Functional Analysis,
Approximation and
Computation
7 (2) (2015), 75–87

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

A survey on the perturbation classes problem for semi-Fredholm and Fredholm operators

Manuel González[a], **Antonio Martínez-Abejón**[b], **Javier Pello**[c]

[a]*Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, E-39071 Santander, España*
[b]*Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, E-33007 Oviedo, España*
[c]*Departamento de Matemática Aplicada, Escuela Superior de Ciencias Experimentales y Tecnología, Universidad Rey Juan Carlos, E-28933 Móstoles, España*

Abstract. We describe the main partial positive answers and counterexamples to the perturbation classes problem for semi-Fredholm and Fredholm operators, and point out several related problems that remain open.

1. Introduction

The study of the perturbation classes for semi-Fredholm and Fredholm operators stemmed from the abstract approach to integral equations at the beginning of the twentieth century. The main problems it considers can be described as follows:

(A) Identify the perturbation classes $P\Phi_+$, $P\Phi_-$ and $P\Phi$ for the upper semi-Fredholm, lower semi-Fredholm and Fredholm operators.

(B) Find intrinsic characterizations for each of the perturbation classes $P\Phi_+$, $P\Phi_-$ and $P\Phi$.

(C) Describe the properties of the components $P\Phi_+(X,Y)$, $P\Phi_-(X,Y)$ and $P\Phi(X,Y)$.

With respect to problem (B), observe that in order to check whether an operator K satisfies the definition of a perturbation class, we have to study the properties of $T+K$ for T in a family of operators, which can be unwieldly. An *intrinsic characterization* is one given in terms of the action of the operator K itself, like that of the strictly singular operators (Definition 4.4). For instance, problem (A) is solved for the class of Fredholm operators after the identification of its

2010 *Mathematics Subject Classification.* Primary: 47A53, 47A55.
Keywords. Semi-Fredholm operators; Perturbation classes.
Received: 5 December 2014; Accepted: 10 January 2015
Communicated by Dragan S. Djordjević
Supported in part by DGI (Spain), Grant MTM2013-45643.
Email addresses: manuel.gonzalez@unican.es (Manuel González), ama@uniovi.es (Antonio Martínez-Abejón), javier.pello@urjc.es (Javier Pello)

perturbation class with the inessential operators. However, this is not totally satisfactory because the definition of an inessential operator (Definition 3.5) is not intrinsic. As we will see, these three problems are interdependent.

In this survey we describe the answers to these problems that have been obtained so far, specially in the last twenty years, and we point out several problems that remain open. Section 2 provides some historical background for the problem, and Section 3 contains the definitions and main properties of Fredholm and semi-Fredholm operators, and of their perturbation classes. In Section 4 we introduce the most important intrinsically-defined classes of operators in connection with the perturbation classes in Fredholm theory: compact, strictly singular, strictly cosingular and improjective operators. In Section 5 we show many examples of pairs of Banach spaces for which the components of the perturbation classes coincide with the corresponding components of strictly singular, strictly cosingular, or improjective operators. Section 6 collects a selection of open problems.

Notation: $\mathcal{L}$ denotes the class of all bounded linear operators (henceforth, operators) acting between Banach spaces; $\mathcal{L}(X,Y)$ is the class of all operators into Y whose domain is X. Given a class of operators $\mathcal{A}$, we refer to $\mathcal{A}(X,Y) := \mathcal{A}\cap\mathcal{L}(X,Y)$ as the component of $\mathcal{A}$ in $\mathcal{L}(X,Y)$; we write $\mathcal{A}(X)$ instead of $\mathcal{A}(X,X)$. The dual of a Banach space X is denoted by X^*, and the conjugate of an operator T is denoted by T^*. The range of $T\in\mathcal{L}(X,Y)$ is denoted by $\operatorname{Ran} T$, its kernel by $\operatorname{Ker} T$, and its co-kernel is $\operatorname{Coker} T := Y/\overline{R(T)}$. Given a class $\mathcal{A}$ of operators, we denote $\mathcal{A}^d := \{\, T : T^*\in\mathcal{A}\,\}$. If Z is a closed subspace of X, then $J_Z\in\mathcal{L}(Z,X)$ is the subspace embedding of Z into X and $Q_Z\in\mathcal{L}(X,X/Z)$ is the associated quotient operator. The identity operator on X is denoted by I_X or simply I.

2. Some historical remarks

As it is well explained in [15], the unified solution given by Fredholm to the integral equations of Abel and Volterra, together with further contributions by Hilbert, led to the formulation of the famous Fredholm integral equations

$$\int_a^b K(x,y)\,u(y)\,dy = f(x) \qquad \text{(first class)} \tag{1}$$

$$u(x) = f(x) + \int_a^b K(x,y)\,u(y)\,dy \quad \text{(second class)} \tag{2}$$

where $K(x,y)$ is a kernel function, $f(x)$ is a given function with integrable square and $u(x)$ is the unknown function. Roughly speaking, the original purpose of integral equations is the study of the stability of certain physical systems under small perturbations. In the case of the Fredholm integral equations, those small perturbations are represented by the kernel function: if $K(x,y)=0$ for all (x,y), then the system represented in (2) is trivial, while if the kernel $K(x,y)$ is small enough (in a certain unspecific sense), then $u(x)$ is close enough to $f(x)$ (in the same unspecific sense) and the system is declared to be stable, in which case the perturbation represented by K is said to be admissible. In the language of functional analysis, the Fredholm integral equations can be written as

$$Ku = f \quad\text{ and }\quad u = f + Ku, \tag{3}$$

where $K\colon H\longrightarrow H$ is a compact operator, H is a Hilbert space, f is a given element of H and u is the unknown. In the second part of equation (3), f is the image of u by the Fredholm operator $I-K$, that is, the identity operator perturbed by K. In fact, research on the Fredholm equations meant the beginning of Functional Analysis.

Once the duality relation in L_p spaces was well understood, the theory of Fredholm operators was extended to the Banach space setting in a natural way, and this last step introduced different ways to measure the smallness of a perturbation other than the one provided by the Euclidean norm. In this way, the theory of Fredholm and semi-Fredholm operators on Banach spaces and the problem of their perturbation classes came hand in hand, and although this last question is totally solved for Fredholm operators on Hilbert spaces, it is still challenging in the context of Banach spaces.

