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Analytically Riesz operators and Weyl and Browder type theorems

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Abstract. Several spectra of analytically Riesz operators will be characterized. These results will led to prove Weyl and Browder type theorems for the aforementioned class of operators.

1. Introduction

Recall that given a Banach space X and an operator $T \in \mathcal{L}(\mathcal{X})$, T is said to be polynomially compact (respectively polynomially Riesz), if there exists a complex polynomial P such that $P(T)$ is a compact operator (respectively a Riesz operator). The structure and the spectrum of a polynomially compact operators is well known; in fact, it was characterized in [14]. In addition, these results were extended to polynomially Riesz bounded and linear maps in [13].

More generally, an operator T is said to be analytically Riesz, if there exists an analytical function f defined on a neighbourhood of the spectrum of T such that $f(T)$ is Riesz. The structure and the spectrum of analytically Riesz operators were studied in [15]. In particular, according to [15, Theorem 1], such an operator can be decomposed as $T = T_0 \oplus S$, where f is locally zero at each point of the spectrum of T_0 but it is not locally zero at any point of the spectrum of S. In addition, S either acts on a finite dimensional space or it has the decomposition $S = T_1 \oplus \cdots \oplus T_n$, and in this last case the spectra of T_i are disjoint and there are λ_i with the property that $T_i - \lambda_i$ are Riesz and $f(\lambda_i) = 0$ $(i = 1, \ldots, n)$.

The first objective of this article is to characterize several spectra of analytically Riesz operators. These spectra are defined in the infinite dimensional context and among others they are the Fredholm spectrum, the Weyl spectrum, the Browder spectrum, the upper Fredholm spectrum, the Weyl essential approximate point spectrum and the Browder essential approximate point spectrum. Moreover, the spectrum of T will be fully described. This will be done in section 3. To this end, however, some restrictions need to be considered.

In fact, according to [15, Theorem 1], since f is locally zero at each point of the spectrum of T_0, no information concerning the spectrum of T_0 can be obtained. Actually, note that $f(T) = f(T_0) \oplus f(S)$ and $f(T_0) = 0$. What is more, if S is defined on a finite dimensional Banach space, then it has no sense to study the aforementioned spectra. As a result, the analytically Riesz operators T that will be considered in this article will be bounded and linear maps defined on infinite dimensional Banach spaces and the holomorphic

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function f with the property that $f(T)$ is Riesz will be assumed not to be locally zero at any point of the spectrum of T. Naturally, polynomially Riesz and polynomially compact operators belong to this subclass of analytically Riesz operators.

In addition, the results obtained in section 3 will be applied in section 4 to prove Weyl and Browder type theorems for analytically Riesz operators.

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2. Preliminary definitions and facts

Fron now on $\mathcal{X}$ will denote an infinite dimensional complex Banach space and $\mathcal{L}(\mathcal{X})$ the algebra of all bounded and linear maps defined on and with values in $\mathcal{X}$. If $T \in \mathcal{L}(\mathcal{X})$, then $N(T)$ and $R(T)$ will stand for the null space and the range of T, respectively. Note that $I \in \mathcal{L}(\mathcal{X})$ will be the identity map defined on $\mathcal{X}$. In addition, $\mathcal{K}(\mathcal{X}) \subset \mathcal{L}(\mathcal{X})$ will denote the closed ideal of compact operators defined on $\mathcal{X}$, $C(\mathcal{X})$ the Calkin algebra of $\mathcal{X}$ and $\pi\colon \mathcal{L}(\mathcal{X}) \to C(\mathcal{X})$ the quotient map. Furthermore, $\mathcal{X}^*$ will stand for the dual space of $\mathcal{X}$ and if $T \in \mathcal{L}(\mathcal{X})$, then $T^* \in \mathcal{L}(\mathcal{X}^*)$ will denote the dual operator. Moreover, given $T \in \mathcal{L}(\mathcal{X})$, $\sigma(T)$ will stand for the spectrum of T and if $K \subset \mathbb{C}$, then acc K will be the set of limit points of K and iso $K = K\setminus$ acc K the set of isolated points of K.

