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L_1-biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space

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Abstract. The matter of biharmonic surfaces of the 3-dimensional Euclidean space has been studied (firstly) from a differential geometric point of view by Bang-Yen Chen and others, who has showed that the only biharmonic surfaces in $\mathbb{E}^3$ are minimal ones. In general, the biharmonicity condition on any hypersurface $x : M^n \to \mathbb{E}^{n+1}$ is defined by $\Delta^2 x = 0$, where Δ is the Laplace operator on M^n. Many people have paid attention to various extensions of Chen's theorem. In this paper, we approve an advanced version of the theorem, replacing Δ by the operator L_1, which stands for the linearized operator of the first variation of the 2-th mean curvature arising from the normal variations of M^n in $\mathbb{E}^{n+1}$. In the case $n = 4$, for any L_1-biharmonic hypersurface $x : M^4 \to \mathbb{E}^5$, having assumed that it has three distinct principal curvatures and constant ordinary mean curvature, we prove that, M^4 has to be 1-minimal.

1. Introduction

The study of biharmonic maps has several physical and geometric motivations. For instance, one can find the role of biharmonic maps in the theory of elastics and fluid mechanics in [1, 12]. The theory of biharmonic maps plays a central role in various fields in differential geometry, computational geometry and the theory of Partial differential equations. In eighteen decade, Bang Yen Chen initiated to investigate the differential geometric properties of biharmonic submanifolds in the Euclidean spaces. He introduced some open problems and conjectures (in [6]), among them, a longstanding conjecture says that a biharmonic submanifold in a Euclidean space is a minimal one. Chen himself has proved the conjecture for surfaces in $\mathbb{E}^3$. Later on, I. Dimitrić has verified Chen conjecture in several different cases such as special curves, submanifolds of constant mean curvature and also, hypersurfaces of the Euclidean spaces with at most two distinct principal curvatures. T. Hasanis and T. Vlachos ([10]) has verified the conjecture for hypersurfaces in $\mathbb{E}^4$. Having assumed the completeness, Akutagawa and Maeta ([2]) gave an affirmative answer to the global version of Chen's conjecture for biharmonic submanifolds in Euclidean spaces. Recently, in [9], it is proved that the only biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^5$ are minimal ones.

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The biharmonicity condition on any hypersurface $x : M^n \to \mathbb{E}^{n+1}$ is defined by $\Delta^2 x = 0$, where Δ is the The Laplace operator which can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_r stands for the linearized operator of the first variation of the $(r+1)$th mean curvature arising from normal variations of the hypersurface (see, for instance, [14]). These operators are given by $L_r(f) = tr(P_r \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_r denotes the rth Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f. From this point of view, as an extension of finite type theory, S.M.B. Kashani ([11]) introduced the notion of L_1-finite type hypersurface in the Euclidean space, which has been followed in the first author in her doctoral thesis (see [5], chapter 11).

In this paper, we pay attention to a generalized version of the concept of biharmonic hypersurfaces by replacing Δ by L_1. In [13], we proved that every L_1-biharmonic surface in $\mathbb{E}^3$ is flat and every L_r-biharmonic hypersurface in $\mathbb{E}^4$ with at most two distinct principal curvatures is r-minimal, $r \leq 2$. In this paper, we study the L_1-biharmonic hypersurfaces having at most three distinct principal curvatures in $\mathbb{E}^5$. We prove that, each L_1-biharmonic hypersurface in $\mathbb{E}^5$ with constant mean curvature and at most three distinct principal curvatures is 1-minimal.

Here is our main result:

Theorem 1.1. *Every L_1-biharmonic hypersurfaces in $\mathbb{E}^5$ with constant mean curvature and three distinct principal curvatures is 1-minimal.*

2. Preliminaries

In this section, we recall preliminary concepts from [4, 9, 13]. Let $x : M^4 \to \mathbb{E}^5$ be an isometrically immersed hypersurface in the Euclidean 4-space, with the Gauss map N. We denote by ∇^0 and ∇ the Levi-Civita connections on $\mathbb{E}^5$ and M^4, respectively, then, the basic Gauss and Weingarten formulae of the hypersurface are written as $\nabla^0_X Y = \nabla_X Y + < SX, Y > N$ and $SX = -\nabla^0_X N$, for all tangent vector fields $X, Y \in \chi(M^4)$, where $S : \chi(M^4) \to \chi(M^4)$ is the shape operator (or Weingarten endomorphism) of M^4 with respect to the Gauss map N.