In 1954, in the context of Banach algebras, Kleinecke introduced the ideal $\mathcal{I}n$ of inessential operators as the class of Φ-admissible perturbations, although his results were published almost ten years later [32]. In 1958, Kato introduced the ideal $\mathcal{SS}$ of strictly singular operators as a natural class of Φ_+-admissible perturbations that includes the compact operators [30], and Gohberg, Markus and Feldman asked for intrinsic characterizations of $P\Phi_+$, $P\Phi_-$ and $P\Phi$ [18]; in particular, they asked if $P\Phi_+ = \mathcal{SS}$. Pełczyński introduced the class $\mathcal{SC}$ of strictly cosingular operators as a kind of dual class of $\mathcal{SS}$ [37] in 1965, and two years later Vladimirskii proved that $\mathcal{SC}$ is a class of Φ_--admissible perturbations [47]. In 1972, Tarafdar introduced the intrinsically-defined class $\mathcal{I}mp$ of improjective operators and proved that $\mathcal{SS} + \mathcal{SC} \subset \mathcal{I}n \subset \mathcal{I}mp$ [44]. He also proved that many components of $\mathcal{I}mp$ coincide with those of $\mathcal{SS}$ and $\mathcal{SC}$. Finally, the discovery of the hereditarily indecomposable spaces allowed to show in [20] that $\mathcal{SS} \neq P\Phi_+$ and $\mathcal{SC} \neq P\Phi_-$ (see Proposition 4.8).

Let us end this section by observing that the movement from the original framework of Hilbert spaces to more general Banach spaces is not merely philosophical: the measurement of smallness of a perturbation by means of norms other than the hilbertian one is very important for technical branches like signal theory [35].

3. Semi-Fredholm operators and their perturbation classes

We begin with the definition of the three classes of semi-Fredholm operators. We refer to [2] for a detailed study of these classes.

Definition 3.1. *An operator $T \in \mathcal{L}(X, Y)$ is said to be:*

(i) upper semi-Fredholm if $\operatorname{Ker} T$ *is finite-dimensional and* $\operatorname{Ran} T$ *is closed;*
(ii) lower semi-Fredholm if $\operatorname{Coker} T$ *is finite-dimensional and* $\operatorname{Ran} T$ *is closed;*
(iii) Fredholm if T is both upper semi-Fredholm and lower semi-Fredholm.

It is not difficult to prove that if $\operatorname{Ran} T$ is finite-codimensional, then it must also be closed, which makes T lower semi-Fredholm.

The classes of upper semi-Fredholm, lower semi-Fredholm and Fredholm operators will be respectively denoted by Φ_+, Φ_- and Φ; hence $\Phi = \Phi_+ \cap \Phi_-$. We will also denote $\Phi_\pm := \Phi_+ \cup \Phi_-$. It follows from the basic duality results for operators that $T \in \Phi_\pm$ if and only if $T^* \in \Phi_\pm$, and that in this case $\dim \operatorname{Coker} T = \dim \operatorname{Ker} T^*$ and $\dim \operatorname{Ker} T = \dim \operatorname{Coker} T^*$.

The *index* of an operator $T \in \Phi_\pm$ is defined by

$$\operatorname{ind} T := \dim \operatorname{Ker} T - \dim \operatorname{Coker} T.$$

Note that $\operatorname{ind} T$ is finite if and only if $T \in \Phi$. Similarly, $T \in \Phi_+$ if and only if $\operatorname{ind} T < \infty$, and $T \in \Phi_-$ if and only if $\operatorname{ind} T > -\infty$. The index is a continuous function on each space $\Phi_\pm(X, Y)$ when it is endowed with the norm topology; as such, it is constant on each connected component of $\Phi_\pm(X, Y)$.

Given an operator T in $\Phi_\pm(X, Y)$, there exists $\varepsilon_T > 0$ such that $T + K \in \Phi_\pm$ and $\operatorname{ind}(T + K) = \operatorname{ind} T$ for all $K \in \mathcal{L}(X, Y)$ such that $\|K\| < \varepsilon_T$. The task of looking for the largest possible value of ε_T is a local question investigated by the theory of operational quantities (see [23], [24] and references therein). However, the concept of a perturbation class is of global character:

Definition 3.2. *The perturbation class $P\mathcal{S}$ of $\mathcal{S} \in \{\Phi_+, \Phi_-, \Phi\}$ is defined by its components in $\mathcal{L}(X,Y)$, when $\mathcal{S}(X,Y) \neq \emptyset$, as follows:*

$$P\mathcal{S}(X,Y) := \{\, K \in \mathcal{L}(X,Y) : T + K \in \mathcal{S}(X,Y) \text{ for all } T \in \mathcal{S}(X,Y) \,\}.$$

*Every operator in $P\mathcal{S}$ is called an $\mathcal{S}$-*admissible perturbation*, and every subclass of $P\mathcal{S}$ is called a* class of $\mathcal{S}$-admissible perturbations.

Note that Definition 3.2 is not intrinsic because, in order to determine if $K \in P\mathcal{S}(X,Y)$, it is not enough to check the action of K, but we have to study the properties of all the operators in the set $\{\, T + K : T \in \mathcal{S}(X,Y) \,\}$. Obviously, an intrinsic characterization for the operators in $P\mathcal{S}$ in terms of its sole action would provide a positive answer to problem (B) in the Introduction.

Let us see the relationship between the three classes $P\Phi_+$, $P\Phi_-$ and $P\Phi$.

Proposition 3.3. *The inclusion $P\Phi_+ \cup P\Phi_- \subset P\Phi$ holds.*

The proof of this inclusion is a direct application of the constancy of the index $\operatorname{ind} T$ on the connected components of $\Phi_\pm(X,Y)$ [14]. Indeed, observe that given $T \in \Phi_\pm(X,Y)$ and $K \in \mathcal{L}(X,Y)$ in the corresponding perturbation class, we have $T + tK \in \Phi_\pm$ for each $t \in [0,1]$, hence $\operatorname{ind}(T+K) = \operatorname{ind} T$.

A class $\mathcal{A}$ of operators is said to be *injective* if $T \in \mathcal{A}$ whenever $LT \in \mathcal{A}$ for one (and then for all) isomorphic embedding L; roughly speaking, $\mathcal{A}$ is injective when the fact that $T \in \mathcal{A}$ does not depend upon the target space of T. In a similar way, $\mathcal{A}$ is said to be *surjective* if $T \in \mathcal{A}$ whenever $TQ \in \mathcal{A}$ for any surjective operator Q.

