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COMMON FIXED POINTS BY TWO STEP ITERATIVE SCHEME FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce an iteration scheme for approximating common fixed points of two asymptotically nonexpansive mappings in the framework of a uniformly convex Banach spaces and established weak and strong convergence results for common fixed points of asymptotically nonexpansive mappings. The results obtained in this paper are generalizations of Khan [9]. Our result also illustrated with help of an example.

1. Introduction

Let E be a real Banach space, K be a nonempty, closed and convex subset of E. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $F(T) := \{x : Tx = x\}$ is the set of fixed point of T. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, if $\| T^n x - T^n y \| \leq k_n \| x - y \|$, for all $x, y \in K$ and for all $n \in \mathbb{N}$. This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [7] in 1972. They proved that if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of K has a fixed point. The fixed point iteration process for asymptotically nonexpansive mapping in Banach spaces including Mann and Ishikawa iterations processes have been studied extensively by many authors; see ([1]-[20]).

The Picard and Mann [21] introduced the following iteration process: $T : K \to K$ are defined by

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = T^n x_n \end{cases} \tag{1}$$

for all $n \in \mathbb{N}$ is called the Picard iteration process and

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, n \in \mathbb{N}, \end{cases} \tag{2}$$

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where $\{\alpha_n\}$ is in $(0,1)$ is called the Mann iteration process.

Recently Khan [9] defined two-step iteration procedure as:

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = T^n[(1-\beta_n)x_n + \beta_n T^n x_n], n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{\beta_n\} \in (0,1)$.

The aim of this paper is to establish a new two-step iterative process and compute the common fixed points for two asymptotically nonexpansive mappings. Let $S, T : K \to K$ be two asymptotically nonexpansive mappings. Then, our process read as follows:

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = T^n[(1-\beta_n)S^n x_n + \alpha_n T^n x_n], n \geq 1, \end{cases} \tag{4}$$

where $\{\beta_n\} \in [0,1]$. However, iteration process (4) reduce to iteration process (3) when $S = I$, that is, the identity mapping.

Our purpose in the rest of the paper is to use the scheme (4) to prove weak and strong convergence results for approximating common fixed points of two asymptotically nonexpansive mappings.

2. Preliminaries

Let $X = \{x \in E : \|x\| = 1\}$ and E^* be the dual of E. The space E has :
(i) Gâteaux differentiable norm if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists for each $x, y \in K$;
(ii) Frèchet differentiable norm (see e.g. [23]) for each x in S, the above limit exists and is attained uniformly for y in S and in this case, it is also well-known that

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x+h\|^2 \leq \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|) \tag{5}$$

for all $x, h \in E$, where J is the Frèchet derivative of the function $\frac{1}{2}\|.\|^2$ at $x \in E$, $\langle ., .\rangle$ is the dual pairing between E and E^*, and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\to 0} \frac{b(t)}{t} = 0$;
(iii) Opial's condition [24] if for any sequence $\{x_n\}$ in $E, x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for all $y \in E$ with $y \neq x$.

Let us recall the following definitions.

Definition 2.1. *Let K be a nonempty, closed and convex subset of Banach space E and $T : K \to K$ be a mapping. Then, T is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}_0^\infty$ in $[1,+\infty)$ with $\lim_{n\to+\infty} k_n = 1$ such that*

$$\|Tx_n - p\| \leq k_n \|x - p\|$$

for all $x \in K$ and for all $q \in F(T)$ ($F(T)$ denotes the set of fixed points of T) and $n \geq 1$.

Definition 2.2. *[11] . Let E be a Banach space, K be a nonempty, closed and convex subset of Banach space E, and $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is said to be demi-closed at 0, if $x_n \to x$ (converges weakly) and $x_n - Tx_n \to 0$ (converges strongly), then it is implies that $x \in K$ and $Tx = x$.*

Definition 2.3. *[6] . Let two mappings $S, T : K \to K$, where K is a subset of a normed space E, said to be satisfy* ***condition*** (A') *if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \geq f(d(x, F))$ or $\|x - Tx\| \geq f(d(x, F))$ for all $x \in K$ where $d(x, F) = inf\{\|x - p\| : p \in F = F(S) \cap F(T)\}$.*

Now, we state the following useful lemma to prove our main results.

