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Fractional elliptic operators from a generalized Glaeske-Kilbas-Saigo-Mellin transform

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Abstract. We show that the deformation of the canonical spectral triples over the n-dimensional torus which is characterized by a conjectured elliptic operator $D_\beta = \not{D}(1+|\not{D}|^2)^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty \tau^{\beta-1}e^{-\tau(1+\not{D}^2)}\not{D}d\tau$ with $\beta \geq 0$ and by a discrete dimension spectrum with fractional values less than n may be obtained if the elliptic operator is defined by means of the fractional Glaeske-Kilbas-Saigo-Mellin transform.

Fractional field theory is a new successful branch of theoretical physics used to treat many important problems in particle physics [**3,6,7,10-15,25,26**]. This new field is in fact characterized by fractional dimensions and fractional differential operators. In reality, the appealing and attractive concept of fractional dimensions plays a crucial and leading role in almost all branches of sciences since it was first introduced by Mandelbrot about three decades ago [**19**]. Actually, fractional operators are considered to be an effective tool for describing dynamical systems displaying algebraic scale-invariant properties with non-integer exponent that is relevant in data analysis, dissipation and long-range interactions in space and/or time (memory) that cannot be illustrated using traditional analytic functions and ordinary differential operators. Due to their obvious scale-invariant features, fractional operators provide, in addition, a practical tool for dealing more precisely with complex dynamics having multiple scales, generated in the deep ultraviolet (UV) regime of quantum field theory [**13,14**].

Fractional elliptic operators were introduced in literature through different contexts [**20-23** and references therein] yet most of them were done by hand with no mathematical background. These operators are useful to define non-integer dimensional deformations of the canonical spectral triples $(\mathrm{A}, \mathrm{H}, \not{D})$. Ais the commutative C*algebra of smooth functions over the n-dimensional torus $T^n, n \in \mathbb{N}$, H is the Hilbert space of square integrable sections of a spinor bundle over T^n and $\not{D}$ is an unbounded elliptic operator acting on $\mathrm{H}=L^2(\mathrm{M},\mathrm{S})$ of square-integrable spinors with positive-definite signature specifying the metric and $\mathrm{C}^\infty(\mathrm{M})$acts on 'H' by multiplication operators with $|||[\not{D},\pi(\mathrm{x})]||| = \left\|grad\pi(x)\right\|_\infty, \pi \in \mathrm{C(M)}$. Besides, an algebra of functions defined on a manifold is replaced by an abstract associative pre-C*algebra $\mathrm{A}=\mathrm{C}^\infty(\mathrm{M})$ of smooth functions on an orientable, connected, compact, N-dimensional differentiable unbounded manifold M with respect to the C^0-norm acting in H by multiplication operators as follows: $(fg)(x) = f(x)g(x), \forall x \in \mathrm{M}$ [**9**]. It is notable that fractional dimensions arise in quantum gravity [**6**] and within the framework of

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dimensional regularization technique [8]. One can therefore correlate spectral triples to certain fractional sets and estimate their spectra.