Proposition 3.4. *Given a pair X, Y of Banach spaces, the following statements hold:*

(i) *$P\Phi_+$ is injective but not surjective;*
(ii) *$P\Phi_-$ is surjective but not injective;*
(iii) *$P\Phi_-(X,Y)$ and $P\Phi_+(X,Y)$ are closed subspaces of $\mathcal{L}(X,Y)$;*
(iv) *$P\Phi_+(X)$ and $P\Phi_-(X)$ are two-sided ideals of $\mathcal{L}(X)$.*
(v) *the components of $P\Phi_+$ and $P\Phi_-$ do not determine operator ideals: there is no operator ideal $\mathcal{A}$ such that $P\Phi_+(X) = \mathcal{A}(X)$ for every Banach space X, and neither is there for $P\Phi_-$.*

The proof of the positive parts of (i) and (ii) is straightforward, and the negative parts are a consequence of the same properties for the strictly singular and strictly cosingular operators. Proofs of (iii) and (iv) can be found in [14, page 97], and (v) follows from the results of [20] and [40, Remark after 26.6.12].

Part (v) of Proposition 3.4 means that problems (A) and (B) have a negative answer in general. However, they can be answered in the positive for many pairs of Banach spaces as we will see in Section 5. Note also that $\mathcal{SS}$ (resp. $\mathcal{SC}$) is the largest proper injective (surjective) operator ideal [40, 4.6.14].

Definition 3.5. *An operator $T \in \mathcal{L}(X,Y)$ is said to be* inessential *if $I_X - AT \in \Phi(X)$ for every $A \in \mathcal{L}(Y,X)$; equivalently, if $I_Y - TA \in \Phi(Y)$ for every $A \in \mathcal{L}(Y,X)$. The class of all inessential operators will be denoted $\mathcal{I}n$.*

Inessential operators were introduced by Kleinecke [32], who proved the following result.

Proposition 3.6. *The class $\mathcal{I}n$ is a closed operator ideal, but it is neither injective nor surjective.*

Note that $\mathcal{I}n^d \subset \mathcal{I}n$, but $\mathcal{I}n$ is not self-dual: the inclusion operator $J \colon c_0 \longrightarrow \ell_\infty$ is inessential because, by a result of Rosenthal [42], any operator on ℓ_∞ is either weakly compact or an isomorphism on some subspace isomorphic to ℓ_∞, but $c_0^\perp$ is complemented in ℓ_∞^*, so the conjugate operator J^* is not inessential.

Theorem 3.7. *The identity $P\Phi(X) = \mathcal{I}n(X)$ holds for every Banach space X.*

Theorem 3.7 solves questions (A) and (C) for $P\Phi$, but the definition of inessential operator is obviously not intrinsic. Aiena obtained some spectral characterizations for the components $P\Phi(X)$ [1] which constitute an important advance towards the solution of problem (B) for $P\Phi$. Note that the study of the components $P\Phi(X)$ is sufficient to know the behavior of $P\Phi(X,Y)$ because the existence of a Fredholm operator $T\colon X \longrightarrow Y$ implies that X is isomorphic to Y up to a finite dimensional subspace. In this sense, the class $P\Phi$ is very different to the classes $P\Phi_+$ and $P\Phi_-$.

Theorem 3.7 and Proposition 3.6 also shed some light upon questions (A) and (B). For instance, it has led to disprove conjectures like the possible identification of $P\Phi$ with the intrinsically-defined class $\mathcal{I}mp$. Also, the identification of $P\Phi$ with an operator ideal shows a difference between the classes $P\Phi_+$ and $P\Phi_-$ and the class $P\Phi$.

4. Intrinsically-defined classes of admissible perturbations

Many operator classes admitting intrinsic descriptions have been proposed as possible solutions to questions (A) and (B). The most remarkable of them are enumerated in this section.

4.1. The compact operators

Recall that an operator $T \in \mathcal{L}(X,Y)$ is said to be *compact* if (Tx_n) contains a convergent subsequence whenever (x_n) is bounded.

Compact operators already occur at the beginning of the early Fredholm theory as explained in the introduction, where we pointed out that

$$P\Phi(H) = P\Phi_+(H) = P\Phi_-(H) = \mathcal{K}(H)$$

if H is a Hilbert space. The main properties of $\mathcal{K}$ are given in the following result.

Proposition 4.1. [40] *The class $\mathcal{K}$ is a self-dual, injective, surjective, closed operator ideal consisting of $\mathcal{S}$-admissible perturbations for all $\mathcal{S} \in \{\Phi_+, \Phi_-, \Phi\}$.*

Although the class $\mathcal{K}$ is properly contained in the other classes of admissible perturbations, it is still interesting in Fredholm theory. For instance, the following characterizations are a very useful tool:

Proposition 4.2. [34] *Let $T \in \mathcal{L}(X,Y)$.*

(i) $T \in \Phi_+$ if and only if $\mathrm{Ker}(T+K)$ *is finite-dimensional for all $K \in \mathcal{K}(X,Y)$.*

(ii) $T \in \Phi_-$ if and only if $\mathrm{Coker}(T+K)$ *is finite-dimensional for all $K \in \mathcal{K}(X,Y)$.*

Also the following observation is useful:

Remark 4.3. *In the previous result we can replace "compact operators" by "nuclear operators with arbitrarily small norm".*

4.2. The strictly singular and the strictly cosingular operators

Definition 4.4. *An operator $T \in \mathcal{L}(X, Y)$ is said to be* strictly singular *if for every subspace Z of X, TJ_Z is an isomorphic embedding only if Z is finite-dimensional.*

T is said to be strictly cosingular *if for every subspace Z of Y, $Q_Z T$ is surjective only if Z is finite-codimensional.*

Proposition 4.5. *The classes $\mathcal{SS}$ and $\mathcal{SC}$ have the following properties:*

(i) $\mathcal{SS}$ and $\mathcal{SC}$ are closed operator ideals;
(ii) $\mathcal{SS}$ is injective but not surjective, and $\mathcal{SC}$ is surjective but not injective;
(iii) $\mathcal{SS} \subset P\Phi_+$ and $\mathcal{SC} \subset P\Phi_-$.

In the previous proposition, the results for $\mathcal{SS}$ are due to Kato [30] and the results for $\mathcal{SC}$ are due to Pełczyński [37] and Vladimirskii [47].

Proposition 4.6. *The following statements hold:*

(i) $\mathcal{K} \subsetneq \mathcal{SS} \cap \mathcal{SC}$;
(ii) $\mathcal{SC} \setminus \mathcal{SS} \neq \emptyset$;
(iii) $\mathcal{SS} \setminus \mathcal{SC} \neq \emptyset$.

