A survey on the perturbation classes problem for semi-Fredholm and Fredholm operators

Manuel González, Antonio Martínez-Abejón, Javier Pello

Abstract. We describe the main partial positive answers and counterexamples to the perturbation classes problem for semi-Fredholm and Fredholm operators, and point out several related problems that remain open.

1. Introduction

The study of the perturbation classes for semi-Fredholm and Fredholm operators stemmed from the abstract approach to integral equations at the beginning of the twentieth century. The main problems it considers can be described as follows:

(A) Identify the perturbation classes $P\Phi_+$, $P\Phi_-$ and $P\Phi$ for the upper semi-Fredholm, lower semi-Fredholm and Fredholm operators.

(B) Find intrinsic characterizations for each of the perturbation classes $P\Phi_+$, $P\Phi_-$ and $P\Phi$.

(C) Describe the properties of the components $P\Phi_+(X,Y)$, $P\Phi_-(X,Y)$ and $P\Phi(X,Y)$.

With respect to problem (B), observe that in order to check whether an operator $K$ satisfies the definition of a perturbation class, we have to study the properties of $T + K$ for $T$ in a family of operators, which can be unwieldy. An intrinsic characterization is one given in terms of the action of the operator $K$ itself, like that of the strictly singular operators (Definition 4.4). For instance, problem (A) is solved for the class of Fredholm operators after the identification of its
perturbation class with the inessential operators. However, this is not totally satisfactory because the
definition of an inessential operator (Definition 3.5) is not intrinsic. As we will see, these three
problems are interdependent.

In this survey we describe the answers to these problems that have been obtained so far,
specially in the last twenty years, and we point out several problems that remain open. Section 2
provides some historical background for the problem, and Section 3 contains the definitions and
main properties of Fredholm and semi-Fredholm operators, and of their perturbation classes. In
Section 4 we introduce the most important intrinsically-defined classes of operators in connection
with the perturbation classes in Fredholm theory: compact, strictly singular, strictly cosingular
and improjective operators. In Section 5 we show many examples of pairs of Banach spaces for
which the components of the perturbation classes coincide with the corresponding components
of strictly singular, strictly cosingular, or improjective operators. Section 6 collects a selection of
open problems.

**Notation:** \( \mathcal{L} \) denotes the class of all bounded linear operators (henceforth, operators) acting
between Banach spaces; \( \mathcal{L}(X, Y) \) is the class of all operators into \( Y \) whose domain is \( X \). Given
a class of operators \( \mathcal{A} \), we refer to \( \mathcal{A}(X, Y) := \mathcal{A} \cap \mathcal{L}(X, Y) \) as the component of \( \mathcal{A} \) in \( \mathcal{L}(X, Y) \);
we write \( \mathcal{A}(X) \) instead of \( \mathcal{A}(X, X) \). The dual of a Banach space \( X \) is denoted by \( X^* \), and the
conjugate of an operator \( T \) is denoted by \( T^* \). The range of \( T \in \mathcal{L}(X, Y) \) is denoted by \( \text{Ran} \ T \),
its kernel by \( \text{Ker} \ T \), and its co-kernel is \( \text{Coker} \ T := Y/\text{R}(T) \). Given a class \( \mathcal{A} \) of operators, we
denote \( \mathcal{A}_d := \{ T : T^* \in \mathcal{A} \} \). If \( Z \) is a closed subspace of \( X \), then \( J_Z \in \mathcal{L}(Z, X) \) is the subspace
embedding of \( Z \) into \( X \) and \( Q_Z \in \mathcal{L}(X, X/Z) \) is the associated quotient operator. The identity
operator on \( X \) is denoted by \( I_X \) or simply \( I \).