Recall that $T \in L(\mathcal{X})$ is said to be a *Fredholm* operator if $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim \mathcal{X}/R(T)$ are finite dimensional, in which case its *index* is given by

$$\mathrm{ind}(T) = \alpha(T) - \beta(T).$$

If $\alpha(T)$ and $\beta(T)$ are finite and equal, so that the index is zero, T is said to be a *Weyl* operator. The set of Fredholm operators will be dented by $\Phi(\mathcal{X})$.

The *ascent* (respectively *descent*) of $T \in \mathcal{L}(\mathcal{X})$ is the smallest non-negative integer a (respectively d) such that $N(T^a) = N(T^{a+1})$ (respectively $R(T^d) = R(T^{d+1})$); if such an integer does not exist, then $asc(T) = \infty$ (respectively $dsc(T) = \infty$). The operator T will be said to be *Browder*, if it is Fredholm and its ascent and descent are finite.

These classes of operators generate the Fredholm or essential spectrum, the Weyl spectrum and the Browder spectrum, which will be denoted by $\sigma_e(T)$, $\sigma_w(T)$ and $\sigma_b(T)$ respectively ($T \in \mathcal{L}(\mathcal{X})$). It is well known that

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{ acc } \sigma(T) \subseteq \sigma(T).$$

Recall that $T \in \mathcal{L}(\mathcal{X})$ is said to be a *Riesz operator*, if $\sigma_e(T) = \{0\}$. The set of all Riesz operators defined on $\mathcal{X}$ will be denoted by $\mathcal{R}(\mathcal{X})$. More generally, T will be said to be an *analytically Riesz operator*, if there exists a holomorphic function f defined on an open neighbourhood of $\sigma(T)$ such that $f(T) \in \mathcal{R}(\mathcal{X})$ ($\mathcal{H}(\sigma(T))$ will denote the algebra of germs of analytic functions defined on open neighbourhoods of $\sigma(T)$). In particular, T will be said to be *polynomially Riesz* (respectively *polynomially compact*), if there exists $P \in C[X]$ such that $P(T) \in \mathcal{R}(\mathcal{X})$ (respectively $P(T) \in \mathcal{K}(\mathcal{X})$).

The concept of Fredholm operator has been generalized. An operator $T \in \mathcal{L}(\mathcal{X})$ will be said to be *B-Fredholm*, if there exists $n \in \mathbb{N}$ for which $R(T^n)$ is closed and the induced operator $T_n \in \mathcal{L}(R(T^n))$ is Fredholm. Note that if for some $n \in \mathbb{N}$, $T_n \in \mathcal{L}(R(T^n))$ is Fredholm, then $T_m \in \mathcal{L}(R(T^m))$ is Fredholm for all $m \geq n$; moreover $\mathrm{ind}(T_n) = \mathrm{ind}(T_m)$, for all $m \geq n$. Therefore, it makes sense to define the index of T by $\mathrm{ind}(T) = \mathrm{ind}(T_n)$. Recall that T is said to be *B-Weyl*, if T is B-Fredholm and $\mathrm{ind}(T) = 0$. Moreover, T is said to be a *B-Browder* operator, if there is some $n \in \mathbb{N}$ such that $R(T^n)$ is closed, $T_n \in L(R(T^n))$ is Fredholm and $asc(T_n)$ and $dsc(T_n)$ are finite. Naturally, from these classes of operators, the B-Fredholm spectrum, the B-Weyl spectrum and the B-Browder spectrum of $T \in \mathcal{L}(\mathcal{X})$ can be derived, which will be denoted by $\sigma_{BF}(T)$, $\sigma_{BW}(T)$ and $\sigma_{BB}(T)$, respectively. Clearly,

$$\sigma_{BF}(T) \subseteq \sigma_{BW}(T) \subseteq \sigma_{BB}(T) \subseteq \sigma(T).$$

