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Residual quotients

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1. Ideal quotients

Ideal quotients are, in the first instance, discussed in commutative rings [8]; our interest is however in semigroups and general rings, including linear and Banach algebras:

1. Definition *If A is a semigroup, with identity 1 ("monoid"), or more generally a category, then for subsets $H \subseteq A$ and $K \subseteq A$ the left and right residual quotients of H by K are given by*

1.1 $$K^{-1}H = \{x \in A : Kx \subseteq H\}\ ,$$

and

1.2 $$HK^{-1} = \{x \in A : xK \subseteq H\}\ .$$

For example, with $A = \mathbb{Z}$ and $An = [n]$, $[0] = \mathbb{O} = \{0\}$, $[1] = [-1] = A$, and generally

$$k = \mathrm{hcf}(m, n) \Longrightarrow [m]^{-1}[n] = [k^{-1}n]\ .$$

General statements about left obviously imply general statements about right residuals, and vice versa: formally "reverse products". In this note we shall primarily focus on *left* residual quotients: **2. Lemma** *If H, K, H', K' and L are subsets of a monoid A then there is implication*

1.3 $$H' \subseteq H\ ,\ K \subseteq K' \implies K'^{-1}H' \subseteq K^{-1}H$$

and

1.4 $$KL \subseteq H \Longleftrightarrow L \subseteq K^{-1}H\ ,$$

and inclusion

1.5 $$L((KL)^{-1}H) \subseteq K^{-1}H$$

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and

1.6 $$(K^{-1}H)L \subseteq K^{-1}(HL)\ .$$

In a ring or additive category we have also

1.7 $$K^{-1}H + K^{-1}H' \subseteq K^{-1}(H + H')\ .$$

Proof. This is not rocket science: (1.3) and (1.4) are visible, while

$$KLx \subseteq H \Longleftrightarrow Lx \subseteq K^{-1}H$$

and

$$Kx \subseteq H \Longrightarrow KxL \subseteq HL\ ,$$

giving (1.5) and (1.6), with for (1.7)

$$Kx \subseteq H\ ,\ Kx' \subseteq H' \implies K(x + x') \subseteq H + H' \bullet$$

For example left ideals $J \subseteq A$ can be characterized residually:

1.8 $$J \in LI(A) \Longleftrightarrow AJ \subseteq J \Longleftrightarrow J \subseteq A^{-1}J\ .$$

Two special cases are of interest; a *primitive ideal* in a ring is the right residual quotient JA^{-1} of a "maximal" (= maximal proper) left ideal $J \in MLI(A)$ by the whole ring, while the left residual $J^{-1}J$ of a left ideal by itself is a subring of which J is a two-sided ideal:

3. Theorem *If H and K are subsets of the monoid A then there is implication*

1.9 $$H \subseteq K \Longrightarrow (K^{-1}H)(K^{-1}H) \subseteq K^{-1}H\ ,$$

1.10 $$K \subseteq H \Longleftrightarrow 1 \in K^{-1}H\ ,$$

and

1.11 $$1 \in K \Longrightarrow K^{-1}H \subseteq H\ .$$

Proof. If $Kx \subseteq H$, $Ky \subseteq H$ then

$$K(xy) \subseteq Hy \subseteq H(K^{-1}H)\ ,$$

If $K \subseteq H$ then

$$1 \in H^{-1}H \subseteq K^{-1}H\ .$$

Finally if $1 \in K$ then

$$Kx \subseteq H \Longrightarrow x = 1x \in H \bullet$$

If $H \in LI(A)$ is a left ideal of the semigroup A then it follows from (1.5) and (1.6) that $JA^{-1} \subseteq J$ is a two-sided ideal of A, by (1.11) contained in J. If in particular $J \in MLI(A)$ is a "maximal left ideal" then this is what is meant by a "primitive ideal", the true non commutative analogue of "maximal ideals":

4. Definition *A maximal left ideal $J \in MLI(A)$ is a proper left ideal $A \neq J \in LI(A)$ with the property that for an arbitrary left ideal $J' \subseteq A$ there is implication*

1.12 $$J \subseteq J' \in LI(A) \Longrightarrow J' \in \{J, A\}\ .$$

A primitive ideal $K \subseteq A$ is the right residual quotient of a maximal left ideal by the whole ring:

1.13 $$K = JA^{-1} \text{ with } J \in MLI(A) .$$

It would be equally logical to work with left residuals of maximal right ideals, and it is apparently quite delicate to see why the two definitions are different. Maximal left ideals $J \in MLI(A)$ however furnish "irreducible representations" $A \to L(A/J, A/J)$ via left multiplication, whose kernels are the induced primitive ideals:

5. Theorem *Primitive ideals of rings are "prime": if $J \in MLI(A)$ is a maximal left ideal and if H, H' are left ideals of A then there is implication*

1.14 $$HH' \subseteq JA^{-1} \Longrightarrow (H \subseteq JA^{-1} \text{ or } H' \subseteq JA^{-1}) ,$$

and implication

1.15 $$JA^{-1} \subseteq J' \in LI(A) \Longleftrightarrow JA^{-1} \subseteq J'A^{-1} .$$

The intersection of the primitive ideals is the Jacobson radical:

1.16 $$\bigcap\{JA^{-1} : J \in MLI(A)\} = \bigcap\{J : J \in MLI(A)\} ,$$

and hence invertibility can be tested with primitive ideals:

1.17 $$A^{-1} = \{x \in A : \forall J \in MLI(A) , x + JA^{-1} \in \left(A/JA^{-1}\right)^{-1}\} .$$

Proof. Towards (1.14) observe that

1.18 $$HA \not\subseteq J \Longleftrightarrow 1 \in J + AHA ;$$

thus for left ideals H, H' for which $HH' \subseteq JA^{-1}$ there is implication

$$H' \not\subseteq JA^{-1} \Longrightarrow H'A \not\subseteq J \Longrightarrow H'A + J = A$$

giving

$$HA = H(H'A + J) \subseteq HH'A + HJ \subseteq JA^{-1} + J = J$$

and hence $H \subseteq JA^{-1}$. For (1.15) it is clear that if $JA^{-1} \subseteq J'$ then also $JA^{-1} = (JA^{-1})A^{-1} \subseteq J'A^{-1} \subseteq J'$. Towards (1.16), is clear that the left hand side of (1.16) is included in the right; since the radical is also a two sided ideal the reverse inclusion follows. It is also clear, towards (1.17), that if $a \in A^{-1}_{left}$ is left invertible then $a + JA^{-1} \in (A/JA^{-1})^{-1}_{left}$ so that $a \in A$ is left invertible modulo JA^{-1}, and similarly right. If conversely $a \in A$ is not left invertible then there is $J \in MLI(A)$ for which $a \in J$, and we claim that also

1.19 $$(1 - Aa) \cap JA^{-1} = \emptyset \ :$$

else we would have

$$1 \in J + JA^{-1} = J .$$

Now if $a \in A^{-1}_{left} \setminus A^{-1}_{right}$ is left but not right invertible, so that there is $b \in A$ with $ba = 1 \neq ab$, and since $c = 1 - ab$ is not in the radical of A, there must be a primitive ideal $K = JA^{-1}$ for which

$$ca + K = 0 + K \neq c + K \bullet$$

2. Left spectral theory

Multivariable spectral theory exploits the situation of Definition 1 in which $K = H$: here a proper left ideal $N = J$ of the original ring is converted into a proper two-sided ideal of the left residual quotient $M = J^{-1}J$. We become interested in conditions under which a certain "projection property" holds:

6. Theorem *If A is a complex Banach algebra and* $1 \notin N = \mathrm{cl}(N) \in LI(A)$ *then*

2.1 $$a \in M = N^{-1}N \Longrightarrow 1 \notin \bigcap_{\lambda \in \mathbb{C}} N + A(a - \lambda) .$$

Proof. We recall that N is a proper closed two-sided ideal of the closed subalgebra $M \subseteq A$, and hence the quotient $B = M/N$ is a non trivial Banach algebra: looking at the spectrum in this algebra of the coset $[a]_N = a + N$, observe [3],[4],[5]

2.2 $$\partial\sigma_B[a]_N \subseteq \tau_B^{left}[a]_N \subseteq (\sigma_A^{left})_{N=0}(a) \equiv \{\lambda \in \mathbf{C} : 1 \notin N + A(a - \lambda)\} .$$

Here τ^{left} denotes the approximate point spectrum [2],[4],[5]; the first inclusion is familiar for Banach algebras. For the second observe that if $\lambda \in \mathbb{C}$ is not in the "restricted left spectrum" of $a \in A$ relative to the left ideal $N \in A$, so that there is $a'_\lambda \in A$ for which