### 2. Some historical remarks

As it is well explained in [15], the unified solution given by Fredholm to the integral equations
of Abel and Volterra, together with further contributions by Hilbert, led to the formulation of the
famous Fredholm integral equations

\[
\begin{align*}
\int_a^b K(x, y) u(y) \, dy &= f(x) \quad \text{(first class)} \quad (1) \\
u(x) &= f(x) + \int_a^b K(x, y) u(y) \, dy \quad \text{(second class)} \quad (2)
\end{align*}
\]

where \( K(x, y) \) is a kernel function, \( f(x) \) is a given function with integrable square and \( u(x) \) is the
unknown function. Roughly speaking, the original purpose of integral equations is the study of
the stability of certain physical systems under small perturbations. In the case of the Fredholm
integral equations, those small perturbations are represented by the kernel function: if \( K(x, y) = 0 \)
for all \( (x, y) \), then the system represented in (2) is trivial, while if the kernel \( K(x, y) \) is small enough
(in a certain unspecified sense), then \( u(x) \) is close enough to \( f(x) \) (in the same unspecified sense) and
the system is declared to be stable, in which case the perturbation represented by \( K \) is said to be
admissible. In the language of functional analysis, the Fredholm integral equations can be written
as

\[
Ku = f \quad \text{and} \quad u = f + Ku, \quad (3)
\]

where \( K : H \to H \) is a compact operator, \( H \) is a Hilbert space, \( f \) is a given element of \( H \) and \( u \)
is the unknown. In the second part of equation (3), \( f \) is the image of \( u \) by the Fredholm operator
\( I - K \), that is, the identity operator perturbed by \( K \). In fact, research on the Fredholm equations
meant the beginning of Functional Analysis.

Once the duality relation in \( L_p \) spaces was well understood, the theory of Fredholm operators
was extended to the Banach space setting in a natural way, and this last step introduced different
ways to measure the smallness of a perturbation other than the one provided by the Euclidean
norm. In this way, the theory of Fredholm and semi-Fredholm operators on Banach spaces and the problem of their perturbation classes came hand in hand, and although this last question is totally solved for Fredholm operators on Hilbert spaces, it is still challenging in the context of Banach spaces.

In 1954, in the context of Banach algebras, Kleinecke introduced the ideal $\mathcal{I}_n$ of inessential operators as the class of $\Phi$-admissible perturbations, although his results were published almost ten years later [32]. In 1958, Kato introduced the ideal $SS$ of strictly singular operators as a natural class of $\Phi_+$-admissible perturbations that includes the compact operators [30], and Gohberg, Markus and Feldman asked for intrinsic characterizations of $P\Phi_+$, $P\Phi_-$ and $P\Phi$ [18]; in particular, they asked if $P\Phi_+ = SS$. Pełczyński introduced the class $SC$ of strictly cosingular operators as a kind of dual class of $SS$ [37] in 1965, and two years later Vladimirskii proved that $SC$ is a class of $\Phi_-$-admissible perturbations [47]. In 1972, Tarafdar introduced the intrinsically-defined class $Imp$ of improjective operators and proved that $SS + SC \subset \mathcal{I}_n \subset Imp$ [44]. He also proved that many components of $Imp$ coincide with those of $SS$ and $SC$. Finally, the discovery of the hereditarily indecomposable spaces allowed to show in [20] that $SS \neq P\Phi_+$ and $SC \neq P\Phi_-$ (see Proposition 4.8).

Let us end this section by observing that the movement from the original framework of Hilbert spaces to more general Banach spaces is not merely philosophical: the measurement of smallness of a perturbation by means of norms other than the hilbertian one is very important for technical branches like signal theory [35].

3. Semi-Fredholm operators and their perturbation classes

We begin with the definition of the three classes of semi-Fredholm operators. We refer to [2] for a detailed study of these classes.

**Definition 3.1.** An operator $T \in L(X,Y)$ is said to be:

(i) upper semi-Fredholm if $\text{Ker} \, T$ is finite-dimensional and $\text{Ran} \, T$ is closed;

(ii) lower semi-Fredholm if $\text{Coker} \, T$ is finite-dimensional and $\text{Ran} \, T$ is closed;

(iii) Fredholm if $T$ is both upper semi-Fredholm and lower semi-Fredholm.

It is not difficult to prove that if $\text{Ran} \, T$ is finite-codimensional, then it must also be closed, which makes $T$ lower semi-Fredholm.

