Functional Analysis, Approximation and Computation
7 (1) (2015), 77–84

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Browder-type theorems for direct sums of operators

Abdelmajid Arroud[a], Hassan Zariouh[b]

[a]*Lab. of Analysis, Geometry and Applications, Department of Mathematics, Faculty of Science, Mohammed I University, PO Box 524, Oujda 60000 Morocco*
[b]*Centre régional des métiers de l'éducation et de la formation, B.P 458, Oujda, Morocco et Equipe de la Théorie des Opérateurs, Université Mohammed I, Faculté des Sciences d'Oujda, Dépt. de Mathématiques, Morocco*

Abstract. In this paper we study the stability of Browder-type theorems for direct sums. Counterexamples show that in general the properties (Bw), (Bb), (Baw) and (Bab) are not preserved under direct sums. Moreover, we characterize the stability of the property (Bb) under direct sum via union of B-Weyl spectra of its summands. We also obtain analogous results for the properties (Baw), (Bab) and (Bw) with extra assumptions. The theory is exemplified in the case of some special classes of operators.

1. Introduction

We begin by setting the terminology used in this paper. Let X and Y be complex Banach spaces, let $L(X,Y)$ denote the set of bounded linear operators from X to Y, and abbreviate the Banach algebra $L(X,X)$ to $L(X)$. For $T \in L(X)$, let $\ker(T)$, $n(T)$, $\mathcal{R}(T)$, $d(T)$, $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$ and $\sigma_p^0(T)$ denote respectively, the null space, the nullity, the range, the defect, the spectrum, the approximate point spectrum, the point spectrum (the set of all eigenvalues of T) and the set of all eigenvalues of finite multiplicity of T. If $\mathcal{R}(T)$ is closed and $n(T) < \infty$ then T is called an *upper semi-Fredholm* operator, while T is called a *lower semi-Fredholm* operator if $\mathcal{R}(T)$ is closed and $d(T) < \infty$. If $T \in L(X)$ is either upper or lower semi-Fredholm, then T is called a *semi-Fredholm* operator, and the *index* of T is defined by $\mathrm{ind}(T) = n(T) - d(T)$. $T \in L(X)$ is said to be a Fredholm operator if $n(T)$ and $d(T)$ are both finite. For a bounded linear operator T and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an *upper (resp. a lower) semi-B-Fredholm* operator. In this case the index $\mathrm{ind}(T)$ of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [4], [8]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm* operator, and $T \in L(X)$ is called *B-Weyl* if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined: $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}$.

The *ascent* $a(T)$ of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N} : \ker(T^n) = \ker(T^{n+1})\}$, and the *descent* $\delta(T)$ of T is defined by $\delta(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. According to [16], a complex number $\lambda \in \sigma(T)$ is a *pole* of the resolvent of T if $T - \lambda I$ has a finite ascent and finite descent, and in this case they

2010 *Mathematics Subject Classification*. Primary 47A53, 47A10, 47A11.
Keywords. Property (Baw); property (Bab); B-Weyl spectrum; direct sum.
Received: 6 March 2015; Accepted: 25 April 2015
Communicated by Dragan S. Djordjević
Email addresses: arroud@hotmail.com (Abdelmajid Arroud), h.zariouh@yahoo.fr (Hassan Zariouh)

are equal. According to [7], a complex number $\lambda \in \sigma_a(T)$ is a *left pole* of T if $a(T-\lambda I) < \infty$ and $R(T^{a(T-\lambda I)+1})$ is closed.

An operator $T \in L(X)$ is called *upper semi-Browder* if it is upper semi-Fredholm operator of finite ascent, and is called *Browder* if it is Fredholm of finite ascent and descent. The *upper semi-Browder spectrum* $\sigma_{ub}(T)$ of T is defined by $\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not an upper semi-Browder$\}$, and the *Browder spectrum* $\sigma_b(T)$ of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Browder$\}$. The following property named SVEP has relevant role in local spectral theory. For more details see the recent monographs [1] and [17].

Definition 1.1. [17] An operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \longrightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in L(X)$ has SVEP at every isolated point of the spectrum. We also have

$$a(T - \lambda_0 I) < \infty \Longrightarrow \text{ T has SVEP at } \lambda_0,$$

and dually

$$\delta(T - \lambda_0 I) < \infty \Longrightarrow T^* \text{ has SVEP at } \lambda_0,$$

where T^* denotes the dual of T, see [1, Teorem 3.8]. Furthermore, if $T - \lambda_0 I$ is an upper semi-Fredholm then the implications above are equivalences.

Definition 1.2. [1] Let $T \in L(X)$. Then
(i) T is said to be *relatively regular* if there exists an operator $S \in L(X)$ for which $T = TST$ and $STS = S$.
(ii) T is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T; while T is said to be *reguloid* if for every isolated point λ of $\sigma(T)$ the operator $T - \lambda I$ is relatively regular. T is said to be *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent of T.