For part (ii), the natural inclusion operator $U\colon L_2[0,1] \longrightarrow L_1[0,1]$ belongs to $\mathcal{SC} \setminus \mathcal{SS}$, and for part (iii), the inclusion operator $V\colon L_\infty[0,1] \longrightarrow L_2[0,1]$ belongs to $\mathcal{SS} \setminus \mathcal{SC}$. Thus, part (i) is proved by the natural inclusion operator $W := UV\colon L_\infty[0,1] \longrightarrow L_1[0,1]$, which belongs to $\mathcal{SS} \cap \mathcal{SC} \setminus \mathcal{K}$.

Indeed, the restriction of U to the subspace generated by the Rademacher functions is an isomorphic injection, so $U \notin \mathcal{SS}$. On the other hand, if N is a subspace of $L_1[0,1]$ such that $Q_N U$ is surjective, then Q_N is a surjection onto a reflexive subspace, hence Q_N is weakly compact. Since $L_1[0,1]$ has the Dunford-Pettis property, Q_N is completely continuous, so $Q_N U$ is compact, hence the quotient $L_1[0,1]/N$ is finite-dimensional and therefore, $U \in \mathcal{SC}$.

Now, $V = U^*$, so $U \notin \mathcal{SS}$ implies $V \notin \mathcal{SC}$. Moreover, since $L_\infty[0,1]$ has the Dunford-Pettis property, any restriction $V|_R$ with R reflexive is completely continuous, so it cannot be an isomorphic embedding, which proves $V \in \mathcal{SS}$. Finally, W is not compact because the sequence (f_n), where $f_n(t) := \sin 2n\pi t$, is bounded in $L_\infty[0,1]$ but does not have any convergent subsequence in $L_1[0,1]$. For additional details, see [4].

For a while, it was thought that the admissible classes $\mathcal{SS}$ and $\mathcal{SC}$ might be identified with the perturbation classes $P\Phi_+$ and $P\Phi_-$, but the discovery of the hereditarily indecomposable spaces denied this possibility.

Definition 4.7. *A Banach space X is said to be* indecomposable *if for each decomposition $X = X_1 \oplus X_2$ with X_1 and X_2 closed subspaces, X_1 or X_2 is finite-dimensional.*

The space X is said to be hereditarily indecomposable *if all its closed subspaces are indecomposable, and it is said to be* quotient indecomposable *if all its quotients are indecomposable.*

Hereditarily indecomposable spaces, unlike Hilbert spaces, support very few operators. In fact, Weis proved [49] that for every space Y we have $\mathcal{L}(X, Y) = \Phi_+(X, Y) \cup \mathcal{SS}(X, Y)$ if and only if X is hereditarily indecomposable; and similarly, for every space Y we have $\mathcal{L}(Y, X) = \Phi_-(Y, X) \cup \mathcal{SC}(Y, X)$ if and only if X is quotient indecomposable.

Proposition 4.8. *There exists a separable reflexive space Z for which $\mathcal{SS}(Z) \subsetneq P\Phi_+(Z)$ and $\mathcal{SC}(Z^*) \subsetneq P\Phi_-(Z^*)$.*

There are at least two different constructions of such a space Z [17, 20], and both of them involve the hereditarily indecomposable spaces discovered in [27] and [12]. Let us succinctly describe the construction given in [20]:

Let X be a reflexive, hereditarily indecomposable space and let Y be an infinite-dimensional closed subspace of X such that $\dim X/Y = \infty$. Then the operator

$$A\colon Y \times X \longrightarrow Y \times X \text{ defined by } A(y,x) = (0,y)$$

is inessential, but clearly not strictly singular. Applying the stability under perturbations of the index of semi-Fredholm operators, we can show that the space $Z := X \times Y$ satisfies $\Phi_+(Z) = \Phi(Z)$. Thus

$$P\Phi_+(Z) = P\Phi(Z) = \mathcal{I}n(Z) \neq \mathcal{SS}(Z).$$

A duality argument shows that the conjugate operator $A^*\colon Z^* \longrightarrow Z^*$ is inessential, but not strictly singular, and that $\Phi_-(Z^*) = \Phi(Z^*)$. Thus

$$P\Phi_-(Z^*) = P\Phi(Z^*) = \mathcal{I}n(Z^*) \neq \mathcal{SC}(Z^*).$$

Since $\mathcal{SS}$ and $\mathcal{SC}$ are closed operator ideals, it is straightforward that so are $\mathcal{SS}^d$ and $\mathcal{SC}^d$. The following result describes the duality between $\mathcal{SS}$ and $\mathcal{SC}$.

Proposition 4.9. [50] *The ideals $\mathcal{SS}$ and $\mathcal{SC}$ satisfy the following statements:*

(i) $\mathcal{SS}^d \subsetneq \mathcal{SC}$;

(ii) $\mathcal{SC}^d \subsetneq \mathcal{SS}$;

An example of an operator in $\mathcal{SC} \setminus \mathcal{SS}^d$ is the natural inclusion $J_{c_0} \in \mathcal{L}(c_0, \ell_\infty)$; and $\mathcal{SS} \setminus \mathcal{SC}^d$ contains the operator $Q \in \mathcal{L}(\ell_1(\ell_2^n), \ell_2)$ defined by

$$Q\big((x_i^n)_{i=1}^{n}{}_{n=1}^{\infty}\big) := (\lim_{n\to\mathcal{U}} x_i^n)_{i\in\mathbb{N}}$$

where $\mathcal{U}$ is a fixed non-trivial ultrafilter on $\mathbb{N}$. In fact, Q is surjective and weakly compact, and as $\ell_1(\ell_2^n)$ has the Schur property (weakly convergent sequences in $\ell_1(\ell_2^n)$ are norm convergent), then Q is strictly singular. But the range of Q^* is complemented, hence $Q^* \notin \mathcal{SC}$ [4].

4.3. The improjective operators

As we have seen, the ideals $\mathcal{K}$, $\mathcal{SS}$ and $\mathcal{SC}$ are intrinsically-defined subclasses of $P\Phi_+$, $P\Phi_-$ and $P\Phi$. In turn, these perturbation classes are subclasses of the following class:

Definition 4.10. *An operator $T \in \mathcal{L}(X,Y)$ is said to be* improjective *if for every subspace Z of X, TJ_Z is an isomorphic embedding and $T(Z)$ is complemented in Y only if Z is finite-dimensional.*

Proposition 4.11. [44, 45] *The class $\mathcal{I}mp$ is an operator quasi-ideal and satisfies the following statements:*

(i) $\mathcal{I}mp^d \subset \mathcal{I}mp$;

(ii) $P\Phi \subset \mathcal{I}mp$.