In the work done in [24], in order to obtain a dimension spectrum with non-integer real values, deformations of the canonical spectral triples over the n-dimensional torus are considered where $(\mathrm{A}, \mathrm{H}, \not{D})$is replaced by $(\mathrm{A}, \mathrm{H}, \mathrm{D}_\beta)$where $\mathrm{D}_\alpha : \mathrm{H} \to \mathrm{H}$is a self-adjoint linear operator with compact resolvent defined by means of the Mellin transform $\mathrm{D}_\beta = \not{D}(1 + |\not{D}|^2)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-\tau(1+\not{D}^2)} \not{D} d\tau$ with $\beta \geq 0$ and its differential is bounder $\forall a \in \mathrm{A}$. $\not{D} = i\gamma_\mu \partial_\mu, \gamma_\mu = \gamma_\mu^+, \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \mu, \nu = 1, ..., n$is the usual Dirac operator where $\gamma_\mu$are Dirac matrices. It is noteworthy that the form $\mathrm{D}_\beta = \not{D}(1 + |\not{D}|^2)^{-\beta}$was introduced by hand in [24] without any mathematical derivation. However, it is well-known from the discrete dimension spectrum definition that a spectral triple has discrete dimension spectrum Sd if $Sd \subset \mathbb{C}$and for any element $b \in$ algebraB[9,24] the zeta function $\zeta_b^{\not{D}}(z) = \mathrm{Tr}[\pi(b) |\not{D}|^{-z}]$extends holomorphically to $\mathbb{C}/Sd$and each of these poles gives the dimension of a certain region of the whole space. In this work, we will show that the form $\mathrm{D}_\beta = \not{D}(1 + |\not{D}|^2)^{-\beta}$may be obtained by means of the generalized Glaeske-Kilbas-Saigo-Mellin fractional integral transform [5,18] and we will prove that fractional elliptic operators be obtained accordingly for some specific values of the free parameters introduced in the theory.

Construction of the fractional elliptic operator:

Definition 1: *The generalized Glaeske-Kilbas-Saigo-Mellin fractional integral is defined by*:

$$({}_\alpha I^{b,f}_{\beta,\gamma,\sigma})(\lambda) = \frac{\beta\lambda^\alpha}{\Gamma\left(1 + \gamma - \frac{1}{\beta}\right)\Gamma\left(\frac{\sigma}{\beta} + \frac{1}{\beta}\right)} \int_1^\infty (t^\beta - 1)^{\gamma-1/\beta} t^\sigma e^{-(t^\beta-1)^{2f}(\lambda^2+b^2)} dt. \tag{1}$$

Here$\alpha, \beta, f \in \mathbb{R}^+, (b, \gamma, \sigma) \in \mathbb{R}$,$\lambda$*may be real or complex and if* $\lambda \in \mathbb{C}$, *then* $\Re(\lambda) > 0$.

Remark 1: For integral corresponds to $f = 1/2, b = \pm 1, \lambda \in C, \beta = 1$, equation (1) is reduced to the Glaeske-Kilbas-Saigo fractional integral [5,18]:

$$({}_\alpha I^{\pm 1,1/2}_{1,\gamma,\sigma})(\lambda) = \frac{\lambda^\alpha}{\Gamma(\gamma)\Gamma(\sigma + 1)} \int_1^\infty (t - 1)^{\gamma-1} t^\sigma e^{-(t-1)(\lambda^2+1)} dt.$$

Lemma 1: *The following property holds*:

$$({}_\alpha I^{0,1/2}_{1,\gamma,\sigma})(\lambda) = \frac{\lambda^\alpha}{(\gamma - 1)\Gamma(\sigma + 1)} \mathrm{U}(\gamma, \sigma + \gamma + 1; \lambda^2), \tag{2}$$

where $\mathrm{U}(\gamma, \sigma + \gamma + 1; \lambda^2)$is the Tricomi's confluent hypergeometric function defined by [1,2]:

$$\mathrm{U}(\gamma, \sigma + \gamma + 1; \lambda^2) = \frac{1}{\Gamma(\gamma - 1)} \int_1^\infty (t - 1)^{\gamma-1} t^\sigma e^{-\lambda^2(t-1)} dt. \tag{3}$$

Proof: By performing the change of variable $T^\beta = t^\beta - 1$, equation (1) is reduced to:

$$({}_\alpha I^{b,f}_{\beta,\gamma,\sigma})(\lambda) = \frac{\beta\lambda^\alpha}{\Gamma\left(\gamma + 1 - \frac{1}{\beta}\right)\Gamma\left(\frac{\sigma}{\beta} + \frac{1}{\beta}\right)} \int_0^\infty T^{\beta(\gamma-1/\beta)+\beta-1} (T^\beta + 1)^{\sigma/\beta-1+1/\beta} e^{-T^{2f\beta}(\lambda^2+b^2)} dT.$$

