) + C\mathcal{M}(\lambda,\mu). \tag{10}$$

(*i*) Since M is bounded then $R_{b,M}(A,\lambda)M(R_{b,M}(A,\mu)B)$ is bounded. According of Proposition 3.2 the operator $[(A-(\lambda M+1))P_{\lambda,M} - (A-(\mu M+1))P_{\mu,M}]$ is bounded, thus $\mathcal{M}(\lambda,\mu)B$ has finite dimensional range, then $R_{b,M}(A,\lambda)B - R_{b,M}(A,\mu)B$ is bounded, hence $R_{b,M}(A,\mu)B$ is closable for some $\mu \in \rho_{b,M}(A)$ if and only if it is closable for all $\mu \in \rho_{b,M}(A)$.

(*ii*) If $CR_{b,M}(A,\lambda)$ is bounded for some $\lambda \in \rho_{b,M}(A)$, then clearly $CR_{b,M}(A,\mu)$ is also bounded for any μ and it follows from the Eq.(10) that $CR_{b,M}(A,\mu)$ is bounded for any μ. The well-known fact that C is A-bounded if and only if $C(A-\mu M)^{-1}$ is bounded for some $\lambda \in \rho_{b,M}(A)$.

(*iii*) According of Proposition 3.2 the operator $\mathcal{M}(\lambda,\mu)$ is a finite rank operator, so, $C\mathcal{M}(\lambda,\mu)$ and $\mathcal{M}(\lambda,\mu)B$ are a finite rank operator, hence, it is clear that $\overline{\mathcal{M}(\lambda,\mu)B}$ is of finite rank if B is densely defined. Since,

$$C\mathcal{M}(\lambda,\mu)B = (CR_{b,M}(A,\mu))[(A-\lambda M)(I-P_{\lambda,M}) + P_{\lambda,M}](R_{b,M}(A,\mu)B)$$

and if B and C satisfy the conditions (*i*) and (*ii*), respectively, then $\overline{C\mathcal{M}(\lambda,\mu)B}$ will again be continuous and densely defined with finite-dimensional range. □

The purpose of this section is to discuss the M-essential spectra $\sigma_{eap,M}(.)$ and $\sigma_{e\delta,M}(.)$ of the 2×2 matrix operator L act on the space $X \times Y$ where M is a bounded operator formally defined on the product space $X \times Y$ by a matrix

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

and L is given by

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where where the operator A acts on X and has domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space Y, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts on X (resp. Y).
In what follows, we will assume that the following conditions hold:

($\mathcal{H}_1$) A is closed, densely defined linear operator on X with non empty M_1-resolvent set $\rho_{M_1}(A)$.

($\mathcal{H}_2$) The operator B is densely defined linear operator on X and for some (hence for all) $\mu \in \rho_{b,M_1}(A)$, the operator $R_{b,M_1}(A,\mu)B$ is closable (in particular, if B is closable, then $R_{b,M_1}(A,\mu)B$ is closable).

($\mathcal{H}_3$) The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence for all) $\mu \in \rho_{b,M_1}(A)$, the operator $CR_{b,M_1}(A,\mu)$ is bounded (in particular, if C is closable, then $CR_{b,M_1}(A,\mu)$ is bounded).

($\mathcal{H}_4$) The lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in Y, and for some (hence for all) $\mu \in \rho_{b,M_1}(A)$, the operator $D - CR_{b,M_1}(A,\mu)B$ is closable, we will denote by $S(\mu)$ its closure.