2.3 $$1 \in a'_\lambda(a - \lambda) + N \subseteq A ,$$

then for arbitrary $x \in M$ there is equality

$$x + N = a'_\lambda(a - \lambda)x + N \subseteq A ,$$

and hence

$$\mathrm{dist}(x, N) \leq \|a'_\lambda\| \, \mathrm{dist}((a - \lambda)x, N) \bullet$$

We note that (2.1) corrects a foolish misprint in (5.3.5) of [5]. We might remark that if in (2.3) we could always arrange that the "regularizer" a'_λ of the "Fredholm" element $a - \lambda$ was in M then in (2.2) we could replace the topological boundary by the whole of the left spectrum $\sigma_B^{left}[a]_N$:

2.4 $$\sigma_B^{left}[a]_N \subseteq (\sigma_M^{left})_{N=0}(a) \equiv \{\lambda \in \mathbf{C} : 1 \notin N + M(a - \lambda)\} .$$

For more general topological algebras the first leg of (2.2) is [2] liable to fail; this can happen even in a "Waelbroeck algebra" A, for which the invertible group A^{-1} is an open set and the inverse mapping $z^{-1} : A^{-1} \to A^{-1}$ continuous. Wawrzyńczyk [12] has succeeded in bypassing this, to establish the spectral mapping theorem for commuting systems in this environment:

7. Theorem *If A is a locally convex Waelbroeck algebra and* $1 \notin N = \mathrm{cl}(N) \in LI(A)$ *then*

2.5 $$a \in \mathrm{comm}_A(N) \Longrightarrow 1 \notin \bigcap_{\lambda \in \mathbf{C}} N + A(a - \lambda) .$$

Proof. We are writing

2.6 $$\mathrm{comm}_A(K) = \{a \in A : \forall b \in K, ab = ba\} , \ \mathrm{comm}_A^2(K) = \mathrm{comm}_A\mathrm{comm}_A(K) ,$$

and of course

2.7 $$N \in LI(A) \Longrightarrow \mathrm{comm}_A(N) \subseteq M = N^{-1}N .$$

With

2.8 $$B = \mathrm{comm}^2_{M/N}([a]_N) \subseteq M/N \ ,$$

where $[a]_N = a + N$ is the coset, observe

2.9 $$\sigma_B[a]_N \subseteq \sigma_A(a) \ .$$

Now if (2.5) fails then there is $(a'_\lambda)_{\lambda\in\mathbb{C}}$ for which

$$\forall \lambda \in \mathbb{C} \ , \ a'_\lambda(a-\lambda) \in 1 + N \ ,$$

and hence (no need for Allan's theorem [1]), for each $\lambda \in \mathbf{C}$ the holomorphic function

2.10 $$\alpha_\lambda = (1 + (\lambda - z)a'_\lambda)^{-1} a'_\lambda$$

with $\alpha_\lambda(z)(a-\lambda) \in 1 + N$ on

$$U_\lambda = \left(1 + (\lambda - z)a'_\lambda\right)^{-1}(A^{-1}) \equiv \{\mu \in \mathbb{C} : \exists (1 + (\lambda - \mu)a'{}_\lambda)^{-1} \in A\} \ .$$

Here $z : \mathbb{C} \to \mathbb{C}$ is the complex coordinate. In particular if $\lambda \notin \sigma_A(a)$ then

$$[a'_\lambda]_N = [a'_\lambda]_N[(a-\lambda)(a-\lambda)^{-1}]_N = [a'_\lambda(a-\lambda)]_N[(a-\lambda)^{-1}]_N = [(a-\lambda)^{-1}]_N \in B \ .$$

We claim

2.11 $$D(a) = \{\lambda \in \mathbf{C} : \exists [a-z]_N^{-1} \in \mathrm{Holo}(\lambda, B)\} \Longrightarrow \partial D(a) = \emptyset \ .$$