Obviously for$\beta = 1, f = 1/2$ and $b = 0$, we get straightforwardly:

$$({}_\alpha I^{0,1/2}_{1,\gamma,\sigma})(\lambda) = \frac{\lambda^\alpha}{\Gamma(\gamma)\Gamma(\sigma + 1)} \int_0^\infty T^{\gamma-1} (T + 1)^\sigma e^{-\lambda^2 T} dT,$$

$$= \frac{\lambda^\alpha}{\Gamma(\gamma)\Gamma(\sigma + 1)} \int_1^\infty (t - 1)^{\gamma-1} t^\sigma e^{-\lambda^2(t-1)} dt = \frac{\lambda^\alpha}{(\gamma - 1)\Gamma(\sigma + 1)} \mathrm{U}(\gamma, \sigma + \gamma + 1; \lambda^2). \blacksquare$$

Motivated by the previous definition, we can now generalize the work of [24] by introducing first the following definition:

Definition 2: *Let* M *be an oriented compact Riemannian manifold of dimension n where we associate for* ${}_\alpha \mathrm{D}^{b,f}_{\beta,\gamma,\sigma}$ *the eigenvalue* $\not{D}$. *We define the generalized fractional elliptic operator* $\mathrm{D} : \mathrm{C}^\infty(\mathrm{M}, \mathrm{S}) \to \mathrm{C}^\infty(\mathrm{M}, \mathrm{S})$ *by means of the Glaeske-Kilbas-Saigo-Mellin fractional integral transform*:

$$\left|{}_{\alpha}\mathrm{D}^{b,f}_{\beta,\gamma,\sigma}\right|^{\varepsilon} = \frac{\beta\lambda^{\alpha}}{\Gamma\left(\gamma+1-\frac{1}{\beta}\right)\Gamma\left(\frac{\sigma}{\beta}+\frac{1}{\beta}\right)}\int_{1}^{\infty}(t^{\beta}-1)^{\gamma-1/\beta}t^{\sigma}\,|\not{D}|^{\varepsilon}\,e^{-(t^{\beta}-1)^{2f}(|\not{D}|^{2}+b^{2})}dt. \tag{4}$$

Lemma 2: *For very large values of the eigenvalues of the elliptic operator, the following property holds*:

$$\left|{}_{\alpha}\mathrm{D}^{1,1/2}_{1,\gamma,\sigma}\right|^{\varepsilon} = \frac{\lambda^{\alpha}}{\Gamma(\gamma)\Gamma(\sigma+1)}\,|\not{D}|^{\varepsilon}\left(|\not{D}|^{2}+1\right)^{-\gamma}. \tag{5}$$

Proof: By performing the change of variable $T^{\beta} = t^{\beta}-1$, the generalized fractional elliptic operator is written as:

$$\left|{}_{\alpha}\mathrm{D}^{b,f}_{\beta,\gamma,\sigma}\right|^{\varepsilon} = \frac{\beta\lambda^{\alpha}}{\Gamma\left(\gamma+1-\frac{1}{\beta}\right)\Gamma\left(\frac{\sigma}{\beta}+\frac{1}{\beta}\right)}\int_{0}^{\infty}T^{\beta(\gamma-1/\beta)+\beta-1}(T^{\beta}+1)^{\sigma/\beta-1+1/\beta}\,|\not{D}|^{\varepsilon}\,e^{-T^{2f\beta}(|\not{D}|^{2}+b^{2})}dT.$$