Note that if $T \in L(X)$ is reguloid then T is isoloid. To see this, suppose that T is reguloid and let $\lambda \in \text{iso}\sigma(T)$. If $n(T - \lambda I) = 0$ then $T - \lambda I$ is an upper semi-Fredholm, since $\mathcal{R}(T - \lambda I)$ is closed. But T^* has SVEP at λ, so $\delta(T - \lambda I) < \infty$ and consequently $\lambda \notin \sigma(T)$, a contradiction. Hence λ is an eigenvalue of T. Note also that an isoloid operator may not be reguloid. Let T be defined on the Hilbert space $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, ...) = (x_2/2, x_3/3, ...)$, then T is isoloid since 0 is the unique isolated point and eigenvalue in $\sigma(T)$. But T is not reguloid since T is not relatively regular. Observe that $\mathcal{R}(T)$ is not closed.

Definition 1.3. [9] Let $T \in L(X)$ and $S \in L(Y)$. We will say that T and S have *a shared stable sign index* if for each $\lambda \notin \sigma_{SBF}(T)$ and $\mu \notin \sigma_{SBF}(S)$, $\text{ind}(T - \lambda I)$ and $\text{ind}(S - \mu I)$ have the same sign, where $\sigma_{SBF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not a semi-B-Fredholm operator$\}$.

For example, from [5, Proposition 2.3] two hyponormal operators T and S acting on a Hilbert space have a shared stable sign index, since $\text{ind}(S - \lambda I) \leq 0$ and $\text{ind}(T - \mu I) \leq 0$ for every $\lambda \notin \sigma_{SBF}(S)$ and $\mu \notin \sigma_{SBF}(T)$. Recall that $T \in L(\mathcal{H})$, $\mathcal{H}$ Hilbert space, is said to be *hyponormal* if $T^*T - TT^* \geq 0$ (or equivalently $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathcal{H}$). The class of hyponormal operators includes also *subnormal* operators and *quasinormal* operators, see [10].

We summarize in the following list the usual notations and symbols needed later.
$\Pi(T)$: poles of T,
$\Pi^0(T)$: poles of T of finite rank,
$\Pi_a^0(T)$: left poles of T of finite rank,
$E(T)$: eigenvalues of T that are isolated in $\sigma(T)$,
$E^0(T)$: eigenvalues of T of finite multiplicity that are isolated in $\sigma(T)$,
$E_a^0(T)$: eigenvalues of T of finite multiplicity that are isolated in $\sigma_a(T)$,
$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T) \Longleftrightarrow$ generalized Browder's theorem holds for T,
iso$A$ (resp., acc$A$) is the set of all isolated (resp., accumulation) points of a given subset A of $\mathbb{C}$.

In this paper, we focus on the problem of giving conditions on the direct summands to ensure that Browder-type properties (introduced very recently in [19]) hold for the direct sum. More recently, several authors have worked in this direction, see for examples [9], [14], [13], [18]. The results obtained are summarized as follows. In the second section, we prove that in general the property (Bw) is not transmitted from the direct summands to the direct sum. Moreover, we prove that if $S \in L(X)$ and $T \in L(Y)$ are isoloid and satisfy property (Bw), then $S \oplus T$ satisfies property (Bw) if and only if $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$, and with no restrictions on S and T we obtain an analogous characterization for property (Bb). In the third section, we give counterexamples which show that property (Bab) is not stable under direct sum $S \oplus T$. Nonetheless, and under the assumption that $\Pi_a^0(S) \cap \rho_a(T) = \Pi_a^0(T) \cap \rho_a(S) = \emptyset$ with T and S both satisfy property (Bab), then $S \oplus T$ satisfies property (Bab) if and only if $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. We also characterize the stability of property (Baw) under direct sum via union of B-Weyl spectra of its components, and under the assumption of equality of their point spectrum.

2. Properties (Bw) and (Bb) for direct sums of operators

We recall that an operator $T \in L(X)$ is said to satisfy property (Bw) if $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$ and is said to satisfy property (Bb) if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$. The properties (Bw) and (Bb) have been introduced very recently in [15] and [19] respectively, as variants of Weyl's theorem and Browder's theorem. We show in the next example (Example 2.3) that property (Bw) may or may not hold for a direct sum of operators for which this property holds. Before that, we include the following two lemmas in order to give a global overview of the subject.

Lemma 2.1. [9] Let $S \in L(X)$ and $T \in L(Y)$. Then $\sigma_{BW}(S \oplus T) \subseteq \sigma_{BW}(S) \cup \sigma_{BW}(T)$.
Moreover, if S and T have a shared stable sign index then $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.

Lemma 2.2. [9] Let $S \in L(X)$ and $T \in L(Y)$. If $S \oplus T$ satisfies generalized Browder's theorem then $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.