On the other hand, recall that a Banach space operator $T \in \mathcal{L}(\mathcal{X})$ is said to be Drazin invertible, if there exists a necessarily unique $S \in \mathcal{L}(\mathcal{X})$ and some $m \in \mathbb{N}$ such that

$$T^m = T^m ST, \quad STS = S, \quad ST = TS.$$

Recall that necessary and sufficient for T to be Drazin invertible is that $asc(T)$ and $des(T)$ are finite ([16, Theorem 4]). If $DR(\mathcal{L}(\mathcal{X})) = \{A \in \mathcal{L}(\mathcal{X})\colon A \text{ is Drazin invertible}\}$, then the Drazin spectrum of $T \in \mathcal{L}(\mathcal{X})$ is the set $\sigma_{DR}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \notin DR(\mathcal{L}(\mathcal{X}))\}$, see [9, 10]. Note that according to [7, Theorem 3.6] and [16, Theorem 4], $\sigma_{BB}(T) = \sigma_{DR}(T)$. In particular, $\sigma_{BW}(T) \subseteq \sigma_{DR}(T) \subseteq \sigma(T)$.

In order to state the (generalized) Weyl's and the (generalized) Browder's theorems, some sets need to be recalled.

Let $T \in \mathcal{L}(\mathcal{X})$ and denote by $E(T) = \{\lambda \in \text{iso}\,\sigma(T)\colon 0 < \alpha(A - \lambda)\}$ (respectively by $E_0(T) = \{\lambda \in E(T)\colon \alpha(A - \lambda) < \infty\}$) the set of eigenvalues of T that are isolated in the spectrum of T (respectively the eigenvalues of finite multiplicity of T isolated in $\sigma(T)$). In addition, denote by $\Pi(T) = \{\lambda \in \sigma(T)\colon 0 < asc(T - \lambda) = dsc(T - \lambda) < \infty\}$ (respectively $\Pi_0(T) = \{\lambda \in \Pi(T)\colon \alpha(T - \lambda) < \infty\}$) the set of poles of T (respectively the set of poles of finite rank of T). Note that $\sigma(T) \setminus \sigma_{DR}(T) = \Pi(T)$ ([16, Theorem 4]) and $\sigma(T) \setminus \sigma_b(T) = \Pi_0(T)$ ([5, Proposition 2]). In particular, $\sigma_{DR}(T) \subseteq \sigma_b(T)$.

Definition 2.1. *Consider a Banach space $\mathcal{X}$ and $T \in \mathcal{L}(\mathcal{X})$. Then, it will be said that*

(i)*Weyl's theorem (Wt) holds for T, if $\sigma_w(T) = \sigma(T) \setminus E_0(T)$.*

(ii) *Generalized Weyl's theorem (gWt) holds for T, if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$.*

(iii) *Browder's theorem (Bt) holds for T, if $\sigma_w(T) = \sigma(T) \setminus \Pi_0(T)$.*

(iv) *Generalized Browder's theorem (gBt) holds for T, if $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$.*

Next the definitions of (generalized) a-Weyl and (generalized) a-Browder theorems will be recalled. To this end, however, some preparation is needed first.

Recall that $T \in \mathcal{L}(\mathcal{X})$ is said to be *bounded below*, if $N(T) = 0$ and $R(T)$ is closed. Denote the *approximate point spectrum* of T by $\sigma_a(T) = \{\lambda \in \mathbb{C}\colon A - \lambda I \text{ is not bounded below}\}$. In addition, if $R(T)$ is closed and $\alpha(T)$ is finite, then $T \in \mathcal{L}(\mathcal{X})$ is said to be *upper semi-Fredholm*. Note that in this case ind (T) is well defined. Let $\sigma_{\Phi_+}(T) = \{\lambda \in \mathbb{C}\colon A - \lambda I \text{ is not upper semi-Fredholm}\}$ denote the upper Fredholm spectrum.