The inclusion $\mathcal{I}mp(X,Y) \subset \mathcal{I}mp^d(X,Y)$ holds if X is reflexive [44]. Unfortunately, the behaviour of $\mathcal{I}mp$ is not as good as that of other classes. Let X_{GM} be the separable, indecomposable space constructed by Gowers and Maurey [28] after their discovery of the hereditarily indecomposable spaces in [27].

Proposition 4.12. [6] *The following statements hold:*

(i) $\mathcal{I}mp(X_{\mathrm{GM}})$ *is not a linear subspace of* $\mathcal{L}(X_{\mathrm{GM}})$*; hence* $\mathcal{I}mp$ *is not an operator ideal;*

(ii) $\mathcal{I}n(X_{\mathrm{GM}}) \subsetneq \mathcal{I}mp(X_{\mathrm{GM}})$.

Using the Continuum Hypothesis, Koszmider constructed a connected compact space K_0 such that every operator $T \in \mathcal{L}\big(C(K_0)\big)$ is of the form $T = gI + S$ with $g \in C(K_0)$ and S a weakly compact operator on $C(K_0)$ [33]. As a consequence, the space $C(K_0)$ is indecomposable. Another $C(K)$ space with the same characteristics but whose construction does not depend on the Continuum Hypothesis was found by Plebanek [41].

Theorem 4.13. [8] *Let* K_0 *be the compact of Koszmider. Then*

$$\mathcal{SS}\big(C(K_0)\big) = \mathcal{I}n\big(C(K_0)\big) \neq \mathcal{I}mp\big(C(K_0)\big).$$

It follows from the original proof of Theorem 4.13 that $\mathcal{I}mp\big(C(K_0)\big)$ is not a linear subspace of $\mathcal{L}\big(C(K_0)\big)$.

The classes of operators that we have considered in connection with the perturbation classes for semi-Fredholm operators and the relationships among them are represented in the following diagram, where $\mathcal{A} \to \mathcal{C}$ means that class $\mathcal{A}$ is properly contained in class $\mathcal{C}$.

$$\begin{tikzcd}
 & \mathcal{SC}^d \arrow[r] & \mathcal{SS} \arrow[r] & P\Phi_+ \arrow[dr] & & & \\
\mathcal{K} \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{SS}\cap\mathcal{SC} \arrow[ur] \arrow[dr] & & & P\Phi_+ + P\Phi_- \arrow[r] & P\Phi = \mathcal{I}n \arrow[r] & \mathcal{I}mp \\
 & \mathcal{SS}^d \arrow[r] & \mathcal{SC} \arrow[r] & P\Phi_- \arrow[ur] & & &
\end{tikzcd}$$

5. Positive answers to the perturbation classes problem

Originally, the perturbation classes problem asked if $\mathcal{SS} = P\Phi_+$ and $\mathcal{SC} = P\Phi_-$. But once that it is known that $\mathcal{SS} \neq P\Phi_+$ and $\mathcal{SC} \neq P\Phi_-$, the problem turns into the following, weaker one: find pairs X, Y of Banach spaces for which $\Phi_+(X,Y) \neq \emptyset$ and $\mathcal{SS}(X,Y) = P\Phi_+(X,Y)$, or $\Phi_-(X,Y) \neq \emptyset$ and $\mathcal{SC}(X,Y) = P\Phi_-(X,Y)$. Note that a positive answer for a given pair of spaces X, Y provides an intrinsic characterization of the corresponding component of the perturbation class. For similar reasons we are interested in spaces X for which $\mathcal{I}n(X) = \mathcal{I}mp(X)$. In this section we shall see that there are many examples among the classical Banach spaces for which this is true.

Since $\mathcal{K}$ is the simplest class of admissible perturbations for Φ, Φ_+ and Φ_-, we start with the following result, which mixes the beginnings of Fredholm perturbation theory with some recent discoveries of exotic Banach spaces.

Proposition 5.1. *The identities* $P\Phi(X) = P\Phi_+(X) = P\Phi_-(X) = \mathcal{K}(X)$ *hold in two very different cases:*

(a) $X = \ell_p$ *for* $1 \leq p < \infty$*;*

(b) $X = X_{\mathrm{AH}}$*, the hereditarily indecomposable space of Argyros and Haydon [13].*

Indeed, the symmetrical structure of the ℓ_p spaces lets them support many operators but, at the same time, helps to prove the equality easily. Part (b) holds by the opposite reason: the space of operators $\mathcal{L}(X_{\mathrm{AH}})$ is very small, up to the point that every $T \in \mathcal{L}(X_{\mathrm{AH}})$ is of the form $T = \lambda I + K$ for some $\lambda \in \mathbb{R}$ and some compact operator K. In particular, $\Phi(X_{\mathrm{AH}}) = \Phi_\pm(X_{\mathrm{AH}})$.

5.1. When $P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$ or $P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$

The perturbation classes $P\Phi_+(X,Y)$ and $P\Phi_-(X,Y)$ admit intrinsic characterizations when they coincide with $\mathcal{SS}(X,Y)$ and $\mathcal{SC}(X,Y)$ respectively. Here we describe some pairs of spaces for which one of the equalities holds.

A Banach space X is said to be *subprojective* if every closed, infinite-dimensional subspace Y of X contains an infinite-dimensional subspace Z complemented in X; if moreover we can always find Z with complement isomorphic to X, then X is said to be *strongly subprojective.*

Theorem 5.2. *Suppose $\Phi_+(X,Y) \neq \emptyset$. Then the identity $P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$ holds in the following cases:*

(a) Y is subprojective;
(b) X is hereditarily indecomposable [7];
(c) X is separable and Y contains a complemented copy of $C[0,1]$ [11];
(d) $X = L_p(0,1)$ for $1 < p < 2$ and Y satisfies the Orlicz property [26];
(e) $X = L_1(0,1)$ and Y is weakly sequentially complete [26];
(f) X is strongly subprojective [22, 43];
(g) X is a Lorentz space $\Lambda_{W,p}(0,1)$ with finite cotype for $1 < p < 2$, or $L_{p,q}(0,1)$ or $L_{p,q}(0,\infty)$ for $1 < p \leq 2$ and $1 \leq q \leq 2$, and $\mathcal{SS}(\ell_2,Y) = \mathcal{K}(\ell_2,Y)$ [25];
(h) X is an Orlicz function space $L_\varphi(0,1)$ with $E_\varphi^\infty \equiv \{t^p\}$ for $1 < p < 2$ and $\mathcal{SS}(\ell_2,Y) = \mathcal{K}(\ell_2,Y)$ [25].