Example 2.3. Let R be the unilateral right shift operator defined on $\ell^2(\mathbb{N})$ and L its adjoint, then property (Bw) holds for R and L, since $\sigma(R) = \sigma_{BW}(R) = D(0,1)$ (here and hereafter, $D(0,1)$ denotes the closed unit disc in $\mathbb{C}$), $E^0(R) = \emptyset$, $\sigma(L) = \sigma_{BW}(L) = D(0,1)$ and $E^0(L) = \emptyset$. But it does not hold for $R \oplus L$. In fact $\sigma(R \oplus L) = D(0,1)$, and as $n(R \oplus L) = d(R \oplus L) = 1$ then $0 \notin \sigma_{BW}(R \oplus L)$. So $\sigma_{BW}(R \oplus L) \subsetneq \sigma(R \oplus L)$. We also remark that $E^0(R \oplus L) = \emptyset$. Thus $\sigma(R \oplus L) \setminus \sigma_{BW}(R \oplus L) \neq E^0(R \oplus L)$ and this proves that $R \oplus L$ does not satisfy property (Bw). Note that S and T are isoloid and $\sigma_{BW}(R \oplus L) \subsetneq \sigma_{BW}(R) \cup \sigma_{BW}(L) = D(0,1)$.

Nonetheless, and under the assumption that S and T are isoloid, we give in the following result a characterization of stability of property (Bw) under direct sum.

Theorem 2.4. *Let $S \in L(X)$ and $T \in L(Y)$. If S and T satisfy property (Bw) and are isoloid, then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property (Bw);
(ii) $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.

Proof. (i) $\Longrightarrow$ (ii) The property (Bw) for $S \oplus T$ implies with no other restriction on either S or T that $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. Indeed, from [15, Theorem 2.4], $S \oplus T$ satisfies generalized Browder's theorem and hence by Lemma 2.2 we have $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.
(ii) $\Longrightarrow$ (i) Suppose that $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. As S and T are isoloid then

$$E^0(S \oplus T) = [E^0(S) \cap \rho(T)] \cup [E^0(T) \cap \rho(S)] \cup [E^0(S) \cap E^0(T)] \text{ (see also [18, equality10.2])},$$

where $\rho(.) = \mathbb{C} \setminus \sigma(.)$. On the other hand, since S and T satisfy property (Bw), i.e. $\sigma(S) \setminus \sigma_{BW}(S) = E^0(S)$ and $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$, we then have

$$\begin{aligned}[]
[\sigma(S) \cup \sigma(T)] \setminus [\sigma_{BW}(S) \cup \sigma_{BW}(T)] &= [(\sigma(S) \setminus \sigma_{BW}(S)) \cap \rho(T)] \cup [(\sigma(T) \setminus \sigma_{BW}(T)) \cap \rho(S)] \\
&\quad \cup [(\sigma(S) \setminus \sigma_{BW}(S)) \cap (\sigma(T) \setminus \sigma_{BW}(T))] \\
&= [E^0(S) \cap \rho(T)] \cup [E^0(T) \cap \rho(S)] \cup [E^0(S) \cap E^0(T)].
\end{aligned}$$

Hence

$$\begin{aligned} E^0(S \oplus T) &= [\sigma(S) \cup \sigma(T)] \setminus [\sigma_{BW}(S) \cup \sigma_{BW}(T)] \\ &= \sigma(S \oplus T) \setminus \sigma_{BW}(S \oplus T), \end{aligned}$$

and this shows that property (Bw) is satisfied by $S \oplus T$. □

Remark 2.5. The assumption "S and T are isoloid" is essential in Theorem 2.4. Let $X = \ell^2(\mathbb{N})$, let $B = \{e_i \mid e_i = (\delta_i^j)_{j\in\mathbb{N}}, i \in \mathbb{N}\}$ be the canonical basis of X. Let E be the subspace of X generated by the set $\{e_i \mid 1 \leq i \leq n\}$. Let P be the operator defined on E by $P(x_1, x_2, x_3, ..., x_{n-1}, x_n) = (0, x_2, x_3, ..., x_{n-1}, x_n)$ and let $T \in L(X)$ given by $T(x_1, x_2, x_3, ...) = (0, x_1, x_2/2, x_3/3, ...)$. Then T satisfies property (Bw), since $\sigma(T) = \sigma_{BW}(T) = \{0\}$ and $E^0(T) = \emptyset$. P satisfies property (Bw), since $\sigma(P) = \{0, 1\}$, $\sigma_{BW}(P) = \emptyset$ and $E^0(P) = \{0, 1\}$. But although that $\sigma_{BW}(T \oplus P) = \sigma_{BW}(T) \cup \sigma_{BW}(P) = \{0\}$, $T \oplus P$ does not satisfy property (Bw), since $\sigma(T \oplus P) = \{0, 1\}$, $\sigma_{BW}(T \oplus P) = \{0\}$ and $E^0(T \oplus P) = \{0, 1\}$. Here P is isoloid, but T is not.