The *Weyl essential approximate point spectrum* of $T \in \mathcal{L}(\mathcal{X})$ is the set ([17])

$$\sigma_{aw}(T) = \{\lambda \in \sigma_a(T)\colon T - \lambda I \text{ is not upper semi-Fredholm or } 0 < \text{ind}(A - \lambda I)\}.$$

In addition, the *Browder essential approximate point spectrum* of $T \in \mathcal{L}(\mathcal{X})$ is the set $\sigma_{ab}(T) = \{\lambda \in \sigma_a(T)\colon \lambda \in \sigma_{aw}(T) \text{ or } asc(T - \lambda I) = \infty\}$ ([17]). It is clear that $\sigma_{aw}(A) \subseteq \sigma_{ab}(A) \subseteq \sigma_a(A)$.

On the other hand, upper semi B-Fredholm operators can be defined in a similar way as B-Fredholm operators. Set $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \text{ is not upper semi B-Fredholm or } 0 < \text{ind}(A - \lambda)\}$ ([8]).

Next denote by $LD(\mathcal{X}) = \{T \in \mathcal{L}(\mathcal{X})\colon\ a = asc(T) < \infty \text{ and } R(T^{a+1}) \text{ is closed}\}$ the set of *left Drazin invertible* operators. Then, given $T \in B(\mathcal{X})$, the *left Drazin spectrum* of T is the set $\sigma_{LD}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \notin LD(\mathcal{X})\}$. Note that according to [8, Lemma 2.12], $\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T) \subseteq \sigma_a(T)$.

Let $T \in \mathcal{L}(\mathcal{X})$ and denote by $E^a(T) = \{\lambda \in \text{iso}\,\sigma_a(T)\colon 0 < \alpha(T - \lambda I)\}$ (respectively by $E_0^a(T) = \{\lambda \in E^a(T)\colon \alpha(T - \lambda T) < \infty\}$) the set of eigenvalues of T that are isolated in $\sigma_a(T)$ (respectively, the eigenvalues of finite multiplicity of T isolated in $\sigma_a(T)$). In addition, denote by $\Pi^a(T) = \{\lambda \in \sigma_a(T)\colon a = asc(T - \lambda I) < \infty \text{ and } R(T - \lambda I)^{a+1} \text{ is closed}\}$ (respectively $\Pi_0^a(T) = \{\lambda \in \Pi^a(T)\colon \alpha(T - \lambda I) < \infty\}$) the set of left poles of T (respectively, the left poles of finite rank of T). Clearly, $\sigma_a(T) \setminus \sigma_{LD}(T) = \Pi^a(T)$. Moreover, according to [17, Corollary 2.2], [12, Corollary 1.3.3] and [12, Corollary 1.3.4], $\sigma_a(T) \setminus \sigma_{ab}(T) = \Pi_0^a(T)$.

Next follows the definitions of (generalized) a-Weyl and (generalized) a-Browder theorems.

Definition 2.2. *Consider a Banach space $\mathcal{X}$ and $T \in \mathcal{L}(\mathcal{X})$. Then, it will be said that*

(i) *a-Weyl's theorem (a-Wt) holds for T, if $\sigma_{aw}(T) = \sigma_a(T) \setminus E_0^a(T)$.*

(ii)*Generalized a-Weyl's theorem (a-gWt) holds for T, if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$.*

(iii) *a-Browder's theorem (a-Bt) holds for T, if $\sigma_{aw}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$.*

(iv)*Generalized a-Browder's theorem (a-gBt) holds for T, if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$.*

Finally, recall that an operator $T \in L(\mathcal{X})$ is said to have the single–valued extension property (SVEP for short), at a (complex) point λ_0, if for every open disc $\mathcal{D}$ centered at λ_0 the only analytic function $f\colon \mathcal{D} \longrightarrow \mathcal{X}$ satisfying $(T-\lambda)f(\lambda) = 0$ is the function $f \equiv 0$. We say that T has SVEP, if it has SVEP at every point of $\mathbb{C}$. Trivially, every operator T has SVEP at points of the resolvent $\rho(A) = C \setminus \sigma(T)$. Also T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. See [1, Chapter 2] for more information on operators with SVEP.