As is well-known, for every point $p \in M^4$, S defines a linear self-adjoint endomorphism on the tangent space T_pM^4, and its eigenvalues $\lambda_1(p)$, $\lambda_2(p)$, $\lambda_3(p)$ and $\lambda_4(p)$ are the principal curvatures of the hypersurface. The characteristic polynomial $Q_S(t)$ of S is defined by

$$Q_S(t) = \det(tI - S) = (t - \lambda_1)(t - \lambda_2)(t - \lambda_3)(t - \lambda_4) = t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4,$$

where the coefficients of $Q_S(t)$ are given by

$$\begin{aligned} &a_1 = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), && a_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \\ &a_3 = -(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4), && a_4 = \lambda_1\lambda_2\lambda_3\lambda_4. \end{aligned} \tag{2.1}$$

The r-th mean curvature H_r or mean curvature of order r of M^4 in $\mathbb{E}^5$ is defined by

$$\binom{4}{r} H_r = (-1)^r a_r, \qquad \text{with} \quad H_0 = 1.$$

If $H_{r+1} = 0$ then we say that M^4 is a r-minimal hypersurface, a 0-minimal hypersurface is nothing but a minimal hypersurface in $\mathbb{E}^5$. The r-th Newton transformation of M^4 is the operator $P_r : \chi(M^4) \to \chi(M^4)$ defined by

$$P_r = \sum_{j=0}^{r} (-1)^j \binom{4}{r-j} H_{r-j} S^j = (-1)^r \sum_{j=0}^{r} a_{r-j} S^j.$$

In particular,

$$P_0 = I, \quad P_1 = 4HI - S, \quad P_2 = 6H_2 I - S \circ P_1.$$

Let us recall that, for every point $p \in M^4$, each $P_r(p)$ is also a self-adjoint linear operator on the tangent hyperplane T_pM^4 which commutes with $S(p)$. Indeed, $S(p)$ and $P_r(p)$ can be simultaneously diagonalized. If $\{e_1, e_2, e_3, e_4\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\{\lambda_1(p), \lambda_2(p), \lambda_3(p), \lambda_4(p)\}$, respectively, then they are also the eigenvectors of $P_r(p)$ with corresponding eigenvalues given by

$$\mu_{i,r} = \sum_{\substack{i_1<\cdots<i_r \\ i_j \neq i}}^{4} \lambda_{i_1} \cdots \lambda_{i_r}. \qquad (i = 1, 2, 3, 4) \tag{2.2}$$

In particular,

$$\begin{aligned} &\mu_{1,1} = \lambda_2 + \lambda_3 + \lambda_4, \quad \mu_{2,1} = \lambda_1 + \lambda_3 + \lambda_4, \quad \mu_{3,1} = \lambda_1 + \lambda_2 + \lambda_4, \quad \mu_{4,1} = \lambda_1 + \lambda_2 + \lambda_3, \\ &\mu_{1,2} = \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \qquad \mu_{2,2} = \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4, \\ &\mu_{3,2} = \lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4, \qquad \mu_{4,2} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3. \end{aligned} \tag{2.3}$$

We have the following formula for the Newton transformations from [4].

$$tr(S^2 \circ P_1) = 12(2HH_2 - H_3).\%\& \tag{2.4}$$

Associated to each Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M^4) \to C^\infty(M^4)$ given by $L_r(f) = tr(P_r \circ \nabla^2 f)$. Here, $\nabla^2 f : \chi(M^4) \to \chi(M^4)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by $< \nabla^2 f(X), Y >=< \nabla_X(\nabla f), Y >$, $X, Y \in \chi(M^4)$. Therefore by considering the local orthonormal frame $\{e_1, e_2, e_3, e_4\}$, $L_r(f)$ is given by

$$L_r(f) = \sum_{i=1}^{4} \mu_{i,r}(e_i e_i f - \nabla_{e_i} e_i f). \tag{2.5}$$

3. L_r-biharmonic hypersurfeces in $\mathbb{E}^5$

Let $x : M^4 \to \mathbb{E}^5$ be a connected orientable hypersurface immersed into the Euclidean 5-space, with Gauss map N. By definition, M^4 is called a L_r-biharmonic hypersurface if its position vector field satiesfies the condition $L_r^2 x = 0$. By the equality $L_r x = c_r H_{r+1} N$ from [4], the condition $L_r^2 x = 0$ has another equivalent expression as $L_r(H_{r+1}N) = 0$. It is clear that, r-minimal hypersurface is L_r-biharmonic. By formulae in [4] page 122, we have

$$L_r^2 x = -2c_r(S \circ P_r)(\nabla H_{r+1}) - c_r\binom{4}{r+1} H_{r+1}\nabla H_{r+1} - c_r(tr(S^2 \circ P_r)H_{r+1} - L_r H_{r+1})N, \tag{3.1}$$

where $c_r = (r+1)\binom{4}{r+1}$.