Before we state our next corollary as an application of Theorem 2.4 to the class of (H)-operators, we recall the definition of this class and definitions of some classes of operators which are contained in the class (H).
According to [1], the *quasinilpotent* part $H_0(T)$ of $T \in L(X)$ is defined as the set $H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n(x)\|^{\frac{1}{n}} = 0\}$. Note that generally, $H_0(T)$ is not closed and from [1, Theorem 2.31] we have if $H_0(T - \lambda I)$ is closed then T has SVEP at λ. We also recall that T is said to belong to the class (H) if for all $\lambda \in \mathbb{C}$ there exists $p := p(\lambda) \in \mathbb{N}$ such that $H_0(T - \lambda I) = \ker((T - \lambda I)^p)$, see [1] for more details about this class of (H)-operators. Of course, every operator T which belongs the class (H) has SVEP, since $H_0(T - \lambda I)$ is closed. Observe also that $a(T - \lambda I) \leq p$, for every $\lambda \in \mathbb{C}$. The class of operators having the property (H) is rather large. Obviously, it contains every operator having the property (H_1). Recall that an operator $T \in L(X)$ is said to have the property (H_1) if $H_0(T - \lambda I) = \ker(T - \lambda I)$ for all $\lambda \in \mathbb{C}$. Although the property (H_1) seems to be rather strong, the class of operators having the property (H_1) is considerably large. In the sequel we give some important classes of operators which satisfy property (H_1). Every *totally paranormal* operator has property (H_1), and in particular every hyponormal operator has property (H_1). Also every *transaloid* operator or *log-hyponormal* has the property (H_1). Some other operators satisfy property (H); for example *M-hyponormal* operators, *p-hyponormal* operators, *algebraically p-hyponormal* operators, *algebraically M-hyponormal* operators, *subscalar* operators and *generalized scalar* operators. For more details about these definitions and comments which we cited above, we refer the reader to [1], [11], [17].

Corollary 2.6. *Let $S \in L(X)$ and $T \in L(Y)$ be isoloid operators and have a shared stable sign index. If S and T satisfy property (Bw), then $S \oplus T$ satisfies property (Bw). In particular, if S and T are (H)-operators satisfying property (Bw) then $S \oplus T$ satisfies property (Bw).*

Proof. Assume that S and T are isoloid and satisfy property (Bw). Since S and T have a shared stable sign index, from Lemma 2.1 we have $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. But this is equivalent by Theorem 2.4, to say that property (Bw) holds for $S \oplus T$. In particular if S and T are (H)-operators, then they are polaroid and consequently isoloid. But every (H)-operator has SVEP. Thus from [6, Theorem 2.5], we conclude that $\text{ind}(T - \lambda I) \leq 0$ and $\text{ind}(S - \mu I) \leq 0$, for each $\lambda \in \rho_{SBF}(T)$ and $\mu \in \rho_{SBF}(S)$. So $S \oplus T$ satisfies property (Bw). □

Generally, the property (Bb) is also not transmitted from the direct summands to the direct sum. For instance, the operators R and L defined in Example 2.3 satisfy property (Bb), but their direct sum $R \oplus L$ does not satisfy this property, since $\sigma(R \oplus L) \setminus \Pi^0(R \oplus L) = D(0, 1)$ and $\sigma_{BW}(R \oplus L) \subsetneq D(0, 1)$. Note that $\sigma_{BW}(R \oplus L) \neq \sigma_{BW}(R) \cup \sigma_{BW}(L) = D(0, 1)$. However, we characterize in the next theorem the stability of property (Bb) under direct sum via union of B-Weyl spectra of its components.

Theorem 2.7. *Let $S \in L(X)$ and $T \in L(Y)$. If S and T satisfy property (Bb), then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property (Bb);
(ii) $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.

Proof. (i) $\Longrightarrow$ (ii) Property (Bb) for $S \oplus T$ implies from [19, Theorem 2.4] that generalized Browder's theorem holds for T. Thus by Lemma 2.2, $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.
(ii) $\Longrightarrow$ (i) Since we know that the Browder spectrum of a direct sum is the union of the Browder spectra of its components, that is, $\sigma_b(S \oplus T) = \sigma_b(S) \cup \sigma_b(T)$, then

$$\begin{array}{rcl}
\Pi^0(S \oplus T) & = & \sigma(S \oplus T) \setminus \sigma_b(S \oplus T) \\
& = & [\sigma(S) \cup \sigma(T)] \setminus [\sigma_b(S) \cup \sigma_b(T)] \\
& = & [(\sigma(S) \setminus \sigma_b(S)) \cap \rho(T)] \cup [(\sigma(T) \setminus \sigma_b(T)) \cap \rho(S)] \\
& & \cup[(\sigma(S) \setminus \sigma_b(S)) \cap (\sigma(T) \setminus \sigma_b(T))] \\
& = & [\Pi^0(S) \cap \rho(T)] \cup [\Pi^0(T) \cap \rho(S)] \cup [\Pi^0(S) \cap \Pi^0(T)].
\end{array}$$

As observed in the proof of Theorem2.4 we have

$$\begin{array}{rcl}
[\sigma(S) \cup \sigma(T)] \setminus [\sigma_{BW}(S) \cup \sigma_{BW}(T)] & = & [(\sigma(S) \setminus \sigma_{BW}(S)) \cap \rho(T)] \cup [(\sigma(T) \setminus \sigma_{BW}(T)) \cap \rho(S)] \\
& & \cup[(\sigma(S) \setminus \sigma_{BW}(S)) \cap (\sigma(T) \setminus \sigma_{BW}(T))].
\end{array}$$