The classes of upper semi-Fredholm, lower semi-Fredholm and Fredholm operators will be respectively denoted by $\Phi_+$, $\Phi_-$ and $\Phi$; hence $\Phi = \Phi_+ \cap \Phi_-$. We will also denote $\Phi_{\pm} := \Phi_+ \cup \Phi_-$. It follows from the basic duality results for operators that $T \in \Phi_{\pm}$ if and only if $T^* \in \Phi_{\pm}$, and that in this case $\text{dim Ker} \, T = \text{dim Ker} \, T^*$ and $\text{dim Coker} \, T = \text{dim Coker} \, T^*$.

The index of an operator $T \in \Phi_{\pm}$ is defined by

$$\text{ind} \, T := \text{dim Ker} \, T - \text{dim Coker} \, T.$$ 

Note that $\text{ind} \, T$ is finite if and only if $T \in \Phi$. Similarly, $T \in \Phi_+$ if and only if $\text{ind} \, T < \infty$, and $T \in \Phi_-$ if and only if $\text{ind} \, T > -\infty$. The index is a continuous function on each space $\Phi_{\pm}(X,Y)$ when it is endowed with the norm topology; as such, it is constant on each connected component of $\Phi_{\pm}(X,Y)$.

Given an operator $T$ in $\Phi_{\pm}(X,Y)$, there exists $\varepsilon_T > 0$ such that $T + K \in \Phi_{\pm}$ and $\text{ind}(T + K) = \text{ind} \, T$ for all $K \in L(X,Y)$ such that $\|K\| < \varepsilon_T$. The task of looking for the largest possible value of $\varepsilon_T$ is a local question investigated by the theory of operational quantities (see [23], [24] and references therein). However, the concept of a perturbation class is of global character:
Definition 3.2. The perturbation class $PS$ of $S \in \{\Phi_+, \Phi_-, \Phi\}$ is defined by its components in $\mathcal{L}(X, Y)$, when $S(X, Y) \neq \emptyset$, as follows:

$$PS(X, Y) := \{ K \in \mathcal{L}(X, Y) : T + K \in S(X, Y) \text{ for all } T \in S(X, Y) \}.$$  

Every operator in $PS$ is called an $S$-admissible perturbation, and every subclass of $PS$ is called a class of $S$-admissible perturbations.

Note that Definition 3.2 is not intrinsic because, in order to determine if $K \in PS(X, Y)$, it is not enough to check the action of $K$, but we have to study the properties of all the operators in the set $\{ T + K : T \in S(X, Y) \}$. Obviously, an intrinsic characterization for the operators in $PS$ in terms of its sole action would provide a positive answer to problem (B) in the Introduction.

Let us see the relationship between the three classes $P\Phi_+, P\Phi_-$ and $P\Phi$.

Proposition 3.3. The inclusion $P\Phi_+ \cup P\Phi_- \subset P\Phi$ holds.

The proof of this inclusion is a direct application of the constancy of the index $\text{ind} T$ on the connected components of $\Phi_+(X, Y)$ [14]. Indeed, observe that given $T \in \Phi_+(X, Y)$ and $K \in \mathcal{L}(X, Y)$ in the corresponding perturbation class, we have $T + tK \in \Phi_+(X)$ for each $t \in [0, 1]$, hence $\text{ind}(T + K) = \text{ind} T$.

A class $\mathcal{A}$ of operators is said to be injective if $T \in \mathcal{A}$ whenever $LT \in \mathcal{A}$ for one (and then for all) isomorphic embedding $L$; roughly speaking, $\mathcal{A}$ is injective when the fact that $T \in \mathcal{A}$ does not depend upon the target space of $T$. In a similar way, $\mathcal{A}$ is said to be surjective if $T \in \mathcal{A}$ whenever $TQ \in \mathcal{A}$ for any surjective operator $Q$.

Proposition 3.4. Given a pair $X, Y$ of Banach spaces, the following statements hold:

(i) $P\Phi_+$ is injective but not surjective;
(ii) $P\Phi_-$ is surjective but not injective;
(iii) $P\Phi_+(X, Y)$ and $P\Phi_+(X)$ are closed subspaces of $\mathcal{L}(X, Y)$;
(iv) $P\Phi_+(X)$ and $P\Phi_-(X)$ are two-sided ideals of $\mathcal{L}(X)$.
(v) the components of $P\Phi_+$ and $P\Phi_-$ do not determine operator ideals: there is no operator ideal $\mathcal{A}$ such that $P\Phi_+(X) = \mathcal{A}(X)$ for every Banach space $X$, and neither is there for $P\Phi_-$.