By using this formula for $L_r^2 x$ and the identifying normal and tangent parts of the L_r-biharmonic condition $L_r^2 x = 0$, one obtains necessary and sufficient conditions for M^4 to be L_r-biharmonic in $\mathbb{E}^5$, namely

$$L_r H_{r+1} = tr(S^2 \circ P_r)H_{r+1} \tag{3.2}$$

and

$$(S \circ P_r)(\nabla H_{r+1}) = -\frac{1}{2}\binom{4}{r+1} H_{r+1}\nabla H_{r+1}. \tag{3.3}$$

From now on, we concentrate on L_1-biharmonic hypersurfaces M^4 in a Euclidean space $\mathbb{E}^5$ with three distinct principal curvatures and constant ordinary mean curvature $H = H_1$.

3.1. Proof of Theorem 1.1

Let $x : M^4 \to \mathbb{E}^5$ be an L_1-biharmonic hypersurfaces with constant ordinary mean curvature and three distinct principal curvatures. Having assumed that the 2th mean curvature of M^4, H_2 is not constant, we will get a contradiction. So, there exists a connected open subset $\mathcal{U}$ of M, on which we have $\nabla H_2(p) \neq 0$. Let $\{e_1, e_2, e_3, e_4\}$ be a local orthonormal frame of principal directions on on $\mathcal{U}$, which are the eigenvectors of the shape operator, S, of M, hence we have $Se_i = \lambda_i e_i$ for real numbers λ_i, and by (2.2) we have $P_2 e_i = \mu_{i,2} e_i$, for $i = 1, 2, 3, 4$. Using the expanded equality

$$H_2 = \frac{1}{6}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4), \tag{3.4}$$

and the inductive definition of P_2, we get

$$P_2(\nabla H_2) = 9H_2\nabla H_2 \qquad on\ \mathcal{U}. \tag{3.5}$$

Observe from (3.5) that ∇H_2 is an eigenvector of P_2 with the corresponding eigenvalue $9H_2$. Without loss of generality, we can choose e_1 such that e_1 is parallel to ∇H_2. Since the shape operator S and P_2 can be simultaneously diagonalized, therefore the shape operator S of M^4 takes the form with respect to a suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}. \tag{3.6}$$

Then we have

$$\mu_{1,2} = 9H_2. \tag{3.7}$$

We can decompose $\nabla H_2 = \sum_{i=1}^{4} e_i(H_2)e_i$. Since e_1 is parallel to ∇H_2, it follows that

$$e_1(H_2) \neq 0, \qquad e_2(H_2) = e_3(H_2) = e_4(H_2) = 0. \tag{3.8}$$

We write

$$\nabla_{e_i} e_j = \sum_{k=1}^{4} \omega_{ij}^k e_k, \quad i, j = 1, 2, 3, 4. \tag{3.9}$$

The compatibility conditions $\nabla_{e_k} < e_i, e_i >= 0$ and $\nabla_{e_k} < e_i, e_j >= 0$ imply respectively that

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.10}$$

for $i \neq j$ and $i, j, k = 1, 2, 3, 4$. Furthermore, it follows from the Codazzi equation that

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{3.11}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j \tag{3.12}$$

for distinct $i, j, k = 1, 2, 3, 4$.
Since $\mu_{1,2} = 9H_2$, from (3.4) we have

$$H_2 = \frac{1}{3}\lambda_1(\lambda_1 - 4H), \tag{3.13}$$

therefore, we get

$$e_1(\lambda_1) \neq 0, \quad e_2(\lambda_1) = e_3(\lambda_1) = e_4(\lambda_1) = 0. \tag{3.14}$$

One can compute that

$$[e_2, e_3](\lambda_1) = [e_3, e_4](\lambda_1) = [e_2, e_4](\lambda_1) = 0,$$

which yields directly

$$\omega_{23}^1 = \omega_{32}^1, \quad \omega_{34}^1 = \omega_{43}^1, \quad \omega_{24}^1 = \omega_{42}^1. \tag{3.15}$$

Now we show that $\lambda_j \neq \lambda_1$ for $j = 2, 3, 4$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, by putting $i = 1$ in (3.11) we have that

$$0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts the first expression of (3.14).