We can now find:

$$\left|{}_{\alpha}\mathrm{D}^{1,1/2}_{1,\gamma,\sigma}\right|^{\varepsilon} = \frac{\lambda^{\alpha}}{\Gamma(\gamma)\Gamma(\sigma+1)}\int_{0}^{\infty}T^{\gamma-1}(T+1)^{\sigma}\,|\not{D}|^{\varepsilon}\,e^{-T(|\not{D}|^{2}+1)}dT,$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\gamma)\Gamma(\sigma+1)}\int_{1}^{\infty}(t-1)^{\gamma-1}t^{\sigma}\,|\not{D}|^{\varepsilon}\,e^{-(t-1)(|\not{D}|^{2}+1)}dt = \frac{\lambda^{\alpha}\,|\not{D}|^{\varepsilon}}{\Gamma(\gamma)\Gamma(\sigma+1)}\mathrm{U}\left(\gamma,\sigma+\gamma+1;|\not{D}|^{2}+1\right).$$

However, when the eigenvalues of the elliptic operator tends to infinity [**4**], we can approximate the Tricomi's function by [**2**]:

$$\mathrm{U}\left(\gamma,\sigma+\gamma+1;|\not{D}|^{2}+1\right)\underset{\substack{\left|\arg|D|^{2}+1\right|\leq\frac{3}{2}-\delta\\ \delta\in\mathrm{R}/\ 0<\delta\ll 1}}{\approx}\left(|\not{D}|^{2}+1\right)^{-\gamma},$$

and then

$$\left|{}_{\alpha}\mathrm{D}^{1,1/2}_{1,\gamma,\sigma}\right|^{\varepsilon} = \frac{\lambda^{\alpha}}{\Gamma(\gamma)\Gamma(\sigma+1)}\,|\not{D}|^{\varepsilon}\left(|\not{D}|^{2}+1\right)^{-\gamma}.\blacksquare$$

Remark 2: As $\sigma,\lambda,\alpha$are free parameters in the theory, we set them all equal to unity for convenience and then:

$$\left|{}_{1}\mathrm{D}^{1,1/2}_{1,\gamma,1}\right| = \frac{1}{\Gamma^{1/\varepsilon}(\gamma)}\,|\not{D}|\left(|\not{D}|^{2}+1\right)^{-\gamma/\varepsilon},$$

with $\gamma/\varepsilon > 0$. This operator leads to a dimension spectrum performing correctly in the ultraviolet and infrared regions. For $\gamma = 1$and $\beta = 1/\varepsilon$we find surprisingly $\left|{}_{1}\mathrm{D}^{1,1/2}_{1,1,1}\right| = |\not{D}|\,(|\not{D}|^{2}+1)^{-\beta}$which is the same obtained in [**24**]. However, for $\beta = 1$, the elliptic operator is not fractional and fractionality occurs merely for fractional values of the parameter β whereas in our approach the elliptic operator depends on two independent parameters and hence for $\gamma = 1$and $\varepsilon \in \mathbb{R}/\{1\}$, fractional elliptic operators are obtained straightforwardly.

Application: The operator $\left|{}_{1}\mathrm{D}^{1,1/2}_{1,\gamma,1}\right|$ is self-adjoint linear operator in H with compact resolvent. In order to apply the discrete dimension spectrum definition to the spectral triples $(\mathrm{A},\mathrm{H},{}_{\alpha}\mathrm{D}^{b,f}_{\beta,\gamma,\sigma})$ for any element $b\in$ algebraB, we follow the arguments of [**24**] and we use the generalized zeta function:

$$\zeta_{b}^{{}_{1}\mathrm{D}^{1,1/2}_{1,\gamma,1}}(z) = \mathrm{Tr}\left[\pi(b)\left|{}_{1}\mathrm{D}^{1,1/2}_{1,\gamma,1}\right|^{-z}\right]. \tag{6}$$

However, using the binomial rule, we can write:

$$\left|{}_{1}\mathrm{D}^{1,1/2}_{1,\gamma,1}\right|^{-z} = \frac{1}{\Gamma^{-z/\varepsilon}(\gamma)}\,|\not{D}|^{-z}\left(|\not{D}|^{2}+1\right)^{z\gamma/\varepsilon} = \frac{1}{\Gamma^{-z/\varepsilon}(\gamma)}\sum_{k=0}^{\infty}\binom{z\gamma/\varepsilon}{k}|\not{D}|^{(\frac{2\gamma}{\varepsilon}-1)z-2k}. \tag{7}$$