Remark 3.4. (*i*) *Under the hypotheses* ($\mathcal{H}_1$) *and* ($\mathcal{H}_4$) *and from Proposition* 3.3 (*ii*) *the following operator*

$$F(\mu) = (C - \mu M_3)R_{b,M_1}(A,\mu)$$

is bounded on X.
(*ii*) *It follows from* ($\mathcal{H}_2$) *and the closed graph theorem that the operator*

$$G(\mu) = \overline{R_{b,M_1}(A,\mu)(B - \mu M_2)}$$

is bounded on Y for every $\mu \in \rho_{b,M_1}(A)$.
(*iii*) *The resolvent identity (8) implies that*

$$\begin{array}{rcl} S(\mu) - S(\mu_0) & = & \overline{(\mu-\mu_0)[M_3G(\mu_0) + F(\mu)M_2 + F(\mu_0)M_1G(\mu)]} \\ & + & \overline{(C - \mu M_3)\mathcal{M}(\mu,\mu_0)(B - \mu M_2)} \end{array}$$

for any $\mu,\ \mu_0 \in \rho_{b,S}(A)$, *where* $\mathcal{M}(\mu,\mu_0)$ *is the finite rank operator given by* (9), *It follows from Remark 3.4* (*i*) *and* (*ii*) *that the difference* $S(\mu) - S(\mu_0)$ *is a bounded operator. Therefore, neither the domain of* $S(\mu)$ *nor the property of being closable depend on* μ.

For each $\mu \in \rho_{b,M_1}(A)$, we define the bounded, lower and upper triangular operator-matrices

$$\mathcal{T}_1(\mu) = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}, \quad \mathcal{T}_2(\mu) = \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix},$$

the finite rank operator-matrix

$$\mathcal{N}(\mu) = \begin{pmatrix} [A - (\mu M_1 + 1)]P_{\mu,M_1} & 0 \\ 0 & 0 \end{pmatrix}$$

and the diagonal operator-matrix

$$\mathcal{D}(\mu) = \begin{pmatrix} A_{\mu,M_1} & 0 \\ 0 & S(\mu) - \mu M_4 \end{pmatrix}.$$

Theorem 3.5. *Under the hypotheses* ($\mathcal{H}_1$) $-$ ($\mathcal{H}_4$), *the matrix operator* $\mathcal{L}_0$ *is closable. Its closure is given by the relation*

$$\mathcal{L} = \overline{\mathcal{L}_0} = \mu M + \mathcal{T}_1(\mu)\mathcal{D}(\mu)\mathcal{T}_2(\mu) + \mathcal{N}(\mu) \tag{11}$$

for all $\mu \in \rho_{b,M_1}(A)$.

Proof. Let $\mu \in \rho_{b,M_1}(A) \cap \rho_{b,M_1}(S(\mu))$ the lower-upper factorization sense

$$\begin{aligned}\mathcal{L} &= \mu M + \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}\begin{pmatrix} A_{\mu,M_1} & 0 \\ 0 & S(\mu)-\mu M_4 \end{pmatrix}\begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} \\ &+ \begin{pmatrix} [A-(\mu M_1+1)]P_{\mu,M_1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \mu M + \begin{pmatrix} A_{\mu,M_1} & A_{\mu,M_1}G(\mu) \\ F(\mu)A_{\mu,M_1} & F(\mu)A_{\mu,M_1}G(\mu)+S(\mu)-\mu M_4 \end{pmatrix} \\ &+ \begin{pmatrix} [A-(\mu M_1+1)]P_{\mu,M_1} & 0 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

or, spelled out,

$$\begin{aligned}\mathcal{D}(\mathcal{L}) &= \big\{(x,y)\in X\times Y,\ x+G(\mu)y\in\mathcal{D}(A),\ y\in\mathcal{D}(S(\mu))\big\} \\ &= \mathcal{D}(A)\times\mathcal{D}(S(\mu))\end{aligned}$$

and

$$\mathcal{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_{\mu,M_1}x + A_{\mu,M_1}G(\mu)y \\ F(\mu)A_{\mu,M_1}x + F(\mu)A_{\mu,M_1}G(\mu)y + S(\mu)y \end{pmatrix}. \quad \square$$

Note that, in view of the previous remark, the description of the operator $\mathcal{L}$ does not depend on the choice of the point $\mu \in \rho_{b,M_1}(A)$.