Part (a) in Theorem 5.2 follows from the fact that if Y is subprojective, then $\mathcal{SS}(X,Y) = \mathcal{I}mp(X,Y)$ [7].

A Banach space X is said to be *superprojective* if every closed, infinite-codimensional subspace Y is contained in a complemented, infinite-codimensional subspace Z of X; if moreover Z can always be obtained isomorphic to X then X is said to be *strongly superprojective.*

Theorem 5.3. *Suppose $\Phi_-(X,Y) \neq \emptyset$. Then the identity $P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$ holds in the following cases:*

(a') X is superprojective;
(b') Y is quotient indecomposable [7];
(c') X contains a complemented copy of ℓ_1 and Y is separable [11];
(d') $Y = L_p(0,1)$ for $2 < p < \infty$ and X^* satisfies the Orlicz property [26];
(e') $Y = L_p(0,1)$ for $1 \leq p \leq 2$ [26];
(f') Y is strongly superprojective [22, 43].
(g') Y is a Lorentz function space $\Lambda_{W,p}(0,1)$ with finite type for $2 < p < \infty$, or $L_{p,q}(0,1)$ or $L_{p,q}(0,\infty)$ for $2 \leq p,q < \infty$ and $\mathcal{SC}(X,\ell_2) = \mathcal{K}(X,\ell_2)$ [25];
(h') Y is an Orlicz function space $L_\varphi(0,1)$ with $E_\varphi^\infty \equiv \{t^p\}$ for $2 < p < \infty$ and $\mathcal{SC}(X,\ell_2) = \mathcal{K}(X,\ell_2)$ [25].

Part (a') in Theorem 5.3 is a consequence of the fact that if X is superprojective, then $\mathcal{SC}(X,Y) = \mathcal{I}mp(X,Y)$ [7].

It is worth noting that Theorems 5.2 and 5.3 hold in two very different cases: on the one hand, for hereditarily indecomposable spaces and quotient indecomposable spaces, which only admit trivial projections; on the other hand, for (strongly) subprojective spaces and (strongly) superprojective spaces, which admit many projections.

There are many classical spaces that are strongly subprojective or strongly superprojective:

Proposition 5.4. *The following Banach spaces are strongly subprojective:*

(1) The sequence spaces ℓ_p for $1 \leq p < \infty$, and c_0.

(2) The James space J.

(3) The Lorentz sequence spaces $d(w,p)$ for $1 \leq p < \infty$ and $w = (w_n)$ a non-increasing null sequence with $\sum_{n=1}^{\infty} w_n$ divergent. This applies to $\ell_{p,q}$ for $1 \leq p, q < \infty$.

(4) The Baernstein spaces B_p for $1 < p < \infty$.

(5) The Tsirelson space T.

(6) The function spaces $L_p(0,1)$ for $2 \leq p < \infty$.

(7) The function spaces $L_p(0,\infty) \cap L_2(0,\infty)$ for $1 \leq p \leq 2$.

(8) The Lorentz spaces $\Lambda_{W,p}(0,1)$, $L_{p,q}(0,\infty)$ and $L_{p,q}(0,1)$ for $2 < p < \infty$ and $1 \leq q < \infty$.

(9) The spaces of continuous functions $C(K)$, with K a scattered compact.

(10) Closed subspaces of the previous examples.

We refer to a recent paper [36] for additional examples of (strongly) subprojective Banach spaces.

Recall that a compact space K is said to be *scattered* (or dispersed) if every non-empty subset of K has an isolated point. Examples of scattered compact spaces are the ordinal intervals $[0,\kappa] = \{\, \alpha \text{ ordinal} : 0 \leq \alpha \leq \kappa \,\}$, endowed with the order topology, and the one-point compactification Γ_∞ of any set Γ endowed with the discrete topology. Note that $C(\Gamma_\infty)$ is isomorphic to $c_0(\Gamma)$.

Proposition 5.5. *The following Banach spaces are strongly superprojective:*

(1') The sequence spaces ℓ_p for $1 < p < \infty$, and c_0.

(2') The dual J^* of James space.

(3') The dual spaces $d(w,p)^*$ of $d(w,p)$ for $1 < p < \infty$ and $w = (w_n)$ a non-increasing null sequence with $\sum_{n=1}^{\infty} w_n$ divergent. This applies to $\ell_{p,q}^*$ for $1 < p, q < \infty$.

(4') The dual spaces B_p^* of Baernstein's spaces for $1 < p < \infty$.

(5') The dual T^* of Tsirelson space.

(6') The function spaces $L_p(0,1)$ for $1 < p \leq 2$.

(7') The function spaces $L_p(0,\infty) + L_2(0,\infty)$ for $2 \leq p < \infty$.

(8') The dual spaces $\Lambda_{W,p}(0,1)^*$, $L_{p,q}(0,\infty)^*$ and $L_{p,q}(0,1)^*$ for $2 < p < \infty$ and $1 < q < \infty$.

(9') The spaces of continuous functions $C(K)$, with K a scattered compact.

(10') Quotients of the previous examples.

Propositions 5.4 and 5.5 were proved in [22]. It is unknown if there are subprojective spaces which are not strongly subprojective, or superprojective spaces which are not strongly superprojective.

5.2. When $P\Phi(X,Y) = \mathcal{I}n(X,Y)$ admits an intrinsic characterization

Here we describe some pairs of spaces for which $\mathcal{I}n(X,Y)$ coincides with $\mathcal{I}mp(X,Y)$, $\mathcal{SS}(X,Y)$ or $\mathcal{SC}(X,Y)$.

Proposition 5.6. [5, 44]

(a) *If X or Y is subprojective or superprojective, then $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$.*

(b) *If Y is subprojective, then $\mathcal{SS}(X,Y) = \mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$.*

(c) *If X is superprojective, then $\mathcal{SC}(X,Y) = \mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$.*

After Theorem 4.13, it is natural to ask for which compacts K the set $\mathcal{I}mp\big(C(K)\big)$ coincides with $\mathcal{SS}\big(C(K)\big)$ or $\mathcal{I}n\big(C(K)\big)$.