Lemma 2.4. *[25]: If $\{r_n\}$, $\{t_n\}$ are two sequences of nonnegative real numbers such that $r_{n+1} \leq (1 + t_n)r_n$, $n \geq 1$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.*

Lemma 2.5. *[14]: Let E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup n \to \infty\|y_n\| \leq r$ and $\lim_{n\to\infty} \|(1 - t_n)x_n + t_n y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Convergence Results

In this section, we prove weak and strong convergence theorems for two asymptotically nonexpansive mappings in the frame work of a uniformly convex Banach spaces.

Theorem 3.1. *Let K be a nonempty, closed and convex subset of a uniformly convex Banach space E. Let $S, T : K \to K$ be two asymptotically nonexpansive mappings with $F(S) \cap F(T) \neq \phi$ and a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{x_n\}$ be the sequence defined by (4), where β_n is a sequence in $[\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$ satisfying:*

$$\|x_n - T^n x_n\| \quad \leq \quad \lambda\|S^n x_n - T^n x_n\|, \tag{6}$$

for all $x, y \in K$, where $\lambda > 1$, then

$$\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|Tx_n - x_n\| = 0$$

.

Proof. Let $p \in F(S) \cap F(T)$ and $F(S) \cap F(T) \neq \phi$. Put, for simplicity, $y_n = (1 - \beta_n)S^n x_n + \beta_n T^n x_n$. From (4), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|T^n y_n - p\| \\ &\leq k_n\|y_n - p\|, \end{aligned} \tag{7}$$

and,

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)S^n x_n + \beta_n T^n x_n - p\| \\ &\leq (1 - \beta_n)\|S^n x_n - p\| + \beta_n\|T^n x_n - p\| \\ &\leq (1 - \beta_n)k_n\|x_n - p\| + \beta_n k_n\|y_n - p\| \\ &\leq k_n\|x_n - p\|. \end{aligned} \tag{8}$$

From (7) and (8), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq k_n^2\|x_n - p\| \\ &\leq [1 + (k_n^2 - 1)\|x_n - p\|. \end{aligned}$$

Hence by using Lemma 2.4, $\lim_{n\to\infty} \|x_n - q\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = c$ and suppose that $c > 0$. Now, from **(8)** , we have

$$\|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq c, \tag{9}$$

Note that,

$$\|y_n - p\| = \|T^n x_n - p\| \le k_n \|x_n - p\| \le c. \tag{10}$$

Combining the estimates in (9) and (10), we have

$$\|y_n - p\| = c. \tag{11}$$

Next, consider

$$\begin{aligned} c = \|y_n - p\| &= \|(1-\beta_n)S^n x_n + \beta_n T^n x_n - p\| \\ &\le (1-\beta_n)\|S^n x_n - p\| + \beta_n \|T^n x_n - p\|. \end{aligned}$$

Applying **Lemma 2.5** , we have

$$\lim_{n\to\infty} \|S^n x_n - T^n x_n\| = 0. \tag{12}$$

Using **(6)** and **(12)**, it follows then that

$$\begin{aligned} \|S^n x_n - x_n\| &= \|S^n x_n - T^n x_n\| + \|T^n x_n - x_n\| \\ &\le \|S^n x_n - T^n x_n\| + \lambda \|S^n x_n - T^n x_n\| \\ &\le (1+\lambda)\|S^n x_n - T^n x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Taking limsup on both sides of the above inequality, we obtain

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0.$$

Now, note that

$$\begin{aligned} \|T^n x_n - x_n\| &\le \|T^n x_n - S^n x_n\| + \|S^n x_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0.$$

Now, by definition of $\{x_n\}$, we have

$$\|x_{n+1} - T^n x_n\| \le k_n \|S^n x_n - T^n x_n\|.$$

Taking limit as $n \to \infty$ in both sides of the above inequality, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - T^n x_n\| = 0. \tag{13}$$

Again note that, $\|x_{n+1} - S^n x_n\| \le \|x_{n+1} - T^n x_n\| + \|T^n x_n - S^n x_n\|$.
Using (12) and (13), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - S^n x_n\| = 0. \tag{14}$$

Also, $\|x_{n+1} - x_n\| \le \|x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\|$, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \tag{15}$$

and

$$\begin{aligned}\|x_{n+1} - Sx_n\| &\le \|x_{n+1} - S^{n+1}x_{n+1}\| + \|S^{n+1}x_{n+1} - S^{n+1}x_n\| \\ &\quad + \|S^{n+1}x_n - Sx_{n+1}\| \\ &\le \|x_{n+1} - S^{n+1}x_{n+1}\| + k_{n+1}\|x_{n+1} - x_n\| \\ &\quad + k_1\|S^n x_n - x_{n+1}\|,\end{aligned}$$