By the assumption, M^4 has three distinct principal curvatures, without loss of generality, we assume that $\lambda_2 = \lambda_3 = \lambda$ and $\lambda_4 \neq \lambda$, hence $\lambda_4 = 4H - \lambda_1 - 2\lambda$.

Consider Eqs. (3.11) and (3.12).

Let $j = 2, i = 3$, and $j = 3, i = 2$ respectively in (3.11). One has

$$e_2(\lambda) = e_3(\lambda) = 0. \tag{3.16}$$

For $j = 1$ and $i \neq 1$ in (3.11), by (3.14) we have $\omega_{1i}^1 = 0$ $(i \neq 1)$. Moreover, by the first expression of (3.10) we have

$$\omega_{1i}^1 = 0, \quad i = 1, 2, 3, 4.$$

For $j = 4, i = 2, 3$ in (3.11), by (3.16) we have

$$\omega_{42}^4 = \omega_{43}^4 = 0.$$

For $i = 1, j = 2, 3, 4$ in (3.11), we obtain

$$\omega_{21}^2 = \omega_{31}^3 = \frac{e_1(\lambda)}{\lambda_1 - \lambda}, \quad \omega_{41}^4 = -\frac{e_1(\lambda_1 + 2\lambda)}{2\lambda_1 + 2\lambda - 4H}. \tag{3.17}$$

For $i = 4, j = 2, 3$ in (3.11), we obtain

$$\omega_{24}^2 = \omega_{34}^3 = \frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda}.$$

For $i = 1$, by choosing $j = 2, k = 3$ or $j = 3, k = 2$ in (3.12), we have

$$\omega_{31}^2 = \omega_{21}^3 = 0.$$

For $i = 4$, by choosing $j = 2, k = 3$ or $j = 3, k = 2$ in (3.12), we get

$$\omega_{34}^2 = \omega_{24}^3 = 0.$$

For $i = 4$ and $j = 1, k = 2, 3$ in (3.12), we have

$$(2\lambda_1 + 2\lambda - 4H)\omega_{24}^1 = (\lambda_1 - \lambda)\omega_{42}^1,$$

$$(2\lambda_1 + 2\lambda - 4H)\omega_{34}^1 = (\lambda_1 - \lambda)\omega_{43}^1,$$

which together with the second and third expressions of (3.15) give

$$\omega_{24}^1 = \omega_{42}^1, = \omega_{34}^1 = \omega_{43}^1 = 0.$$

Similarly, we can also obtain
$$\omega_{12}^4 = \omega_{13}^4 = 0.$$

Let us introduce two smooth functions α and β as follows:

$$\alpha = \frac{e_1(\lambda)}{\lambda_1 - \lambda}, \qquad \beta = \frac{e_1(\lambda_1 + 2\lambda)}{2\lambda_1 + 2\lambda - 4H}. \tag{3.18}$$

Combining the above remarks with (3.10) and summarizing, the covariant derivatives $\nabla_{e_i} e_j$ simplify to

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, \qquad \nabla_{e_2}e_1 = \alpha e_2, \qquad \nabla_{e_3}e_1 = \alpha e_3, \qquad \nabla_{e_4}e_1 = -\beta e_4,\\
&\nabla_{e_1}e_2 = \omega_{12}^3 e_3, \quad \nabla_{e_2}e_2 = -\alpha e_1 + \omega_{22}^3 e_3 - \frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda} e_4, \quad \nabla_{e_3}e_2 = \omega_{32}^3 e_3, \quad \nabla_{e_4}e_2 = \omega_{42}^3 e_3,\\
&\nabla_{e_1}e_3 = \omega_{13}^2 e_2, \quad \nabla_{e_2}e_3 = \omega_{23}^2 e_2, \quad \nabla_{e_3}e_3 = -\alpha e_1 + \omega_{33}^2 e_2 - \frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda} e_4, \quad \nabla_{e_4}e_3 = \omega_{43}^2 e_2,\\
&\nabla_{e_1}e_4 = 0, \quad \nabla_{e_2}e_4 = \frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda} e_2, \quad \nabla_{e_3}e_4 = \frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda} e_3, \quad \nabla_{e_4}e_4 = \beta e_1.
\end{aligned} \tag{3.19}$$