Lemma 3.6. *(i) If $F(\mu) \in \mathcal{F}_+^b(X,Y)$ for some $\mu \in \rho_{b,M_1}(A)$, then $F(\mu) \in \mathcal{F}_+^b(X,Y)$ for all $\mu \in \rho_{b,M_1}(A)$ and $\sigma_{eap,M_1}(S(\mu))$ does not depend on the choice of μ.*

(ii) If $F(\mu) \in \mathcal{F}_-^b(X,Y)$ for some $\mu \in \rho_{b,M_1}(A)$, then $F(\mu) \in \mathcal{F}_-^b(X,Y)$ for all $\mu \in \rho_{b,M_1}(A)$ and $\sigma_{e\delta,M_1}(S(\mu))$ does not depend on the choice of μ.

Proof. Let $\mu,\ \mu_0 \in \rho_{b,M_1}(A)$. Using (8) we have

$$\begin{aligned}F(\mu)-F(\mu_0) &= (\mu-\mu_0)[F(\mu_0)M_1R_{b,S}(A,\mu) + M_3R_{b,M_1}(A,\mu_0)] \\ &+ (C-\mu M_3)\mathcal{M}(\mu,\mu_0).\end{aligned}$$

If we assume that $F(\mu_0) \in \mathcal{F}_+^b(X,Y)$, then it follows from the item (*iii*) Proposition 3.3 that the right-hand side of the previous equality is in $\mathcal{F}_+^b(X,Y)$. Hence $F(\mu) \in \mathcal{F}_+^b(X,Y)$. This proves the first result in (i). Similar reasoning leads to (*ii*). $\square$

In the sequel we will denote by $\mathcal{M}(\mu)$ the matrix-operator defined as follows

$$\mathcal{M}(\mu) = \begin{pmatrix} 0 & M_1G(\mu)-M_2 \\ F(\mu)M_1-M_3 & F(\mu)M_1G(\mu) \end{pmatrix}.$$

We are now in the position to express the main result of this section

Theorem 3.7. *Let the assumptions $(\mathcal{H}_1)-(\mathcal{H}_4)$ hold, then:*

(i) If for some $\mu \in \rho_{b,M_1}(A)$ the operator $F(\mu) \in \mathcal{F}_+^b(X,Y)$ and $\mathcal{M}(\mu) \in \mathcal{F}_+(X\times Y)$, then

$$\sigma_{e1,M}(\mathcal{L}) = \sigma_{e1,M_1}(A) \cup \sigma_{e1,M_4}(S(\mu)),$$

and

$$\sigma_{eap,M}(\mathcal{L}) \subseteq \sigma_{eap,M_1}(A) \cup \sigma_{eap,M_4}(S(\mu)).$$

If in addition we suppose that the sets $\Phi_{M_1,A}$ and $\Phi_{M_4,S(\mu)}$ are connected and the sets $\rho_{M_4}(S(\mu))$ and $\rho_M(\mathcal{L})$ are not empty, then

$$\sigma_{eap,M}(\mathcal{L}) = \sigma_{eap,M_1}(A) \cup \sigma_{eap,M_4}(S(\mu)).$$

(*ii*) *If for some $\mu \in \rho_{b,M_1}(A)$ the operator $F(\mu) \in \mathcal{F}_-^b(X,Y)$ and $\mathcal{M}(\mu) \in \mathcal{F}_-(X \times Y)$, then*

$$\sigma_{e2,M}(\mathcal{L}) = \sigma_{e2,M_1}(A) \cup \sigma_{e2,M_4}(S(\mu)),$$

and

$$\sigma_{e\delta,M}(\mathcal{L}) \subseteq \sigma_{e\delta,M_1}(A) \cup \sigma_{e\delta,M_4}(S(\mu)).$$

If in addition we suppose that the sets $\Phi_{M,\mathcal{L}}$, $\Phi_{M_1,A}$ and $\Phi_{M_4,S(\mu)}$ are connected and the sets $\rho_{M_4}(S(\mu))$ and $\rho_M(\mathcal{L})$ are not empty, then

$$\sigma_{e\delta,M}(\mathcal{L}) = \sigma_{e\delta,M_1}(A) \cup \sigma_{e\delta,M_4}(S(\mu)) \tag{12}$$