If to the contrary $\lambda \in \partial D(a)$ then the holomorphic left inverse $\alpha_\lambda(z)$ of (2.10) coincides on $D(a)\cap U_\lambda$ with $[a-z]_N^{-1} \in B$. Now, with U'_λ the connected component of λ in U_λ,

$$\forall \varphi \in (M/N)^* \ , \ B \subseteq \varphi^{-1}(0) \Longrightarrow \varphi([\alpha_\lambda(z)]_N) = 0 \ on \ U'_\lambda \ ,$$

since the function $\varphi([\alpha_\lambda(z)]_N) : U'_\lambda \to \mathbf{C}$ is holomorphic and vanishes on a nonempty open subset. By the Hahn-Banach theorem it follows

$$[\alpha_\lambda(z)]_N(U'_\lambda) \subseteq B \ .$$

This means that $U'_\lambda \subseteq D(a)$, contradicting the assumption that $\lambda \in \partial D(a)$. Thus (2.11) holds, and hence $D(a) = \mathbf{C}$. By Liouville's theorem it follows that

2.12 $$[a-z]_N^{-1} = 0 \ on \ \mathbf{C} \ :$$

this contradiction means that (2.5) holds •

Alternatively, in place of the double commutant $B \subseteq M/N$ of the coset $[a]_N = a + N$ we can work with a maximal abelian subalgebra. Wawrzyńczyk [12] considers only left ideals N which are both finitely and commutatively generated, and derives the spectral mapping theorem for finite commuting systems of elements. It is clear that an identical argument is valid for proper right ideals, and the extension from finite to infinite systems is a matter of compactness. Where there is a real gap between Waelbroeck and Banach algebras is that the element $a \in A$ has to commute with everything in the ideal N, rather than just to live in the residual quotient $M = N^{-1}N$. At least part of the argument [3],[4],[5] extending the spectral mapping theorem in Banach algebra to "quasi commutative" systems, $a \in A^n$ for which

2.13 $$(a_i a_j - a_j a_i)a_k = a_k(a_i a_j - a_j a_i) \ (i, j, k = 1, 2, \ldots, n) \ ,$$

works with elements which commute with ideals which are not themselves commutatively generated.

Inclusion (2.9) does not extend to arbitrary $a \in M = N^{-1}N$:

8. Example *If*

2.14 $$A = D^{2\times 2} = \begin{bmatrix} D & D \\ D & D \end{bmatrix} \text{ and } N = \begin{bmatrix} D & O \\ D & O \end{bmatrix} = Ap \text{ with } p = p^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then

2.15 $$M = N^{-1}N = \begin{bmatrix} D & O \\ D & D \end{bmatrix} = \{a \in A : pa(1-p) = 0\} \text{ and } M/N \cong D\ .$$

If

2.16 $$a = \begin{bmatrix} v & 0 \\ 1-uv & u \end{bmatrix} \text{ with } vu = 1 \neq uv$$

then

2.17 $$0 \in \sigma_{M/N}([a]_N) \setminus \sigma_A(a)\ .$$

Proof. Evidently

2.18 $$\exists a^{-1} = \begin{bmatrix} u & 1-uv \\ 0 & v \end{bmatrix} \in A \setminus M \bullet$$

For a specific example take u and v to be the forward and the backward unilateral shift in the algebra $D = B(X)$ with $X = \ell_2(\mathbb{N})$, so that a and a^{-1} are essentially the bilateral shifts in $A = B(\ell_2(\mathbb{Z}))$.

3. Taylor split spectral theory

The definition [4],[5] of the "Taylor spectrum" and of the "Taylor split spectrum" is based on *exactness*. In a ring with identity A we can call the pair $(b, a) \in A^2$ *splitting exact* if there is inclusion

3.1 $$1 \in bA + Aa\ ,$$

whether or not the pair (b, a) satisfies the *chain condition*

3.2 $$ba = 0\ .$$

If (3.1) holds with $b = a$ we describe $a \in A$ as *self exact*, and write

3.3 $$A^{-1}_{left,right} = \{a \in A : 1 \in Aa + aA\}\ .$$

Now of course the chain condition (3.2) takes the form $a^2 = 0$. In general

3.4 $$N = bA + Aa$$

is neither a left nor a right ideal, and not even a subring; if however we write

3.5 $$M = N : N = (L_N + R_N)^{-1}(N) \equiv \{c \in A : Nc + cN \subseteq N\}\ ,$$

then, provided

3.6 $$N \cdot N \subseteq N\ ,$$

$1 \in M \subseteq A$ is a subring and N is a two-sided ideal of M, proper if the splitting exact condition (3.1) does not hold. When A is a (complex) linear algebra then we can express exactness and invertibility in terms of "spectrum". For normed and Banach algebra A it becomes a problem that $N \subseteq A$ may not be norm closed; more seriously it does not appear clear that there is implication

3.7 $$1 \in \mathrm{cl}(bA + Aa) \Longrightarrow 1 \in bA + Aa\ .$$

In the presence of (3.7) we will prefer

3.8 $$N = \mathrm{cl}(bA + Aa)\ .$$

According to Theorem 10.7.3 of [4], if A is a Banach algebra then the chains $(b, a) \in A^2$ satisfying (3.1) form a (relatively) open subset of the closed set of chains in A^2.