3. Spectra of analytically Riesz operators

As it was explained in the Introduction, to study several spectra of analytically Riesz operators, it is necessary to consider a subclass of these operatores. To this end, given $T \in \mathcal{L}(\mathcal{X})$, set $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) = \{f \in \mathcal{H}(\sigma(T))\colon f(T) \in \mathcal{R}(\mathcal{X})$ and f is not locally zero at any point of $\sigma(T)\}$. In the following remark some basic facts will be considered.

Remark 3.1. Let $\mathcal{X}$ be an infinte dimensional Banach space and consider $T \in \mathcal{L}(\mathcal{X})$ such that $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) \neq \emptyset$. Let $f \in \mathcal{H}_{\mathcal{R}NLZ}(\sigma(T))$.
(i). According to the proof of [15, Thorem 1],

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq f^{-1}(0).$$

(ii). Recall that acc $\sigma(T) \subseteq \sigma_{DR}(T)$ ([10, Theorem 12(iv)]). In particular, since $\sigma_{DR}(T) \subseteq \sigma_b(T)$,

$$\text{acc}\, \sigma(T) \subseteq \sigma_{DR}(T) \subseteq f^{-1}(0).$$

(iii). Since f is not locally zero at any point of the spectrum of T, the set $f^{-1}_{\sigma(T)}(0) = \{\lambda \in \sigma(T)\colon f(\lambda) = 0\}$ is finite. Define

$$n_T = \min\{\sharp f^{-1}_{\sigma(T)}(0)\colon f \in \mathcal{H}_{\mathcal{R}NLZ}(\sigma(T))\}.$$

The function $h \in \mathcal{H}_{\mathcal{R}NLZ}(\sigma(T))$ will be said to be *a minimal analytical function associated to T*, if $n_T = \sharp h^{-1}_{\sigma(T)}(0)$.
In addition, recall that there exists an analytic function g defined on a neighbourhod of $\sigma(T)$ such that $g(z) \neq 0$, $z \in \sigma(T)$, and $f(z) = (z-\lambda_1)^{k_1} \ldots (z-\lambda_n)^{k_n} g(z)$, where $f^{-1}_{\sigma(T)}(0) = \{\lambda_1, \ldots, \lambda_n\}$.
On the other hand, when $T \in \mathcal{L}(\mathcal{X})$ is a polynomially Riesz operator, a polynomial $Q \in \mathbb{C}[X]$ will be said to a *minimal polynomial associated to T*, if $Q(T) \in \mathcal{R}(\mathcal{X})$, and if $P \in \mathbb{C}[X]$ is such that $P(T) \in R(\mathcal{X})$, then the degree of Q is less or equal to the degree of P.
(iv). Note that $\sigma(T)$ is countable. In fact, it is a consequence of (ii)-(iii) and [11, Theorem 2.2].

Applying Remark 3.1, the following properties can be deduced.

Corollary 3.2. *Let $\mathcal{X}$ be a Banach space and consider $T \in \mathcal{L}(\mathcal{X})$ such that $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) \neq \emptyset$. Then, the following statements hold.*
(i) *T and T^* have the SVEP.*
(ii) *$\sigma(T) = \sigma_a(T)$ and $\sigma_e(T) = \sigma_{\Phi_+}(T)$.*
(iii) *$E(T) = E^a(T)$ and $E_0(T) = E^a_0(T)$.*
(iv) *$\Pi(T) = \Pi^a(T)$ and $\Pi_0(T) = \Pi^a_0(T)$.*

Proof. (i). According to Remark 3.1(iv), $\sigma(T) = \partial\sigma(T)$. In addition, since $\sigma(T) = \sigma(T^*)$, $\sigma(T^*) = \partial\sigma(T^*)$. In particular, T and T^* have the SVEP.
(ii). Apply [1, Corollary 2.45, Chapter 2, Section 3] and [1, Corollary 3.53, Chapter 3, Section 4], respectively.
(iii). It can be derived from statement (ii).
(iv). According to [6, Theorem 2.7] and [10, Theorem 12(i)], $\Pi(T) = \Pi^a(T)$, which in turn implies that $\Pi_0(T) = \Pi^a_0(T)$. □

In the following theorem the Fredholm, Weyl and Browder spectra of analytically Riesz operators will be characterized.

