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A Bound for Imaginary Parts of Eigenvalues of Hilbert - Schmidt Operators

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Abstract. Let A be a Hilbert - Schmidt operator in a separable Hilbert space, A^* is the adjoint to A, and $N_2(A) = [Trace\ (AA^*)]^{1/2}$. It is proved that

$$\sum_{k=1}^{\infty}(Im\ \lambda_k)^2 \le N_2^2(A_I) - \frac{1}{8}\left(\zeta(A) - \sqrt{\zeta^2(A) + 2\sqrt{2}N_2([A,A^*])}\right)^2,$$

where $A_I = (A-A^*)/(2i)$, λ_k $(k = 1,2,...)$ are the eigenvalues of A, $\zeta(A) := \sup_{j,k=1,2,...;\ j\neq k}|\lambda_j - \lambda_k|$ is the spread of the eigenvalues and $[A,A^*] = AA^* - A^*A$. That result refines the classical inequality

$$\sum_{k=1}^{\infty}(Im\ \lambda_k)^2 \le N_2^2(A_I).$$

1. Introduction and statement of the main result

Let H be a separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and unit operator I. For a linear operator A in H, A^* is the adjoint of A; $A_I := (A - A^*)/2i$; λ_k $(k = 1,2,...)$ are the eigenvalues of A taken with their multiplicities and enumerated as $|\lambda_k| \ge |\lambda_{k+1}|$; $s_k(A)$ $(k = 1,2,...)$ are the singular values of A, enumerated with their multiplicities in the non-increasing order. In addition,

$$\zeta(A) := \sup_{j,k=1,2,...;\ j\neq k}|\lambda_j - \lambda_k| \text{ is the spread of the eigenvalues of } A \text{ and } K := [A,A^*] = AA^* - A^*A$$

is the self-commutator. By SN_2 we denote the Hilbert-Schmidt ideal of compact operators A with the finite norm $N_2(A) := [Trace\ \ (AA^*)]^{1/2}$.

The aim of this paper is to prove the following result.

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Theorem 1.1. *Let $A \in SN_2$. Then*

$$\sum_{k=1}^{\infty}(Im\ \lambda_k)^2 \le N_2^2(A_I) - \frac{1}{8}\left(\zeta(A) - \sqrt{\zeta^2(A) + 2\sqrt{2}N_2(K)}\right)^2. \tag{1.1}$$

This theorem is proved in the next section.

Inequality (1.1) refines the well-known inequality

$$\sum_{k=1}^{\infty}(Im\ \lambda_k)^2 \le N_2^2(A_I),$$

cf. [4, Theorem II.6.1], [6, Section III.1.4.2].

Multiplying and deleting the right-hand part of (1.1) by

$$(\zeta(A) + (\zeta^2(A) + 2\sqrt{2}N_2(K))^{1/2})^2,$$

we get

$$\sum_{k=1}^{\infty}(Im\ \lambda_k)^2 \le N_2^2(A_I) - \frac{N_2^2(K)}{\left(\sqrt{\zeta^2(A) + 2\sqrt{2}N_2(K)} + \zeta(A)\right)^2}. \tag{1.2}$$

For any selfadjoint operator $T \in SN_2$ whose entries in an orthogonal normal basis are t_{jk} $(j,k = 1,2,...)$ we have

$$\sum_{k=1}^{\infty}|\lambda_k(T)|^2 \ge \sum_{k=1}^{\infty}|t_{kk}|^2,$$

where $\lambda_k(T)$ are the eigenvalues of T taken with their multiplicities, cf. [4, Section II.4.3]. Therefore, if c_{jk} $(j,k = 1,2,...)$ are the entries of $K = [A, A^*]$ in an orthogonal normal basis, then Theorem 1.1 implies

$$\sum_{k=1}^{\infty}(Im\ \lambda_k)^2 \le N_2^2(A_I) - \frac{1}{8}\left(\zeta(A) - \sqrt{\zeta^2(A) + 2\sqrt{2}\tau(K)}\right)^2, \tag{1.3}$$

where

$$\tau(K) = (\sum_{k=1}^{\infty}|c_{kk}|^2)^{1/2}.$$

Obviously,

$$\zeta(A) \le \sup_{j,k=1,2,...;\ j\ne k}(|\lambda_j| + |\lambda_k|) \le s_1(A) + s_2(A) \le 2\|A\|. \tag{1.4}$$

From (1.2) and (1.4) it follows

$$\sum_{k=1}^{\infty}(Im\ \lambda_k)^2 \le N_2^2(A_I) - \frac{N_2^2(K)}{\left(\sqrt{(s_1(A) + s_2(A))^2 + 2\sqrt{2}N_2(K)} + s_1(A) + s_2(A)\right)^2} \le \tag{1.5}$$

$$N_2^2(A_I) - \frac{N_2^2(K)}{\left(\sqrt{4\|K\|^2 + 2\sqrt{2}N_2(K)} + 2\|K\|\right)^2}.$$

Besides, one can replace $N_2(K)$ by $\tau(K)$.