Recall the definition of the Gauss curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

One can compute the curvature tensor R by (3.19) and apply the Gauss equation for different values of X, Y and Z. After comparing the coefficients with respect to the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ we get the following:

- $X = e_1, \quad Y = e_2, \quad Z = e_1,$

$$e_1(\alpha) + \alpha^2 = -\lambda_1 \lambda; \tag{3.20}$$

- $X = e_1, \quad Y = e_2, \quad Z = e_4,$

$$e_1\left(\frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda}\right) + \alpha \frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda} = 0; \tag{3.21}$$

- $X = e_1, \quad Y = e_4, \quad Z = e_1,$

$$-e_1(\beta) + \beta^2 = -\lambda_1(4H - \lambda_1 - 2\lambda); \tag{3.22}$$

- $X = e_3, \quad Y = e_4, \quad Z = e_1,$

$$e_4(\alpha) + (\alpha + \beta)\frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda} = 0; \tag{3.23}$$

- $X = e_4, \quad Y = e_2, \quad Z = e_4,$

$$-e_4\left(\frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda}\right) + \alpha\beta - \left(\frac{e_4(\lambda)}{4H - \lambda_1 - 3\lambda}\right)^2 = \lambda(4H - \lambda_1 - 2\lambda). \tag{3.24}$$

Now, we consider the L_1-biharmonic equation (3.2). It follows from (2.5) and (3.19) that

$$(\lambda_1 - 4H)e_1 e_1(H_2) + (2(\lambda - 4H)\alpha + (\lambda_1 + 2\lambda)\beta)e_1(H_2) - 12H_2(2HH_2 - H_3) = 0. \tag{3.25}$$

From (3.8) and (3.19), we obtain

$$e_i e_1(H_2) = 0, \qquad i = 2, 3, 4. \tag{3.26}$$

Differentiating α and β along e_4, we get Eqs

$$(\lambda_1 - \lambda)e_4(\alpha) - \alpha e_4(\lambda) = e_4 e_1(\lambda),$$

$$(\lambda_1 + \lambda - 2H)e_4(\beta) + \beta e_4(\lambda) = e_4 e_1(\lambda),$$

respectively and eliminating $e_4 e_1(\lambda)$, we have

$$(\lambda_1 + \lambda - 2H)e_4(\beta) = (\lambda_1 - \lambda)e_4(\alpha) - (\alpha + \beta)e_4(\lambda).$$

Putting the value of $e_4(\alpha)$ from (3.23) in the above equation, we find

$$e_4(\beta) = \frac{e_4(\lambda)(\alpha + \beta)(4\lambda - 4H)}{(\lambda_1 + \lambda - 2H)(4H - \lambda_1 - 3\lambda)}.$$

Differentiating (3.25) along e_4 and using (3.26), (3.23) and $e_4(\beta)$, we get

$$e_4(\lambda)\left[\frac{2(\alpha + \beta)(8H\lambda_1 - \lambda_1{}^2 - 3\lambda\lambda_1 + 12H\lambda - 16H^2)e_1(H_2)}{\lambda_1 + \lambda - 2H} + 6H_2\lambda(4H - \lambda_1 - 3\lambda)^2\right] = 0. \tag{3.27}$$

We claim that $e_4(\lambda) = 0$. Indeed, if $e_4(\lambda) \neq 0$, then

$$\frac{2(\alpha + \beta)Ae_1(H_2)}{\lambda_1 + \lambda - 2H} + 6H_2\lambda(4H - \lambda_1 - 3\lambda)^2 = 0, \tag{3.28}$$

where $A := 8H\lambda_1 - \lambda_1{}^2 - 3\lambda\lambda_1 + 12H\lambda - 16H^2$.

Now, differentiating (3.28) along e_4, we have

$$\frac{2(\alpha + \beta)\,[A(6\lambda - 6H) + B]\,e_1(H_2)}{(\lambda_1 + \lambda - 2H)^2} - 36H_2(4H - \lambda_1 - 3\lambda)^2 = 0, \tag{3.29}$$

where $B := (-3\lambda_1 + 12H)(\lambda_1 + \lambda - 2H)(4H - \lambda_1 - 3\lambda)$.