Since S and T satisfy property (Bb), i.e. $\sigma(S) \setminus \sigma_{BW}(S) = \Pi^0(S)$; $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$, and by hypothesis, $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$, then

$$\begin{array}{rcl}
\sigma(S \oplus T) \setminus \sigma_{BW}(S \oplus T) & = & [\sigma(S) \cup \sigma(T)] \setminus [\sigma_{BW}(S) \cup \sigma_{BW}(T)] \\
& = & [\Pi^0(S) \cap \rho(T)] \cup [\Pi^0(T) \cap \rho(S)] \cup [\Pi^0(S) \cap \Pi^0(T)].
\end{array}$$

Hence $\sigma(S \oplus T) \setminus \sigma_{BW}(S \oplus T) = \Pi^0(S \oplus T)$, i.e. $S \oplus T$ satisfies property (Bb). □

From Theorem 2.7 and Lemma 2.1, we have immediately the following corollary:

Corollary 2.8. *If $S \in L(X)$ and $T \in L(Y)$ have a shared stable sign index and satisfy property (Bb), then $S \oplus T$ satisfies property (Bb). In particular, if S and T are (H)-operators satisfying property (Bb) then $S \oplus T$ satisfies property (Bb).*

3. Properties (Bab) and (Baw) for direct sums of operators

We recall that an operator $T \in L(X)$ is said to satisfy property (Baw) if $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$ and is said to satisfy property (Bab) if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$. The properties (Baw) and (Bab) were introduced very recently in [19], as variants of property (Bw) and property (Bb). Generally, if $T \in L(X)$ and $S \in L(Y)$ satisfy property (Bab), then it is not guaranteed that the direct sum $S \oplus T$ satisfies property (Bab), as we can see in the following example.

Example 3.1. Let $T \in L(\mathbb{C}^n)$ be a quasinilpotent operator and let $R \in L(\ell^2(\mathbb{N}))$ be the unilateral right shift operator. Then $\sigma(T) = \{0\}$, $\sigma_{BW}(T) = \emptyset$, $\Pi_a^0(T) = \{0\}$. Thus $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$, i.e. the property (Bab) is satisfied by T. Moreover, $\sigma(R) = D(0,1)$, $\sigma_{BW}(R) = D(0,1)$, $\Pi_a^0(R) = \emptyset$. So $\sigma(R) \setminus \sigma_{BW}(R) = \Pi_a^0(R)$ and R satisfies property (Bab). But the direct sum $T \oplus R$ defined on the Banach space $\mathbb{C}^n \oplus \ell^2(\mathbb{N})$ does not satisfy property (Bab), because $\sigma(T \oplus R) = D(0,1)$, $\sigma_{BW}(T \oplus R) = D(0,1)$ and $\Pi_a^0(T \oplus R) = \{0\}$. Here $\Pi_a^0(T) \cap \rho_a(R) = \{0\}$ and $\sigma_{BW}(T \oplus R) = \sigma_{BW}(T) \cup \sigma_{BW}(R)$; where $\rho_a(.) = \mathbb{C} \setminus \sigma_a(.)$.

However, and under extra assumptions, we characterize in the following theorem the preservation of property (Bab) under direct sum.

Theorem 3.2. *Suppose that $S \in L(X)$ and $T \in L(Y)$ are such that $\Pi_a^0(S) \cap \rho_a(T) = \Pi_a^0(T) \cap \rho_a(S) = \emptyset$. If S and T satisfy property (Bab), then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property (Bab);
(ii) $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.

Proof. (ii) $\Longrightarrow$ (i) we have

$$\begin{aligned}[\sigma(S)\cup\sigma(T)]\setminus[\sigma_{BW}(S)\cup\sigma_{BW}(T)] &= [(\sigma(S)\setminus\sigma_{BW}(S))\cap\rho(T)]\cup[(\sigma(T)\setminus\sigma_{BW}(T))\cap\rho(S)]\\ &\quad\cup[(\sigma(S)\setminus\sigma_{BW}(S))\cap(\sigma(T)\setminus\sigma_{BW}(T))].\end{aligned}$$

Since S and T satisfy property (Bab), i.e. $\sigma(S)\setminus\sigma_{BW}(S)=\Pi_a^0(S)$ and $\sigma(T)\setminus\sigma_{BW}(T)=\Pi_a^0(T)$ then

$$[\sigma(S)\cup\sigma(T)]\setminus[\sigma_{BW}(S)\cup\sigma_{BW}(T)] = [\Pi_a^0(S)\cap\rho(T)]\cup[\Pi_a^0(T)\cap\rho(S)]\cup[\Pi_a^0(S)\cap\Pi_a^0(T)].$$