It follows from (14) and (15), we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Similarly, we may show that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This proof is completed. □

Example 3.2. *Let E be the real line with the usual norm $|.|$ and suppose $K = [0,1]$. Define $S, T : K \to K$ by*

$$Tx = \frac{2-x}{2}$$

and

$$Sy = \frac{y+2}{4}$$

for all $x, y \in K$. Obviously both S and T are an asymptotically nonexpansive with the common fixed point $\frac{2}{3}$ for all $x, y \in K$. Now we check that our condition $\|x - Sy\| \le \lambda\|Tx - Sx\|$ for all $x, y \in K$ is true. If $x, y \in [0,1]$ and $\lambda > 1$, then

$$\begin{aligned}|x - Sy| &= |x - \frac{(y+2)}{4}| \\ &= |\frac{4x - y - 2}{4}|,\end{aligned}$$

and

$$\begin{aligned}|Tx - Sy| &= |\frac{2-x}{2} - \frac{y+2}{4}| \\ &= |\frac{2x + y - 2}{4}|.\end{aligned}$$

It is clear that $|\frac{4x-y-2}{4}| \le \lambda|\frac{2x+y-2}{4}|$, where $\lambda > 1$, so $|x - Sy| \le \lambda|Tx - Sx|$ exists, for all $x, y \in K$. Now, we check that S and T are quasi-nonexpansive type mappings. In fact, if $x \in [0,1]$ and $p = 0 \in [0,1]$, then

$$\begin{aligned}|Tx - p| &= |\frac{2-x}{2} - 0| = |\frac{2-x}{2}| \\ &= |\frac{2-x}{2}| \le |x| = |x - 0| = |x - p|,\end{aligned}$$

that is

$$|Tx - p| \le |x - p|.$$

Similarly, we can prove that

$$|Sx - p| \le |x - p|.$$

So that S and T are quasi-nonexpansive type mappings.

Lemma 3.3. *: Let K be a nonempty, closed and convex subset of a uniformly convex Banach space E. Let $\{x_n\}$ be the sequence defined in Theorem (3.4) with $F \neq \phi$. Then, for any $p_1, p_2 \in F$, $\lim_{n\to\infty}\langle x_n, J(p_1 - p_2)\rangle$ exist, in particular, $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_\omega(x_n)$.*

Proof. Take $x = p_1 - p_2$, with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in the inequality **(5)** to get:

$$\begin{aligned}\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle &\leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2\\ &\leq \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle\\ &+ b(t\|x_n - p_1\|).\end{aligned}$$

As $sup_{n\geq1}\|x_n - p_1\| \leq M'$ for some $M' > 0$, it follows that

$$\begin{aligned}&\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle\\ &\qquad\leq \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|^2\\ &\qquad\leq \frac{1}{2}\|p_1 - p_2\|^2 + b(tM') + t\liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle.\end{aligned}$$

That is,

$$\limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle \leq \liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle + \frac{b(tM')}{tM'}M'.$$

If $t \to 0$, then $\lim_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle$ exists for all $p_1, p_2 \in F$, in particular, we get

$$\langle p - 1, J(p_1 - p_2)\rangle = 0$$

for all $p, q \in \omega_\omega(x_n)$. □

Theorem 3.4. *Let E be a uniformly convex Banach space satisfying Opial condition and K, T, S and $\{x_n\}$ be taken as* ***Theorem 3.1****. If $F(S) \cap F(T) \neq \phi$, $I - T$ and $I - S$ are demiclosded at zero, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $p \in F(S) \cap F(T)$, then as proved in **Theorem 3.1** $\lim_{n\to\infty}\|x_n - p\|$ exist. Since E is uniformly convex Banach space. Thus there exists subsequences $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z_1 \in K$. From **Theorem 3.1**, we have