Theorem 3.3. *Let $\mathcal{X}$ be a Banach space and consider $T \in \mathcal{L}(\mathcal{X})$ such that $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) \neq \emptyset$. Let $h \in \mathcal{H}_{\mathcal{R}NLZ}(\sigma(T))$ be a minimal analytic function associated to T. Then, the following statements hold.*
(i) $\sigma_e(T) = \sigma_w(T) = \sigma_b(T) = h^{-1}_{\sigma(T)}(0)$.
(ii) $\sigma(T) \setminus h^{-1}_{\sigma(T)}(0) = \Pi_0(T)$.

Proof. (i). According to Remark 3.1(i) ([15, Theorem 1]), it is enough to prove that $h^{-1}_{\sigma(T)}(0) \subseteq \sigma_e(T)$. Recall that there exist $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \sigma(T)$ such that $h^{-1}_{\sigma(T)}(0) = \{\lambda_1, \dots, \lambda_n\}$ and $h(z) = (z-\lambda_1)^{k_1} \dots (z-\lambda_n)^{k_n} g(z)$, where g is an analytic function defined on a neighbourhod of $\sigma(T)$ such that $g(z) \neq 0$, $z \in \sigma(T)$. Suppose that there is j, $1 \leq j \leq n$, such that $\lambda_j \notin \sigma_e(T)$. In particular, $(z-\lambda_j)^{k_j} \in \mathcal{L}(\mathcal{X})$ is a Fredholm operator.

Let $U_i \in \mathcal{L}(\mathcal{X})$ and $K_i \in \mathcal{K}(\mathcal{X})$, $i = 1, 2$, such that $U_1(T-\lambda_j)^{k_j} = I - K_1$ and $(T-\lambda_j)^{k_j} U_2 = I - K_2$. Set $h_1(z) = \Pi_{1 \leq i \leq n, i \neq j}(z-\lambda_i)^{k_i} g(z)$ and consider $\Pi\colon \mathcal{L}(\mathcal{X}) \to C(\mathcal{X})$. Note that $\sigma(\pi(h(T))) = \{0\}$, $\pi(U_1) = \pi(U_2) = (\pi(T-\lambda_j)^{k_j})^{-1}$ and

$$\pi(h(T)) = \pi(h_1(T))\pi(T-\lambda_j)^{k_j} = \pi(T-\lambda_j)^{k_j}\pi(h_1(T)).$$

However, an easy calculaton proves that

$$\pi(U_1)\pi(h(T)) = \pi(h(T))\pi(U_1) = \pi(h_1(T)).$$

In particular, $\sigma(\pi(h_1(T)) = \{0\}$, equivalenty $h_1(T) \in \mathcal{R}(\mathcal{X})$, which is impossible because $n = n_T$.
(ii). Apply statement (i) and [5, Proposition 2]. □

Remark 3.4. Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Suppose that $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) \neq \emptyset$ an let $h \in \mathcal{H}_{\mathcal{R}NLZ}(\sigma(T))$ be a minimal analytic function associated to T.
(i). Recall that $\Pi(T) \subseteq \operatorname{iso} \sigma(T)$. Let $I(T) = \operatorname{iso} \sigma(T) \setminus \Pi(T)$. Then, according to [10, Theorem 12], $\sigma(T) = \sigma_{DR}(T) \cup \Pi(T)$, $\sigma_{DR}(T) \cap \Pi(T) = \emptyset$ and $\sigma_{DR}(T) = \operatorname{acc} \sigma(T) \cup I(T)$. Then, since $\sigma_{DR}(T) \subseteq \sigma_b(T)$, according to Theorem 3.3,

$$h^{-1}_{\sigma(T)}(0) = \sigma_e(T) = \sigma_b(T) = \operatorname{acc} \sigma(T) \cup I(T) \cup (\Pi(T) \setminus \Pi_0(T)).$$

(ii). Suppose that $T \in \mathcal{L}(\mathcal{X})$ is Polynoially Riesz and consider $Q \in C[X]$ a minimal polynomial associated to T. Applying arguments similar to the ones in Remark 3.1(i) and Theorem 3.3 it is not difficult to pove that

$$\sigma_e(T) = \sigma_w(T) = \sigma_b(T) = \{\lambda \in \mathbb{C}\colon Q(\lambda) = 0\}.$$

Therefore, when T is a polynomially Riesz operator and Q is a minimal polynomial associated to T, $n_T = \sharp\{\lambda \in \mathbb{C}\colon Q(\lambda) = 0\}$.