Proof. Let $\mu \in \mathbb{C}$ be such that $\mathcal{M}(\mu) \in \mathcal{F}_+(X \times Y)$. Using the Eq. (11), we have

$$\begin{array}{rcl} \mathcal{L} - \mu M &=& \mathcal{T}_1(\mu)\mathcal{D}(\mu)\mathcal{T}_2(\mu) + \mathcal{N}(\mu) + (\mu - \lambda)M \\ &=& \mathcal{T}_1(\mu)\mathcal{V}(\lambda)\mathcal{T}_2(\mu) + (\mu - \lambda)\mathcal{M}(\mu) - \mathcal{P}(\mu) + \mathcal{N}(\mu). \end{array} \tag{13}$$

where the matrix-operators $\mathcal{V}(\lambda)$ and $\mathcal{P}(\mu)$ are defined by

$$\mathcal{V}(\lambda) = \begin{pmatrix} A - \lambda M_1 & 0 \\ 0 & S(\lambda) - \lambda M_4 \end{pmatrix}$$

and

$$\mathcal{P}(\mu) = \begin{pmatrix} [A - (\mu M_1 + 1)]P_{\mu,M_1} & [A - (\mu M_1 + 1)]P_{\mu,M_1}G(\mu) \\ F(\mu)[A - (\mu M_1 + 1)]P_{\mu,M_1} & F(\mu)[A - (\mu M_1 + 1)]P_{\mu,M_1}G(\mu) \end{pmatrix}.$$

(*i*) Let $\mu \in \rho_{b,M_1}(A)$. As, $\mathcal{M}(\mu) \in \mathcal{F}_+(X \times Y)$ and $\mathcal{N}(\mu)$ and $\mathcal{P}(\mu)$ are finite rank matrix-operators, we have

$$(\mu - \lambda)\mathcal{M}(\mu) - \mathcal{P}(\mu) + \mathcal{N}(\mu) \in \mathcal{F}_+(X \times Y).$$

Then, from Eq. (13), we get $\mathcal{L} - \lambda M \in \Phi_+(X \times Y)$ if and only if $\mathcal{T}_1(\mu)\mathcal{V}(\lambda)\mathcal{T}_2(\mu)$ if and only if $A - \lambda M_1 \in \Phi_+(X)$ and $S(\mu) - \lambda M_4 \in \Phi_+(Y)$, since $\mathcal{T}_1(\mu)$ and $\mathcal{T}_2(\mu)$ are bounded and have bounded inverse, then

$$\sigma_{e1,M}(\mathcal{L}) = \sigma_{e1,M_1}(A) \cup \sigma_{e1,M_4}(S(\mu)).$$

Now, let $\lambda \notin [\sigma_{eap,M_1}(A) \cup \sigma_{eap,M_4}(S(\mu))]$ then, $A - \lambda M_1 \in \Phi_+(X)$, $S(\mu) - \lambda M_4 \in \Phi_+(Y)$ and $i(A - \lambda M_1) \leq 0$, $i(S(\mu) - \lambda M_4) \leq 0$. Since $\mathcal{N}(\mu)$ and $\mathcal{P}(\mu)$ are finite rank matrix-operators, then

$$(\mu - \lambda)\mathcal{M}(\mu) - \mathcal{P}(\mu) + \mathcal{N}(\mu) \in \mathcal{F}_+(X \times Y).$$

As, $\mathcal{T}_1(\mu)$ and $\mathcal{T}_2(\mu)$ are bounded and have bounded inverse, then $\mathcal{L} - \lambda M \in \Phi_+(X \times Y)$ and $i(\mathcal{L} - \lambda M) \leq 0$. Hence $\lambda \notin \sigma_{eap,M}(\mathcal{L})$. We infer that

$$\sigma_{eap,M}(\mathcal{L}) \subseteq \sigma_{eap,M_1}(A) \cup \sigma_{eap,M_4}(S(\mu))$$