The assumption $\Pi_a^0(S)\cap\rho_a(T)=\Pi_a^0(T)\cap\rho_a(S)=\emptyset$ implies that $\Pi_a^0(S)\cap\rho(T)=\Pi_a^0(T)\cap\rho(S)=\emptyset$, and therefore

$$[\sigma(S)\cup\sigma(T)]\setminus[\sigma_{BW}(S)\cup\sigma_{BW}(T)]=\Pi_a^0(S)\cap\Pi_a^0(T).$$

On the other hand, as we know that $\sigma_{ub}(S\oplus T)=\sigma_{ub}(S)\cup\sigma_{ub}(T)$ for any pair of operators, then

$$\begin{aligned}\Pi_a^0(S\oplus T) &= \sigma_a(S\oplus T)\setminus\sigma_{ub}(S\oplus T)\\ &= [\sigma_a(S)\cup\sigma_a(T)]\setminus[\sigma_{ub}(S)\cup\sigma_{ub}(T)]\\ &= [(\sigma_a(S)\setminus\sigma_{ub}(S))\cap\rho_a(T)]\cup[(\sigma_a(T)\setminus\sigma_{ub}(T))\cap\rho_a(S)]\\ &\quad\cup[(\sigma_a(S)\setminus\sigma_{ub}(S))\cap(\sigma_a(T)\setminus\sigma_{ub}(T))]\\ &= [\Pi_a^0(S)\cap\rho_a(T)]\cup[\Pi_a^0(T)\cap\rho_a(S)]\cup[\Pi_a^0(S)\cap\Pi_a^0(T)].\end{aligned}$$

Since we have $\Pi_a^0(T)\cap\rho_a(S)=\emptyset=\Pi_a^0(S)\cap\rho_a(T)$, then it follows that $\Pi_a^0(S\oplus T)=\Pi_a^0(S)\cap\Pi_a^0(T)$. Hence

$$\Pi_a^0(S\oplus T)=[\sigma(S)\cup\sigma(T)]\setminus[\sigma_{BW}(S)\cup\sigma_{BW}(T)].$$

As by hypothesis $\sigma_{BW}(S\oplus T)=\sigma_{BW}(S)\cup\sigma_{BW}(T)$, then $\Pi_a^0(S\oplus T)=\sigma(S\oplus T)\setminus\sigma_{BW}(S\oplus T)$ and $S\oplus T$ satisfies property (Bab).
(i) $\Longrightarrow$ (ii) If $S\oplus T$ satisfies property (Bab) then from [19, Corollary 2.8], $S\oplus T$ satisfies property (Bb). We conclude that $\sigma_{BW}(S\oplus T)=\sigma_{BW}(S)\cup\sigma_{BW}(T)$ as seen in the proof of Theorem 2.7. □

Remark 3.3. Generally, we cannot ensure the transmission of the property (Bab) from two operators S and T to their direct sum even if $\Pi_a^0(S)\cap\rho_a(T)=\Pi_a^0(T)\cap\rho_a(S)=\emptyset$. For this, the shift operators R and L defined in Example 2.3 satisfy property (Bab), because $\sigma(R)=\sigma_{BW}(R)=D(0,1)$, $\Pi_a^0(R)=\emptyset$, $\sigma(L)=\sigma_{BW}(L)=D(0,1)$ and $\Pi_a^0(L)=\emptyset$. But this property is not satisfied by their direct sum, since $\Pi_a^0(R\oplus L)=\emptyset$, $\sigma(R\oplus L)=D(0,1)$ and $\sigma_{BW}(R\oplus L)\subsetneq D(0,1)$. Remark that $\Pi_a^0(R)\cap\rho_a(L)=\Pi_a^0(L)\cap\rho_a(R)=\emptyset$.

A bounded linear operator $A\in L(X,Y)$ is said to be *quasi-invertible* if it is injective and has dense range. Two bounded linear operators $T\in L(X)$ and $S\in L(Y)$ on complex Banach spaces X and Y are *quasisimilar* provided there exist quasi-invertible operators $A\in L(X,Y)$ and $B\in L(Y,X)$ such that $AT=SA$ and $BS=TB$.

Corollary 3.4. *If $S\in L(\mathcal{H})$ and $T\in L(\mathcal{H})$ are quasisimilar hyponormal operators and satisfy property (Bab), then $S\oplus T$ satisfies property (Bab).*

Proof. As S and T are quasisimilar hyponormal, then by [9, Lemma 2.8] we have $\Pi^0(T)=\Pi^0(S)$. The property (Bab) for S and for T entails that $\Pi^0(T)=\Pi_a^0(T)$ and $\Pi^0(S)=\Pi_a^0(S)$, see [19]. So $\Pi_a^0(S)\cap\rho_a(T)=\Pi_a^0(T)\cap\rho_a(S)=\emptyset$. Moreover, since S and T are hyponormal operators then they have a shared stable sign index. This implies from Lemma 2.1 that $\sigma_{BW}(S\oplus T)=\sigma_{BW}(S)\cup\sigma_{BW}(T)$. But this is equivalent by Theorem 3.2, to say that $S\oplus T$ satisfies property (Bab). □

In the next theorem, we characterize the stability of property (Baw) under direct sum via union of B-Weyl spectra of its summands, which in turn are supposed to have the same eigenvalues of finite multiplicity. But before this, we recall that $\sigma_p(S\oplus T)=\sigma_p(S)\cup\sigma_p(T)$ and $n(S\oplus T)=n(S)+n(T)$ for every pair of operators so that $\sigma_p^0(S\oplus T)=\{\lambda\in\sigma_p^0(S)\cup\sigma_p^0(T): n(S-\lambda I)+n(T-\lambda I)<\infty\}$. Moreover, if A and B are bounded subsets of complex plane $\mathbb{C}$ then $\mathrm{acc}(A\cup B)=\mathrm{acc}(A)\cup\mathrm{acc}(B)$.