Next the Weyl approximation essential point spectrum and the Browder approximation essential point spectrum will be characterized.

Theorem 3.5. *Let $\mathcal{X}$ be a Banach space and consider $T \in \mathcal{L}(\mathcal{X})$ such that $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) \neq \emptyset$. Let $h \in \mathcal{H}_{\mathcal{R}NLZ}(\sigma(T))$ be a minimal analytic function associated to T.Then, the following statements hold.*
(i) $\sigma(T) \setminus h^{-1}_{\sigma(T)}(0) = \Pi^a_0(T)$.
(ii) $\sigma_{\Phi_+}(T) = \sigma_{aw}(T) = \sigma_{ab}(T) = h^{-1}_{\sigma(T)}(0)$.

Proof. According to Corollary 3.2(iv) and Theorem 3.3(ii), statement (i) holds.

On the other hand, since $\Pi^a_0(T) = \Pi_0(T)$, according to Theorem 3.3(i) and Corollary 3.2(ii),

$$\sigma_{\Phi_+}(T) \subseteq \sigma_{aw}(T) \subseteq \sigma_{ab}(T) = \sigma_b(T) = \sigma_e(T) = \sigma_{\Phi_+}(T).$$

□

4. Weyl and Browder type theorems

In first place, Browder type theorems will be considered.

Theorem 4.1. *Let $\mathcal{X}$ be a Banach space and consider $T \in \mathcal{L}(\mathcal{X})$ such that $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) \neq \emptyset$. Then, the following statements hold.*
(i) *(Generalized) Browder's theorem holds for T.*
(ii) *(Generalized) a-Browder's theorem holds for T.*
(iii) *Given $f \in \mathcal{H}(\sigma(T))$, generalized a-Browder's theorem holds for $f(T)$ and $f(T^*)$.*

Proof. (i). Theorem 3.3(i) implies that Browder's theorem holds. According to [4, Theorem 2.1], Browder's theorem and generalized Browder's theorem are equivalent.
(ii). Theorem 3.5(ii) implies that a-Browder's theorem holds. According to [4, Theorem 2.2], a-Browder's theorem and generalized a-Browder's theorem are equivalent.
(iii). Apply Corollary 3.2(i) and [3, Theorem 3.2]. □

Recall that generalized a-Browder's theorem is equivalent to a-Browder's theorem ([4, Theorem 2.2]) and it implies generalized Browder's theorem ([8, Theorem 3.8]), which in turn is equivalent to Browder's theorem ([4, Theorem 2.1]). Therefore, under the same hypothesis of Theorem 4.1(iii), (generalized) Browder's theorem and a-Browder's theorem hold for $f(T)$ and $f(T^*)$.

Corollary 4.2. *Let $\mathcal{X}$ be a Banach space and consider $T \in \mathcal{L}(\mathcal{X})$ such that $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) \neq \emptyset$. Then,*

$$\sigma_{BW}(T) = \sigma_{DR}(T) = \sigma_{LD}(T) = \sigma_{SBF_+^-}(T).$$

Proof. Note that generalized Browder's theorem (respectively a-generalized Browder's theorem) is equivalent to $\sigma_{BW}(T) = \sigma_{DR}(T)$ (respectively $\sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$). In adition, since $\sigma(T) = \sigma_a(T)$ and $\sigma(T) = \partial(T)$ (Corollary 3.2), according to [6, Theorem 2.7], $\sigma_{DR}(T) = \sigma_{LD}(T)$. □

Next Weyl type theorems will be considered.