Eliminating $e_1(H_2)$ from (3.28) and (3.29), we obtain

$$2A(3H - \lambda_1 - 2\lambda) = (-\lambda_1 + 4H)(\lambda_1 + \lambda - 2H)(4H - \lambda_1 - 3\lambda). \tag{3.30}$$

Differentiating (3.30) along e_4, we get that $4H = \lambda_1$, which is not possible, since λ_1 is not constant. Consequently, $e_4(\lambda) = 0$. Therefore, (3.24) reduces to

$$\alpha\beta = \lambda(4H - \lambda_1 - 2\lambda). \tag{3.31}$$

Note that (3.13) yields

$$e_1(H_2) = -\frac{4}{3}(\lambda_1 - 2H)e_1(\lambda) + \frac{4}{3}(\lambda_1 + \lambda - 2H)(\lambda_1 - 2H)\beta. \tag{3.32}$$

By using (3.32), (3.31), (3.22) and (3.20), we obtain

$$\begin{aligned} e_1 e_1(H_2) = {} & \frac{4}{3}\lambda_1\lambda(\lambda_1 - \lambda)(\lambda_1 - 2H) + \frac{4}{3}(4H - \lambda_1 - 2\lambda)(\lambda_1 - 2H)(5\lambda_1\lambda + \lambda_1{}^2 - 4H\lambda - 2H\lambda_1) \\ & + \left[-4\alpha + 3\beta + 2\frac{(\lambda_1 + \lambda - 2H)\beta - (\lambda_1 - \lambda)\alpha}{\lambda_1 - 2H}\right]e_1(H_2). \end{aligned} \tag{3.33}$$

Combining (3.25) with (3.33) gives

$$(P_{1,2}\alpha + P_{2,2}\beta)e_1(H_2) = P_{3,6}, \tag{3.34}$$

where $P_{1,2},\ P_{2,2}$ and $P_{3,6}$ are polynomials in terms of λ and λ_1 of degrees 2, 2 and 6 respectively.

Differentiating (3.34) along e_1 and using (3.31), (3.22), (3.20) and (3.34), we get following relation

$$P_{4,8}\alpha + P_{5,8}\beta = P_{6,5}e_1(H_2), \tag{3.35}$$

where $P_{4,8},\ P_{5,8}$ and $P_{6,5}$ are polynomials in terms of λ and λ_1 of degrees 8, 8 and 5 respectively. Also, we have

$$e_1(H_2) = \frac{4}{3}(\lambda_1 - 2H)\Big(\frac{¢}{¢}\beta(\lambda_1 + \lambda - 2H) - \alpha(\lambda_1 - \lambda)\Big). \tag{3.36}$$

Combining (3.35) and (3.36), we obtain

$$\Big(P_{4,8} + \frac{4}{3}P_{6,5}(\lambda_1 - \lambda)(\lambda_1 - 2H)\Big)\alpha + \Big(P_{5,8} - \frac{4}{3}P_{6,5}(\lambda_1 + \lambda - 2H)(\lambda_1 - 2H)\Big)\beta = 0. \tag{3.37}$$

On the other hand, combining (3.36) with (3.34) and using (3.31), we find

$$P_{2,2}(\lambda_1 + \lambda - 2H)(\lambda_1 - 2H)\beta^2 - P_{1,2}(\lambda_1 - \lambda)(\lambda_1 - 2H)\alpha^2 = L, \tag{3.38}$$

where L is given by

$$L = \lambda(4H - \lambda_1 - 2\lambda)(\lambda_1 - 2H)\Big(\frac{¢}{¢}P_{2,2}(\lambda_1 - \lambda) - P_{1,2}(\lambda_1 + \lambda - 2H)\Big) + \frac{3}{4}P_{3,6}.$$

Using (3.37) and (3.31), we get

$$\begin{aligned}
\alpha^2 &= \frac{\frac{4}{3}P_{6,5}(\lambda_1 + \lambda - 2H)(\lambda_1 - 2H) + P_{5,8}}{P_{4,8} + \frac{4}{3}P_{6,5}(\lambda_1 - \lambda)(\lambda_1 - 2H)}\lambda(4H - \lambda_1 - 2\lambda),\\
\beta^2 &= \frac{\frac{4}{3}P_{6,5}(\lambda_1 - \lambda)(\lambda_1 - 2H) - P_{4,8}}{P_{5,8} - \frac{4}{3}P_{6,5}(\lambda_1 + \lambda - 2H)(\lambda_1 - 2H)}\lambda(4H - \lambda_1 - 2\lambda).
\end{aligned} \tag{3.39}$$