Then

$$\zeta_b^{{}_1\mathrm{D}^{1,1/2}_{1,\gamma,1}}(z) = \mathrm{Tr}\left[\pi(b)\left|{}_1\mathrm{D}^{1,1/2}_{1,\gamma,1}\right|^{-z}\right] = \frac{1}{\Gamma^{-z/\varepsilon}(\gamma)}\sum_{k=0}^{\infty}\begin{pmatrix} z\gamma/\varepsilon \\ k \end{pmatrix}\zeta_b^{\not{D}}\left(2k-\left(\tfrac{2\gamma}{\varepsilon}-1\right)z\right), \tag{8}$$

and therefore, since according to the discrete spectrum dimension theorem, the zeta functions for $(\mathrm{A},\mathrm{H},\mathrm{D})$ have a single simple pole at its argument equal to n. It is effortless to check that the zeta function for the fractional triples $(\mathrm{A},\mathrm{H},{}_1\mathrm{D}^{1,1/2}_{1,\gamma,1})$ has simple poles at

$$z = \frac{n-2k}{1-\frac{2\gamma}{\varepsilon}}, k = 0,1,2,.... \tag{9}$$

For $\varepsilon = 2/3$which corresponds for:

$$\left|{}_1\mathrm{D}^{1,1/2}_{1,\gamma,1}\right|^{2/3} = \frac{1}{\Gamma^{3/2}(\gamma)}\left|\not{D}\right|^{2/3}\left(\left|\not{D}\right|^2+1\right)^{-3\gamma/2}, \tag{10}$$

we find for the case of a 4-dimensional torus $z = 2(2-k)/(1-3\gamma)$. For the highest pole $k = 0$and $\gamma = 1$, we obtain $z = -2$ whereas for $\gamma \approx 0.675$ we find $z \approx -3.9$closely to the result obtained in [**25**]. In [**24**], we find for $\left|{}_1\mathrm{D}^{1,1/2}_{1,1,1}\right| = \left|\not{D}\right|\left(\left|\not{D}\right|^2+1\right)^{-1/\varepsilon}$, $z = (n-2k)/(1-2/\varepsilon)$and hence for the highest pole $k = 0$, we get $z = 4/(1-2/\varepsilon)$ and for specific values of εwe find a fractional dimension spectrum yet a fractional elliptic operator can not be obtained as there is merely one parameter $\varepsilon$and not two independent parameters.

Remark 3: Equation (7) is closely similar to the fractional Riesz derivative discussed in [**17**] and accordingly we argue that the Glaeske-Kilbas-Saigo-Mellin fractional integral may be correlated to fractional Riesz derivatives. Some interesting properties of fractional operators were discussed in [**16**]. The following table summarize our results by comparing our result with the [**24**]:

$(\mathrm{A},\mathrm{H},\mathrm{D}_\beta)$	$(\mathrm{A},\mathrm{H},{}_\alpha\mathrm{D}^{b,f}_{\beta,\gamma,\sigma})$
$\left|\mathrm{D}_\beta\right| = \left|\not{D}\right|(1+\left|\not{D}\right|^2)^{-\beta}$ $= \frac{1}{\Gamma(\beta)}\int_0^\infty \tau^{\beta-1}e^{-\tau(1+\not{D}^2)}\ \not{D}d\tau$	$\left|{}_\alpha\mathrm{D}^{1,1/2}_{1,\gamma,\sigma}\right|^{\varepsilon} = \frac{\lambda^\alpha}{\Gamma(\gamma)\Gamma(\sigma+1)}\int_0^\infty T^{\gamma-1}(T+1)^\sigma\left|\not{D}\right|^\varepsilon e^{-T(\left|\not{D}\right|^2+1)}dT$ $= \frac{1}{\Gamma(\beta)}\int_0^\infty \tau^{\beta-1}e^{-\tau(1+\not{D}^2)}\ \not{D}d\tau$ $= \frac{\lambda^\alpha\left|\not{D}\right|^\varepsilon}{\Gamma(\gamma)\Gamma(\sigma+1)}\mathrm{U}\left(\gamma,\sigma+\gamma+1;\left|\not{D}\right|^2+1\right)$ with $\mathrm{U}\left(\gamma,\sigma+\gamma+1;\left|\not{D}\right|^2+1\right) \underset{\left|\arg\left|D\right|^2+1\right| \le \frac{3}{2}-\delta}{\approx} \left(\left|\not{D}\right|^2+1\right)^{-\gamma}$ $\delta \in \mathrm{R}/\ 0 < \delta \ll 1$ then $\left|{}_\alpha\mathrm{D}^{1,1/2}_{1,\gamma,\sigma}\right|^{\varepsilon} = \frac{\lambda^\alpha}{\Gamma(\gamma)\Gamma(\sigma+1)}\left|\not{D}\right|^\varepsilon\left(\left|\not{D}\right|^2+1\right)^{-\gamma}$
$\left|\mathrm{D}_\beta\right|^{-z} = \sum_{k=0}^\infty\begin{pmatrix}\beta z\\ k\end{pmatrix}\left|\not{D}\right|^{2(\alpha-1/2)z-k)}$	$\left|{}_1\mathrm{D}^{1,1/2}_{1,\gamma,1}\right|^{-z} = \frac{1}{\Gamma^{-z/\varepsilon}(\gamma)}\sum_{k=0}^\infty\begin{pmatrix}z\gamma/\varepsilon\\ k\end{pmatrix}\left|\not{D}\right|^{(\frac{2\gamma}{\varepsilon}-1)z-2k}$
$\zeta_b^{\mathrm{D}_\beta}(z) = \mathrm{Tr}[\pi(b)\left|\mathrm{D}_\beta\right|^{-z}]$ $= \sum_{k=0}^\infty\begin{pmatrix}\beta z\\ k\end{pmatrix}\zeta_b^{\not{D}}\left(2k-2\left(\alpha-\frac{1}{2}\right)z\right)$	$\zeta_b^{{}_1\mathrm{D}^{1,1/2}_{1,\gamma,1}}(z) = \mathrm{Tr}\left[\pi(b)\left|{}_1\mathrm{D}^{1,1/2}_{1,\gamma,1}\right|^{-z}\right]$ $= \frac{1}{\Gamma^{-z/\varepsilon}(\gamma)}\sum_{k=0}^\infty\begin{pmatrix}z\gamma/\varepsilon\\ k\end{pmatrix}\zeta_b^{\not{D}}\left(2k-\left(\frac{2\gamma}{\varepsilon}-1\right)z\right)$
$z = \frac{n-2k}{1-2\beta}, k-0,1,2,...$	$z = \frac{n-2k}{1-\frac{2\gamma}{\varepsilon}}, k = 0,1,2,...$

Table 1: Comparing the approach of [**24**] and our approach

In summary, we showed that the elliptic operator $\mathrm{D}_\beta = \not{D}(1+\left|\not{D}\right|^2)^{-\beta}$introduced by hand in [**24**] may be obtained if the elliptic operator is defined by means of the Glaeske-Kilbas-Saigo-Mellin fractional integral transform which deforms the canonical spectral triples from $(\mathrm{A},\mathrm{H},\not{D}) \to (\mathrm{A},\mathrm{H},{}_\alpha\mathrm{D}^{\mathrm{b,f}}_{\beta,\gamma,\sigma})$over the n-dimensional torus. Fractional elliptic operators and a discrete dimension spectrum with fractional values less than n are obtained accordingly. At the end, still more general question: is it possible to build, on the base of the discussed fractional elliptic operator, a meaningful fractional quantum field theory? We hope that proper interpretations will go behind.

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