Theorem 4.3. *Let $\mathcal{X}$ be a Banach space and consider $T \in \mathcal{L}(\mathcal{X})$ such that $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) \neq \emptyset$. Then, the following statements are equivalent.*
(i) *Weyl's theorem holds for T (respectively T^*).*
(ii) *a-Weyl's theorem holds for T (respectively T^*).*
In addition, the following statements are equivalent.
(iii) *Generalized Weyl's theorem holds for T (respectively T^*).*
(iv) *Generalized a-Weyl's theorem holds for T (respectively T^*).*

Proof. Recall that T and T^* have the SVEP (Corollary 3.2(i)). Then apply [2, Theorem 3.6] and [3, Theorem 3.1] □

Theorem 4.4. *Let $\mathcal{X}$ be a Banach space and consider $T \in \mathcal{L}(\mathcal{X})$ such that $\mathcal{H}_{\mathcal{R}NLZ}(\sigma(T)) \neq \emptyset$. Suppose that* iso $\sigma(T) = E(T) = \Pi(T)$. *Then, given $f \in \mathcal{H}(\sigma(T))$, generalized a-Weyl's theorem holds for $f(T)$.*

Proof. Apply Corollary 3.2(i) and [4, Corollary 2.4] □

Since generalized a-Weyl's theorem implies a-Weyl's theorem ([8, Theorem 3.11]), generalized Weyl's theorem ([8, Theorem 3.7]) and Weyl's theorem ([8, Corollary 3.10]), the statement of Theorem 4.4 holds if instead of generalized a-Weyl's theorem the other Weyl type theorems are considered.

References

[1] P. Aiena, Fredholm and Local Spectral Theory with Applications to Multipliers, Kluwer, 2004.
[2] P. Aiena, Classes of operators satisfying a-Weyl's theorem, Studia Math. 169 (2005), 105-122.
[3] M. Amouch, Generalized a-Weyl's theorem and the single-valued extension property, Extracta Math. 21 (2006), 51-65.
[4] M. Amouch and H. Zguitti, On the equivalence of Browder's and generalized Browder's theorem, Glasgow Math. J. 48 (2006), 179-185.
[5] B. A. Barnes, Riesz points and Weyl's theorem, Integral Equations Operator Theory 34 (1999), 187-196.
[6] O. Bel Hadj Fredj, M. Burgos and M. Oudghiri, Ascent spectrum and essential ascent spectrum, Studia Math. 187 (2008), 59-73.
[7] M. Berkani, Restriction of an operator to the range of its powers, Studia Math. 140 (2000), 163-175.
[8] M. Berkani and J. J. Koliha, Weyl type theorems for bounded linear operators, Acta Sci. Math (Szeged) 69 (2003), 359-376.
[9] M. Berkani and M. Sarih, An Atkinson-type theorem for B-Fredholm operators, Studia Math. 148 (2001), 251-257.
[10] E. Boasso, Drazin spectra of Banach space operators and Banach algebra elements, J. Math. Anal. Appl. 359 (2009), 48-55.
[11] E. Boasso, The Drazin spectrum in Banach algebras, An Operator Theory Summer: Timisoara June 29-July 4 2010, International Book Series of Mathematical Texts, Theta Foundation, Bucharest, 2012, 21-28
[12] S. R. Caradus, W. E. Pfaffenberg and B. Yood, Calkin algebras and algebra of operators on Banach space, Marcel Dekker, Inc., New York, 1974.
[13] C. Gheorghe, Some remarks on structure of polynomially Riesz operators. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 54 (1973), 42-45.
[14] F. Gilfeather, The structure and asymptotic behaviour of polynomially compact operators, Proc. Amer. Math. Soc. 25 (1970), 127-134.
[15] M. A. Kaashoek and M. R. F. Smyth, On operators T such that $f(T)$ is Riesz or meromorphic, Proc. Roy. Irish Acad. Sect. A 72 (1972), 81-87.
[16] C. King, A note on Drazin inverses, Pacific J. Math. 70 (1977), 383-390.
[17] V. Rakočević, Approximate point spectrum and commuting compact perturbations, Glasgow Math. J. 28 (1986), 193-198.