Now, suppose that Φ_{M_1} and $\Phi_{M_4,S(\mu)}$ are connected, then $\sigma_{eap,M_1}(A) = \sigma_{e1,M_1}(A)$ and $\sigma_{eap,M_4}(S(\mu)) = \sigma_{e1,M_4}(S(\mu))$. We deduce that

$$\sigma_{eap,M}(\mathcal{L}) = \sigma_{eap,M_1}(A) \cup \sigma_{eap,M_4}(S(\mu)).$$

(*ii*) The proof of (*ii*) is similar. □

References

[1] A. Ammar, B. Boukettaya and A. Jeribi, Stability of the S-left and S-right essential spectra of a linear operator, Acta Math. Sci. Ser. B Engl. Ed. 34 , no. 6, (2014) 1922-1934 .
[2] F. Abdmouleh, A. Ammar and A. Jeribi, Stability of the S-essential spectra on a Banach space, Math. Slovaca 63, no. 2, (2013) 299-320.
[3] I. Gohberg, A. Markus and I. A. Feldman, Normally solvable operators and ideals associated with them, Amer. Math. Soc. Transl. Ser. 2 61, (1967) 63-84.
[4] S. Goldberg, Unbounded Linear Operators. New York: McGraw-Hill, 1966.
[5] K. Gustafson and J. Weidmann, On the essential spectrum, J. Math. Anal. App. 25, (1969)121-127.
[6] A. Jeribi, Quelques remarques sur le spectre de Weyl et applications, C. R. Acad. Sci. Paris Sr. I Math. 327, no. 5,(1998) 485-490.
[7] A. Jeribi, Une nouvelle caractrisation du spectre essentiel et application, C. R. Acad. Sci. Paris Sr. I Math. 331, no. 7,(2000) 525-530.
[8] A. Jeribi, A characterization of the essential spectrum and applications, Boll. Unione Mat. Ital. Sez. B Artic. Ric Mat. (8) 5, (2002)805-825.
[9] A. Jeribi, Some remarks on the Schechter essential spectrum and applications to transport equations, J. Math. Anal. App. 275, (2002) 222-237.
[10] A. Jeribi and N. Moalla, A characterization of some subsets of Schechter's essential spectrum and application to singular transport equation, J. Math. Anal. Appl., (358) no. 2,(2009) 434–444.
[11] T. Kato , Perturbation theory for nullity, deficiency and other quantities of linear operators, J. Anal. Math. 6, 261-322 1958.
[12] T. Kato, Perturbation theory for linear operators, Die Grundlehren der mathematischen Wissenschaften, Band 132 Springer-Verlag New York, Inc., New York 1966.
[13] A. S. Markus, Introduction to the spectral theory of polynomial operator pencils, American Mathematical Society, Providence, RI . iv+250 pp. ISBN: 0-8218-4523-3 (1988).
[14] A. Pelczynski, *Strictly singular and strictly cosingular operators. I. Strictly singular and strictly cosingular operators on* $C(\Omega)$*-spaces,* Bull. Acad. Polon. Sci. 31-36(1965).
[15] V. Rakočević, On one subset of M. Schechter's Essential Spectrum, Math. Vesnuk 5, (1981) 389-391.
[16] V. Rakočević, Approximate point spectrum and commuting compact perturbations, Glasgow Math. J. 28, (1986)193-198.
[17] M. Schechter, Spectra of partial differential operators, North-Holland, Amsterdam. New York. Oxford, 1986.
[18] M. Schechter, Principles of functionnal Analysis, Academic Press, New York, 1971.
[19] M. Schechter, Principles of functionnal Analysis, Grad. Stud. Math., 36, Amer. Math. Soc., Providence, RI, 2002.
[20] A. A. Shkalikov, On the essential spectrum of some matrix operators, Math. Notes, 58, no. 5-6,(1995) 1359-1362 .
[21] C. Schmoeger, The spectral mapping theorem for the essential approximate point spectrum, Colloq. Math. 74, no. 2, (1997)167-176.
[22] Ju. I. Vladimirskii, *Strictly cosingular operators,* Soviet. Math. Dokl. 8,(1967) 739-740.