Theorem 3.5. *Let $S \in L(X)$ and $T \in L(Y)$ be such that $\sigma_p^0(S) = \sigma_p^0(T)$. If S and T satisfy property (Baw), then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property (Baw);
(ii) $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.

Proof. (ii) $\Longrightarrow$ (i) Suppose that $\sigma_{BW}(S \oplus T) = \sigma_{SBW}(S) \cup \sigma_{BW}(T)$. As S and T satisfy property (Baw), i.e. $\sigma(S) \setminus \sigma_{BW}(S) = E_a^0(S)$ and $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$, then as seen in the proof of Theorem 2.4 we have

$$\begin{aligned}\sigma(S \oplus T) \setminus \sigma_{BW}(S \oplus T) &= [\sigma(S) \cup \sigma(T)] \setminus [\sigma_{BW}(S) \cup \sigma_{BW}(T)] \\ &= [E_a^0(T) \cap \rho(S)] \cup [E_a^0(S) \cap \rho(T)] \cup [E_a^0(S) \cap E_a^0(T)].\end{aligned}$$

Since by hypothesis $\sigma_p^0(T) = \sigma_p^0(S)$, then $E_a^0(T) \cap \rho_a(S) = E_a^0(S) \cap \rho_a(T) = \emptyset$ which implies that $E_a^0(T) \cap \rho(S) = E_a^0(S) \cap \rho(T) = \emptyset$. Thus

$$\sigma(S \oplus T) \setminus \sigma_{BW}(S \oplus T) = E_a^0(S) \cap E_a^0(T).$$

On the other hand, since $\sigma_p^0(T) = \sigma_p^0(S)$ then $\sigma_p^0(S \oplus T) = \sigma_p^0(S) = \sigma_p^0(T)$. We then have

$$\begin{aligned}E_a^0(S \oplus T) &= \{\text{iso}\sigma_a(S \oplus T)\} \cap \sigma_p^0(S \oplus T) \\ &= \{\text{iso}[\sigma_a(S) \cup \sigma_a(T)]\} \cap \sigma_p^0(S) \\ &= \{[\sigma_a(S) \cup \sigma_a(T)] \setminus \text{acc}[\sigma_a(S) \cup \sigma_a(T)]\} \cap \sigma_p^0(S) \\ &= \{[\sigma_a(S) \cup \sigma_a(T)] \setminus [\text{acc}\sigma_a(S) \cup \text{acc}\sigma_a(T)]\} \cap \sigma_p^0(S) \\ &= \{[\text{iso}\sigma_a(S) \cap \rho_a(T)] \cup [\text{iso}\sigma_a(T) \cap \rho_a(S)] \cup [\text{iso}\sigma_a(S) \cap \text{iso}\sigma_a(T)]\} \cap \sigma_p^0(S) \\ &= [E_a^0(S) \cap \rho_a(T)] \cup [E_a^0(T) \cap \rho_a(S)] \cup [E_a^0(S) \cap E_a^0(T)] \\ &= E_a^0(S) \cap E_a^0(T), \text{ because } E_a^0(S) \cap \rho_a(T) = E_a^0(T) \cap \rho_a(S) = \emptyset.\end{aligned}$$

Hence $\sigma(S \oplus T) \setminus \sigma_{BW}(S \oplus T) = E_a^0(S \oplus T)$ and $S \oplus T$ satisfies property (Baw).
(i) $\Longrightarrow$ (ii) If $S \oplus T$ satisfies property (Baw), then by [19, Corollary 3.5], $S \oplus T$ satisfies property (Bw). Consequently, we have the equality $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$, as seen in the proof of Theorem 2.4. □

Example 3.6. In general, we cannot expect that property (Baw) will be satisfied by the direct sum $S \oplus T$ for every two operators S and T satisfying property (Baw). For instance, if we consider the operators T and R defined in Example 3.1, then T and R satisfy property (Baw), because $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T) = \{0\}$, $\sigma(R) \setminus \sigma_{BW}(R) = E_a^0(R) = \emptyset$. They also satisfy the equality $\sigma_{BW}(T \oplus R) = \sigma_{BW}(T) \cup \sigma_{BW}(R) = D(0,1)$. But $T \oplus R$ does not satisfy property (Baw), because $\sigma(T \oplus R) \setminus \sigma_{BW}(T \oplus R) = \emptyset \neq E_a^0(T \oplus R) = \{0\}$. Observe that $\sigma_p^0(R) = \emptyset \neq \sigma_p^0(T) = \{0\}$.