Eliminating α^2 and β^2 from (3.38), we obtain

$$\begin{aligned}
&\lambda(4H - \lambda_1 - 2\lambda)(\lambda_1 - 2H)\Big[P_{1,2}(\lambda_1 - \lambda)\Big(P_{5,8} - \frac{4}{3}P_{6,5}(\lambda_1 + \lambda - 2H)(\lambda_1 - 2H)\Big)^2\\
&-P_{2,2}(\lambda_1 + \lambda - 2H)\Big(P_{4,8} + \frac{4}{3}P_{6,5}(\lambda_1 - \lambda)(\lambda_1 - 2H)\Big)^2\Big]\\
&= L\Big(P_{5,8} - \frac{4}{3}P_{6,5}(\lambda_1 + \lambda - 2H)(\lambda_1 - 2H)\Big)\Big(P_{4,8} + \frac{4}{3}P_{6,5}(\lambda_1 - \lambda)(\lambda_1 - 2H)\Big),
\end{aligned} \tag{3.40}$$

which is a polynomial equation of degree 22 in terms of λ and λ_1.

Now consider an integral curve of e_1 passing through $p = \gamma(t_0)$ as $\gamma(t),\ t \in I$. Since $e_i(\lambda_1) = e_i(\lambda) = 0$ for $i = 2, 3, 4$ and $e_1(\lambda_1),\ e_1(\lambda) \neq 0$, we can assume $t = t(\lambda)$ and $\lambda_1 = \lambda_1(\lambda)$ in some neighborhood of $\lambda_0 = \lambda(t_0)$. Using (3.37), we have

$$\begin{aligned}
\frac{d\lambda_1}{d\lambda} &= \frac{d\lambda_1}{dt}\frac{dt}{d\lambda} = \frac{e_1(\lambda_1)}{e_1(\lambda)}\\
&= 2\frac{(\lambda_1 + \lambda - 2H)\beta - (\lambda_1 - \lambda)\alpha}{(\lambda_1 - \lambda)\alpha}\\
&= \frac{2\left(P_{4,8} + \frac{4}{3}P_{6,5}(\lambda_1 - \lambda)(\lambda_1 - 2H)\right)(\lambda_1 + \lambda - 2H)}{\left(\frac{4}{3}P_{6,5}(\lambda_1 + \lambda - 2H)(\lambda_1 - 2H) - P_{5,8}\right)(\lambda_1 - \lambda)} - 2
\end{aligned} \tag{3.41}$$

Differentiating (3.40) with respect to λ and substituting $\frac{d\lambda_1}{d\lambda}$ from (3.41), we get

$$f(\lambda_1, \lambda) = 0, \tag{3.42}$$

another algebraic equation of degree 30 in terms of λ_1 and λ.

We rewrite (3.40) and (3.42) respectively in the following forms

$$\sum_{i=0}^{22} f_i(\lambda_1)\lambda^i, \qquad \sum_{i=0}^{30} g_i(\lambda_1)\lambda^i, \tag{3.43}$$

where $f_i(\lambda_1)$ and $g_j(\lambda_1)$ are polynomial functions of λ_1. We eliminate λ^{30} between these two polynomials of (3.43) by multiplying $g_{30}\lambda^8$ and f_{22} respectively on the first and second equations of (3.43), we obtain a new polynomial equation in λ of degree 29. Combining this equation with the first equation of (3.43), we successively obtain a polynomial equation in λ of degree 28. In a similar way, by using the first equation of (3.43) and its consequences we are able to gradually eliminate λ. At last, we obtain a non-trivial algebraic polynomial equation in λ_1 with constant coefficients. Therefore, we conclude that the real function λ_1 must be a constant, which is a contradiction. Hence H_2 is constant on M^4. If $H_2 \neq 0$, by using (3.2) and (2.4) we obtain that H_3 is constant. Therefore all the mean curvatures H_i are constant functions, this is equivalent to M^4 is isoparametric. An isoparametric hypersurface of Euclidean space can have at most two distinct principal curvatures ([15]), which is a contradiction. So $H_2 \equiv 0$. □

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