Note that in [2] and [3] bounds for sums of the the absolute values of eigenvalues of Schatten - von Neumann operators were established. About other interesting recent investigations of compact operators see [5], [7]-[10].

2. Proof of Theorem 1.1

Lemma 2.1. *For any operator $A \in SN_2$ one has $N_2^2(K) \leq 2N_2^4(A)$.*

Proof. Clearly,

$$(AA^* - A^*A)^2 = (AA^* - A^*A)(AA^* - A^*A) = (AA^*)^2 + (A^*A)^2 - A(A^*)^2A - A^*A^2A^*.$$

But

$$Trace\ A^*A^2A^* = Trace\ A^2(A^2)^* \geq 0, Trace\ A(A^*)^2A = Trace\ A^2(A^2)^* \geq 0.$$

Thus

$$Trace\ (AA^* - A^*A)^2 \leq Trace\ (AA^*)^2 + Trace\ (A^*A)^2 \leq 2N_2^2(A^*A) \leq 2N_2^4(A),$$

as claimed. □

As it is proved in [1, Lemma 6.5.1], for any quasinilpotent Hilbert - Schmidt operator V one has

$$N_2^2(V - V^*) = 2N_2^2(V). \tag{2.1}$$

Replacing V by iV, we obtain

$$N_2^2(V + V^*) = 2N_2^2(V). \tag{2.2}$$

Furthermore, let A be finite dimensional. Then due to the Schur theorem it admits the triangular representation

$$A = D + V, \tag{2.3}$$

where D is a normal matrix and V is a nilpotent (strictly upper triangular) one, having the same invariant subspaces. Besides, V and D are called the nilpotent part and diagonal one of A, respectively.

Lemma 2.2. *For any finite dimensional operator A, whose nilpotent part is V one has*

$$N_2(K) \leq \sqrt{2}(N_2^2(V) + \zeta(A)N_2(V)). \tag{2.4}$$

Proof. According to (2.3) we can write

$$K = (D + V)(D + V)^* - (D + V)^*(D + V) =$$

$$VD^* + DV^* + VV^* - D^*V - V^*D - V^*V.$$

Hence, due to Lemma 2.1

$$N_2(K) \leq N_2(V^*V - VV^*) + N_2(D^*V - VD^* + V^*D - DV^*) \leq \sqrt{2}N_2^2(V) + N_2(C + C^*), \tag{2.5}$$

where $C = D^*V - VD^*$ is a nilpotent operator; so by (2.2) $N_2(C + C^*) = \sqrt{2}N_2(C)$. Now (2.5) implies

$$N_2(K) \leq \sqrt{2}(N_2^2(V) + N_2(C)). \tag{2.6}$$

Let A be reduced to the upper triangular form with the entries a_{jk} $(j, k = 1, ..., n)$, where $n = dim\, range\ A < \infty$. Then

$$N_2^2(C) = \sum_{k=2}^{n} \sum_{j=1}^{k-1} |a_{jk}|^2 |\lambda_k - \lambda_j|^2 \leq \zeta^2(A)N_2^2(V).$$

Therefore (2.6) yields the required inequality. □

Proof of Theorem 1.1: First assume that A is n-dimensional.

We need the equality

$$2\sum_{k=1}^{n}(Im\ \lambda_k)^2 = 2N^2(A_I) - N_2^2(V), \tag{2.7}$$

proved in [1, Lemma 6.5.2], where V is the nilpotent part of A.

Solving (2.4) , we obtain

$$N_2(V) \geq \frac{1}{2}\left(-\zeta(A) + \sqrt{\zeta^2(A) + 2\sqrt{2}N_2(K)}\right).$$

Now (2.7) yields

$$2\sum_{k=1}^{n}(Im\ \lambda_k)^2 \leq 2N_2^2(A_I) - \frac{1}{4}\left(-\zeta(A) + \sqrt{\zeta^2(A) + 2\sqrt{2}N_2(K)}\right)^2.$$

This proves Theorem 1.1 in the finite dimensional case. Now let $A \in SN_2$ be infinite dimensional. Since Hilbert-Schmidt operators are limits of finite dimensional operators in the N_2-norm, we obtain the required result. □

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