Corollary 3.7. *Let $S \in L(X)$ and $T \in L(Y)$ be quasisimilar operators satisfying property (Baw). If S or T has SVEP, then $S \oplus T$ satisfies property (Baw).*

Proof. The quasisimilarity of S and T implies that $\sigma_p^0(S) = \sigma_p^0(T)$. It implies also from [1, Theorem 2.15] that S and T have SVEP. So they have a shared stable sign index and hence $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. But this is equivalent from Theorem 3.5, to say that $S \oplus T$ satisfies property (Baw). □

We finish this section by some illustrating examples.

1. A bounded linear operator $T \in L(\mathcal{H})$ is said to be p-hyponormal, with $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ and is said to be log-hyponormal if T is invertible and satisfies $\log(T^*T) \geq \log(TT^*)$. According to [3], if $T \in L(\mathcal{H})$ is invertible and p-hyponormal, there exists $S \in L(\mathcal{H})$ log-hyponormal quasisimilar to T. Then $\sigma_p^0(S) = \sigma_p^0(T)$. As S and T have a shared stable sign index then $\sigma_{BW}(S \oplus T = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. Moreover, if S and T satisfy property (Baw), then $S \oplus T$ satisfies property (Baw).

2. A bounded linear operator $T \in L(\mathcal{H})$ is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\|\,\|x\|$, for all $x \in \mathcal{H}$. According to [2], every paranormal operator has SVEP. Moreover, paranormal operators are polaroid [12, Lemma 2.3] and hence isoloid. So by Theorem 2.4, if S and T are paranormal operators and satisfy property (Bw), then $S \oplus T$ satisfies property (Bw). We notice that a paranormal operator may not be in the class of (H)-operators, for instance see [2, Example 2.3].

Acknowledgment. The authors would like to thank the referee for his several interesting remarks and suggestions which have improved this paper.

References

[1] P. Aiena, *Fredholm and Local Spectral Theory, with Application to Multipliers*, Kluwer Academic Publishers, (2004).
[2] P. Aiena, J.R. Guillen, *Weyl's theorem for perturbations of paranormal operators*, Proc. Amer. Math. Soc. **135** (2007), 2443–2451.
[3] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integr. Equ. and Oper. Theory, **13** (1990), 307–315.
[4] M. Berkani, *On a class of quasi-Fredholm operators*, Integr. Equ. and Oper. Theory, **34** (1999), no. 2, p. 244–249.
[5] M. Berkani and A. Arroud, *Generalized Weyl's theorem and hyponormal operators*, J. Aust. Math. Soc. **76** (2004), 291–302.
[6] M. Berkani and N. Castro and S. V. Djordjević, *Single valued extension property and generalized Weyl's theorem*, Math. Bohemica, **131** (2006), No. 1, p. 29–38.
[7] M. Berkani, J. J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69** (2003), 359–376.
[8] M. Berkani, M. Sarih, *On semi B-Fredholm operators*, Glasgow Math. J. **43** (2001), 457–465.
[9] M. Berkani, H. Zariouh, *Weyl-type Theorems for direct sums*, Bull. Korean. Math. Soc. **49** (2012), No. 5, pp. 1027–1040.
[10] J. B. Conway, (1990). *The theory of subnormal operators*, Mathematical Surveys and mlonographs, N. 36, (1992). American Mthematical Society, Providence, Rhode Island. Springer-Verlag, New York.
[11] R. E. Curto, Y. M. Han, *Weyl's theorem, a-Weyl's thorem, and local spectral theory*, J. London Math. Soc. **67** (2) (2003), 499–509.
[12] R. Curto and Y.M. Han Weyl's theorem for algebraically paranormal operators. In- tegr. equ. oper. theory **47**, No.3, (2003), 307-314.
[13] S. V. Djordjević and Y. M. Han, *A note on Weyl's theorem for operator matrices*, Proc. Amer. Math. Soc. **131**, No. 8 (2003), pp. 2543–2547.
[14] B. P. Duggal, C. S. Kubrusly, *Weyl's theorem for direct sums*, Studia Sci. Math. Hungar. **44** (2007), 275–290.
[15] A. Gupta and N. Kashyap, *Property (Bw) and Weyl type theorems*, Bull. Math. Anal. Appl. **3** (2) (2011), 1–7.
[16] H. Heuser, *Functional Analysis*, John Wiley & Sons Inc, New York, (1982).
[17] K. B. Laursen and M. M. Neumann, *An introduction to Local Spectral Theory*, Clarendon Press Oxford, (2000).
[18] W. Y. Lee, *Weyl spectra of operator matrices*, Proc. Amer. Math. Soc. **129** (2001), 131–138.
[19] H. Zariouh and Zguitti, *Variations on Browder's Theorem*, Acta Math. Univ. Comenianae Vol **81**, 2 (2012), pp, 255–264.