Proposition 5.7. [8] *The identity $\mathcal{I}n\big(C(K)\big) = \mathcal{I}mp\big(C(K)\big)$ holds in the following cases:*

(i) *K is scattered,*
(ii) *every separable subspace of $C(K)$ is contained in a separable complemented subspace,*
(iii) *every non-weakly compact operator acting on $C(K)$ is an isomorphism on some subspace isomorphic to ℓ_∞.*

An additional condition improves Proposition 5.7. Let us recall that a compact set K is a *Valdivia compact* if it contains a dense subset A for which there exists a set Γ and a homeomorphism $h\colon K \longrightarrow \mathbb{R}^\Gamma$ such that $h(A)$ is the set of elements in $h(K)$ with countable support. Eberlein, Gulko, and Corson compact sets are examples of Valdivia compacts [29].

Proposition 5.8. [8] *The identity $\mathcal{SS}\big(C(K)\big) = \mathcal{I}mp\big(C(K)\big)$ holds when K is a continuous image of a Valdivia compact.*

6. Some open problems

All known counterexamples to the perturbation classes problem for semi-Fredholm operators involve hereditarily indecomposable Banach spaces, whose properties are quite exotic.

Problem 1. *Find examples of classical Banach spaces X and Y such that $P\Phi_+(X,Y) \neq \mathcal{SS}(X,Y)$ or $P\Phi_-(X,Y) \neq \mathcal{SC}(X,Y)$.*

We can consider candidates by lifting some of the restrictions from the results shown in Subsection 5.1.

Problem 2. *Suppose that $1 < p < 2$ and Y is a Banach space containing a subspace isomorphic to $L_p(0,1)$. Is it true that $P\Phi_+(L_p(0,1),Y) \neq \mathcal{SS}(L_p(0,1),Y)$?*

We showed in Theorem 5.2 that the answer is positive when Y satisfies the Orlicz property. We can also consider the following dual problem, whose answer is again known true only under some restrictions.

Problem 3. *Suppose that $2 < p < \infty$ and X is a Banach space admitting a quotient isomorphic to $L_p(0,1)$. Is it true that $P\Phi_-(X,L_p(0,1)) \neq \mathcal{SS}(X,L_p(0,1))$?*

All the examples of spaces Z for which $P\Phi_+(Z) \neq \mathcal{SS}(Z)$ have the form $Z = X \times Y$ with X and Y infinite-dimensional. However, it was proved in [21] that a hereditarily indecomposable space Z_T constructed in [46] satisfies $P\Phi_-(Z_T) \neq \mathcal{SC}(Z_T)$.

Problem 4. *Does there exist an indecomposable space Z for which $P\Phi_+(Z) \neq \mathcal{SS}(Z)$?*

Note that Z in the previous problem cannot be hereditarily indecomposable (see (b) in Theorem 5.2).

Finally, we know the answer to the following problem in some special cases [8], but this could be improved.

Problem 5. *Characterize those compact spaces K for which $\mathcal{I}n\big(C(K)\big) = \mathcal{I}mp\big(C(K)\big)$.*

7. Final remarks

The term *semigroup* in reference to the classes Φ_+, Φ_- and Φ was first used by Lebow and Schechter [34]. Later, the term *operator semigroup* was extended by Aiena et al. [9] to any class of operators closed under composition and cartesian product of operators, and containing all bijective isomorphisms.

Weis proved that $P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$ when $X = Y = L_p(0,1)$ for $1 \leq p < 2$ [48]. This result is a consequence of a more general result included in Theorem 5.2, and it implies that $P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$ when $X = Y = L_p(0,1)$ for $2 < p < \infty$. We refer to [43] for a recent exposition of the properties of $P\Phi_+$ and $P\Phi_-$.

In [16], Friedman introduced a certain condition (C) and showed that

$$T \in \mathcal{SS} \Rightarrow T \text{ satisfies (C)} \Rightarrow T \in P\Phi_+.$$

In [10] it was shown that $T \in P\Phi_+$ does not imply property (C), leaving it open whether the converse of the other implication is valid.

We have considered the perturbation classes problems for bounded operators. The corresponding problems for closed semi-Fredholm and Fredholm operators with dense range were studied by Weis in [49]. We observe that the problems for bounded operators are different from their counterparts for closed operators. Indeed, we showed in Proposition 4.8 that there exists a separable reflexive space Z for which $\mathcal{SS}(Z) \neq P\Phi_+(Z)$ and $\mathcal{SC}(Z^*) \neq P\Phi_-(Z^*)$. However, it was proved in [49] that, for a separable space, the perturbation classes for closed upper and lower semi-Fredholm operators with dense range coincide respectively with the strictly singular and the strictly cosingular operators. We refer to [3] for additional information.

References

[1] P. Aiena. *An internal characterization of inessential operators.* Proc. Amer. Math. Soc. **102**, 3 (1988) 625–626.

[2] P. Aiena. *Fredholm and local spectral theory, with applications to multipliers.* Kluwer Acad. Publ., Dordrecht, 2004.

[3] P. Aiena, M. Chō and M. González. *The perturbation classes problem for closed operators.* Filomat (to appear).

[4] P. Aiena and M. González. *On the perturbation classes of semi-Fredholm operators and Fredholm operators.* Suppl. Rend. Circolo Mat. Ital. Serie II, **40** (1996) 37–46.

[5] P. Aiena and M. González. *On inessential and improjective operators.* Studia Math. 131 (1998) 271–287.

[6] P. Aiena and M. González. *Examples of improjective operators.* Math. Z. 233 (2000) 471–479.

[7] P. Aiena and M. González. *Inessential operators between Banach spaces.* Rendiconti Circ. Mat. Palermo, Ser II Suppl. 68 (2002) 3–26.

[8] P. Aiena and M. González. *Intrinsic characterizations of perturbation classes on some Banach spaces.* Arch. Math. 94 (2010) 373–381.

[9] P. Aiena, M. González and A. Martínez-Abejón. *Operator semigroups in Banach space theory.* Boll. Unione Mat. Ital. (8) 4 (2001) 157–205.

[10] P. Aiena, M. González and A. Martínez-Abejón. *Characterizations of strictly singular and strictly cosingular operators by perturbation classes.* Glasgow Math. J. 54 (2012) 87–96.