$$\lim_{n\to\infty}\|Tx_{n_k} - x_{n_k}\| = 0,$$

and

$$\lim_{n\to\infty}\|Sx_{n_k} - x_{n_k}\| = 0.$$

Since $I - T$ and $I - S$ are demiclosed at zero, therefore $Sz_1 = z_1$. Similarly $Tz_1 = z_1$. Finally, we prove that $\{x_n\}$ converges weakly to z_1. Let on contrary that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ and $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $z_2 \in K$ and $z_1 \neq z_2$. Again in the same way, we can prove that $z_2 \in F(S) \cap F(T)$. From **Theorem 3.1** the limits $\lim_{n\to\infty}\|x_n - z_1\|$ and $\lim_{n\to\infty}\|x_n - z_2\|$ exists. Suppose that $z_1 \neq z_2$, then by the Opial's condition, we get

$$\begin{aligned}\lim_{n\to\infty}\|x_n - z_1\| &= \lim_{n_i\to\infty}\|x_{n_i} - z_1\| < \lim_{n_i\to\infty}\|x_{n_i} - z_2\|\\ &= \lim_{n\to\infty}\|x_n - z_2\| = \lim_{n_j\to\infty}\|x_{n_j} - z_2\|\\ &< \lim_{n_j\to\infty}\|x_{n_j} - z_1\| = \lim_{n\to\infty}\|x_n - z_1\|.\end{aligned}$$

This is a contradiction so $z_1 = z_2$. Hence $\{x_n\}$ converges weakly to a common fixed point of T and S. □

Theorem 3.5. *Let E be a Banach space and $K, S, T, F, \{x_n\}$ be as in Lemma 3.1. If $F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$ where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.*

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. As in the proof of Lemma 3.1, we have

$$\|x_{n+1} - p\| \leq k_n \|x_n - p\|.$$

This gives

$$\|x_{n+1} - F\| \leq k_n \|x_n - F\|.$$

So that $\lim_{n\to\infty} d(x_n, F)$ exists. But by hypothesis $\liminf_{n\to\infty} d(x_n, F)$, so we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. Suppose $\epsilon > 0$ be given. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists n_0 in N such that for all $n \geq n_0$, we get $d(x_n, F) < \frac{\epsilon}{2}$. In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\epsilon}{2}$. There must exist $p^* \in F$ such that $\|x_{n_0} - p^*\| < \frac{\epsilon}{2}$. Now for $n, m \geq n_0$, we have

$$\begin{aligned}
\|n_{n+m} - x_n\| &\leq \|n_{n+m} - p^*\| + \|p^* - x_n\| \\
&\leq 2\|p^* - x_n\| \\
&\leq 2\frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E, therefore it must converge in K. Suppose $\lim_{n\to\infty} x_n = q$. Now $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(x, F) = 0$. It is well-known that F is closed and so $q \in F$. □

Using **Theorem 3.5**, we obtain a strong convergence theorem of the iteration scheme (4) under the **condition** (A') as below:

Theorem 3.6. *Let E be a uniformly convex Banach space and $K, S, T, , F, \{x_n\}$ be as in **Theorem 3.1**. Let S, T satisfy the **condition** (A') and $F \neq \phi$. Then $\{x_n\}$ converges strongly to a point of F.*

Proof. We proved in **Theorem 3.1**, i.e.

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$$

Then from the definition of **condition** (A'), we obtain

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|Tx_n - x_n\| = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

In above cases, we get

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

But $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$, so that we get

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

□

All the conditions of **Theorem 3.5** are satisfied, therefore by its conclusion $\{x_n\}$ converges to strongly to a fixed point of F.

We now state two strong convergence theorems. The mapping $T : K \to K$ with $F(T) \neq \phi$ is said to satisfy condition (A) [22] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in K$, $\|x - Tx\| \geq d(x, F(T))$.

Theorem 3.7. *Let E be a uniformly convex Banach space and K a nonempty, closed, convex subset of E which is also a asymptotically nonexpansive retract of E. Let $T : K \to K$ be a asymptotically nonexpansive mapping with $F(T) \neq \phi$. Let $\{\beta_n\}$ be sequences in $[\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (3). Suppose T satisfies condition (A). Then $\{x_n\}$ converges strongly to some fixed point of T.*

Theorem 3.8. *Let E be a uniformly convex Banach space and K a nonempty, closed, convex subset of E which is also a asymptotically nonexpansive retract of E. Let $T : K \to K$ be a asymptotically nonexpansive mapping with $F(T) \neq \phi$. Let $\{\beta_n\}$ be sequences in $[\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (3). Suppose that $T(K)$ is contained in a compact subset of E. Then $\{x_n\}$ converges strongly to some fixed point of T.*

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