One situation where we might expect (3.7) would be with

3.9 $$b = a = \Lambda_c = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\ ,\ A = D^{2\times 2}$$

for a Banach algebra D; there is two-way implication

3.10 $$c \in D^{-1} \Longleftrightarrow 1 \in aA + Aa\ :$$

observe

3.11 $$\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\begin{bmatrix} D & D \\ D & D \end{bmatrix} + \begin{bmatrix} D & D \\ D & D \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} Dc & 0 \\ cD + Dc & cD \end{bmatrix},$$

and evidently the implication (3.7) holds here. Generalizing (3.11), and looking for an induction [12], we observe ([4] Theorems 10.9.5, 10.9.6; [5] §5.10)

3.12 $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \begin{bmatrix} b & 0 \\ c & -b \end{bmatrix}\begin{bmatrix} D & D \\ D & D \end{bmatrix} + \begin{bmatrix} D & D \\ D & D \end{bmatrix}\begin{bmatrix} b & 0 \\ c & -b \end{bmatrix} = \begin{bmatrix} bD + Db + Dc & bD + Db \\ cD - bD + Db + Dc & cD - bD + Db \end{bmatrix},$$

if and only if

3.13 $$1 \in (Db + bD + Dc) \cap (Db + bD + cD)\ .$$

Generalizing (3.9), if

$$a = (b, c) \in A^k \times A = A^{k+1}$$

inductively define [7], with a block diagonal $\triangle_c$,

3.14 $$\Lambda_a = \begin{bmatrix} \Lambda_b & 0 \\ \triangle_c & -\Lambda_b \end{bmatrix}\ :$$

In search of a "projection property", if $1 \notin \Lambda_b D + D\Lambda_b$ we look for $\lambda \in \mathbb{C}$ for which

3.15 $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin \Lambda_{b,c-\lambda} D^{2\times 2} + D^{2\times 2}\Lambda_{b,c-\lambda}\ .$$

4. Approximation theory

The same ideas, in particular the trick (3.5), are involved in a version [6],[9],[10] of approximation theory: Suppose A is a ring with identity 1, with

4.1 $$1 \notin J \subseteq A$$

where now J is only a subring, and then again define

4.2 $$J : J \equiv (L_J + R_J)^{-1}(J) = \{c \in A : Jc + cJ \subseteq J\} \ :$$

with $N = J$ and $M = J : J$ we have $c \in M$ iff

4.3 $$\{d, d'\} \subseteq N \Longrightarrow dc + cd' \in N \ .$$

Evidently

4.4 $$1 \in M \subseteq A$$

is a subring, and then

4.5 $$1 \notin MN + NM \subseteq N \subseteq M$$

is a proper two-sided ideal.

We might for example [6] call $d \in N$ "compact" and then $c \in M$ "compatible"; now $c \in M$ would be "Fredholm" if it had a *regularizer*, $c' \in A$ for which

4.6 $$\{1 - c'c, 1 - cc'\} \subseteq N \ .$$

Of interest, *cf* (2.3), is the *Fredholm permanence* condition that says that

4.7 $$(4.6) \Longrightarrow c' \in M \ .$$

When this happens then the Fredholm condition reverts to coset invertibility:

4.8 $$c + N \in (M/N)^{-1} \ .$$

In the context [6] of "quasi-banded operators" the subspace $N \subseteq A = B(X)$ of "p-compact operators" is derived from a sequence $p = (p_n)$ of projections, by setting

4.9 $$N = \{a \in A : \|a(1 - p_n)\| + \|(1 - p_n)a\| \to 0\} \ .$$

When the sequence $p_n \to 1$ converges strongly to the identity then $N \subseteq A$ coincides with the usual compact operators, and forms a two-sided ideal of A and hence $M = A$. Implication (4.7) was eventually established ([10] Theorem 1.16) for $X = L_p(\mathbb{R})$ and $p_n = L_{a_n}$ with $a_n = \chi_{[-n,n]}$.

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