[11] P. Aiena, M. González and A. Martinón. *On the perturbation classes of continuous semi-Fredholm operators.* Glasgow Math. J. 45 (2003) 91–95.

[12] S. A. Argyros and V. Felouzis. *Interpolating hereditarily indecomposable Banach spaces.* J. Amer. Math. Soc. 13 (2000) 243–294.

[13] S. A. Argyros and R. G. Haydon. *A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact problem.* Acta Math. 206 (2011) 1–54.

[14] S. Caradus, W. Pfaffenberger and B. Yood. *Calkin algebras and algebras of operators in Banach spaces.* M. Dekker Lecture Notes in Pure & Appl. Math. New York, 1974.

[15] J. Dieudonné. *History of Functional Analysis.* North-Holland, 1981.

[16] T. L. Friedman. *Relating strictly singular operators to the condition $X < Y$ mod $(\mathcal{S},\mathcal{T})$ and resulting perturbations.* Analysis (Munich) 22 (2002) 347–354.

[17] J. Giménez, M. González and A. Martínez-Abejón. *Perturbation of semi-Fredholm operators on products of Banach spaces.* J. Operator Theory 68 (2012) 501–514.

[18] I. C. Gohberg, A. S. Markus and I. A. Feldman. *Normally solvable operators and ideals associated with them.* Bul. Akad. Štiince RSS Moldoven 10 (76) (1960) 51–70. Translation: Amer. Math. Soc. Transl. (2) 61 (1967) 63–84.

[19] S. Goldberg. *Unbounded linear operators: Theory and applications.* McGraw-Hill, New York, 1966. Reprint: Dover, 1985.

[20] M. González. *The perturbation classes problem in Fredholm theory.* J. Funct. Anal. 200 (2003) 65–70.

[21] M. González. *Duality results for perturbation classes of semi-Fredholm operators.* Arch. Math. 97 (2011) 345–352.

[22] M. González, A. Martínez-Abejón and M. Salas-Brown. *Perturbation classes for semi-Fredholm operators on subprojective and superprojective spaces.* Ann. Acad. Sci. Fennicae Math. 36 (2011) 481–491.

[23] M. González and A. Martinón. *Operational quantities characterizing semi-Fredholm operators.* Studia Math. 114 (1995) 33–27.

[24] M. González and A. Martinón. *Quantities characterizing semi-Fredholm operators and perturbation radii.* J. Math. Anal. Appl. 390 (2012) 362–367.

[25] M. González, J. Pello and M. Salas-Brown. *Perturbation classes for semi-Fredholm operators in some Banach lattices.* J. Math. Anal. Appl. 420 (2014) 792–800.

[26] M. González and M. Salas-Brown. *Perturbation classes for semi-Fredholm operators in $L_p(\mu)$-spaces.* J. Math. Anal. Appl. 370 (2010) 11-17.

[27] W. T. Gowers and B. Maurey. *The unconditional basic sequence problem.* J. Amer. Math. Soc. 6 (1993) 851–874.

[28] W. T. Gowers and B. Maurey. *Banach spaes with small spaces of operators.* Math. Ann. 307 (1997) 543–568.

[29] O. F. K. Kalenda. *Valdivia compact spaces in topology and Banach space theory.* Extracta Math. 15 (2000) 1–85.

[30] T. Kato. *Perturbation theory for nullity, deficiency and other quantities of linear operators.* J. d'Analyse Math. 6 (1958) 261–322.

[31] T. Kato. *Perturbation theory for linear operators.* Reprint of the 1980 edition. Classics in Math. Springer-Verlag, Berlin, 1995.

[32] D. Kleinecke. *Almost finite, compact, and inessential operators.* Proc. Amer. Math. Soc. 14 (1963) 863–868.

[33] P. Koszmider. *Banach spaces of continuous functions with few operators.* Math. Ann. 330 (2004) 151–183.

[34] A. Lebow and M. Schechter. *Semigroups of operators and measures of noncompactness.* J. Funct. Anal. 7 (1971) 1–26.

[35] M. Martins, F. Acero. *Fredholm, ecuaciones integrales y álgebra lineal.* Ciencia y Tecnología, **7** (2008) 93–112.

[36] T. Oikhberg and E. Spinu. *Subprojective Banach spaces.* J. Math. Anal. Appl. 424 (2015) 613–635.

[37] A. Pełczyński. *On strictly singular and strictly cosingular operators I. Strictly singular and strictly cosingular operators in $C(S)$-spaces.* Bull. Acad. Polon. Sci. 13 (1965) 31–36.

[38] A. Pełczyński. *On strictly singular and strictly cosingular operators II. Strictly singular and strctly cosingular operators in $L(\nu)$-spaces.* Bull. Acad. Polon. Sci. 13 (1965) 37–41.

[39] A. Pietsch. *Inessential operators in Banach spaces.* Integral Equations Operator Theory 1 (1978) 589–591.

[40] A. Pietsch. *Operator ideals.* North-Holland, Amsterdam, 1980.

[41] G. Plebanek. *A construction of a Banach space $C(K)$ with few operators.* Topology Appl. 143 (2004) 217–239.

[42] H. P. Rosenthal. *On relatively disjoint families of measures, with some applications to Banach space theory.* Studia Math. 37 (1970) 13–36.

[43] M. Salas-Brown. *Clases de perturbación en teoría de Fredholm.* Ph. D. Thesis, Universidad Central de Venezuela, 2012.

[44] E. Tarafdar. *Improjective operators and ideals in a category of Banach spaces.* J. Austral. Math. Soc. 14 (1972) 274–292.

[45] E. Tarafdar. *On further properties of improjective operators.* J. Austral. Math. Soc. 14 (1972) 352-363.

[46] M. Tarbard. *Hereditarily indecomposable, separable $\mathcal{L}_\infty$-spaces with ℓ_1 dual having few operators, but not very few operators.* J. London Math. Soc. 85 (2012) 737–764.

[47] J. I. Vladimirskii. *Strictly cosingular operators.* Soviet Math. Doklady 8 (1967) 739–740.

[48] L. Weis. *On perturbation of Fredholm operators in $L_p(\mu)$-spaces.* Proc. Amer. Math. Soc. 67 (1977) 287–292.

[49] L. Weis. *Perturbation classes of semi-Fredholm operators.* Math. Z. 178 (1981) 429–442.

[50] R. J. Whitley. *Strictly singular operators and their conjugates.* Trans. Amer. Math. Soc. 113 (1964) 252–261.