The proof of the positive parts of (i) and (ii) is straightforward, and the negative parts are a consequence of the same properties for the strictly singular and strictly cosingular operators. Proofs of (iii) and (iv) can be found in [14, page 97], and (v) follows from the results of [20] and [40, Remark after 26.6.12].

Part (v) of Proposition 3.4 means that problems (A) and (B) have a negative answer in general. However, they can be answered in the positive for many pairs of Banach spaces as we will see in Section 5. Note also that $SS$ (resp. $SC$) is the largest proper injective (surjective) operator ideal [40, 4.6.14].

Definition 3.5. An operator $T \in \mathcal{L}(X, Y)$ is said to be inessential if $I_X - AT \in \Phi(X)$ for every $A \in \mathcal{L}(Y, X)$; equivalently, if $I_Y - TA \in \Phi(Y)$ for every $A \in \mathcal{L}(Y, X)$. The class of all inessential operators will be denoted $\mathcal{I}_n$.

Inessential operators were introduced by Kleinecke [32], who proved the following result.

Proposition 3.6. The class $\mathcal{I}_n$ is a closed operator ideal, but it is neither injective nor surjective.

Note that $\mathcal{I}_n^d \subset \mathcal{I}_n$, but $\mathcal{I}_n$ is not self-dual: the inclusion operator $J: c_0 \rightarrow \ell_\infty$ is inessential because, by a result of Rosenthal [42], any operator on $\ell_\infty$ is either weakly compact or an isomorphism on some subspace isomorphic to $\ell_\infty$, but $c_0^d$ is complemented in $\ell_\infty^*$, so the conjugate operator $J^*$ is not inessential.
Theorem 3.7. The identity $P \Phi(X) = \mathcal{I}n(X)$ holds for every Banach space $X$.

Theorem 3.7 solves questions (A) and (C) for $P \Phi$, but the definition of inessential operator is obviously not intrinsic. Aiena obtained some spectral characterizations for the components $P \Phi(X)$ [1] which constitute an important advance towards the solution of problem (B) for $P \Phi$. Note that the study of the components $P \Phi(X)$ is sufficient to know the behavior of $P \Phi(X, Y)$ because the existence of a Fredholm operator $T: X \rightarrow Y$ implies that $X$ is isomorphic to $Y$ up to a finite dimensional subspace. In this sense, the class $P \Phi$ is very different to the classes $P \Phi_+$ and $P \Phi_-$. Theorem 3.7 and Proposition 3.6 also shed some light upon questions (A) and (B). For instance, it has led to disprove conjectures like the possible identification of $P \Phi$ with the intrinsically-defined class $\mathcal{I}mp$. Also, the identification of $P \Phi$ with an operator ideal shows a difference between the classes $P \Phi_+$ and $P \Phi_-$ and the class $P \Phi$.

4. Intrinsically-defined classes of admissible perturbations

Many operator classes admitting intrinsic descriptions have been proposed as possible solutions to questions (A) and (B). The most remarkable of them are enumerated in this section.

4.1. The compact operators

Recall that an operator $T \in \mathcal{L}(X, Y)$ is said to be compact if $(Tx_n)$ contains a convergent subsequence whenever $(x_n)$ is bounded.

Compact operators already occur at the beginning of the early Fredholm theory as explained in the introduction, where we pointed out that

$$P \Phi(H) = P \Phi_+(H) = P \Phi_-(H) = K(H)$$

if $H$ is a Hilbert space. The main properties of $K$ are given in the following result.

Proposition 4.1. [40] The class $K$ is a self-dual, injective, surjective, closed operator ideal consisting of $S$-admissible perturbations for all $S \in \{ \Phi_+, \Phi_-, \Phi \}$.

Although the class $K$ is properly contained in the other classes of admissible perturbations, it is still interesting in Fredholm theory. For instance, the following characterizations are a very useful tool:

Proposition 4.2. [34] Let $T \in \mathcal{L}(X, Y)$.

(i) $T \in \Phi_+$ if and only if $\text{Ker}(T + K)$ is finite-dimensional for all $K \in K(X, Y)$.

(ii) $T \in \Phi_-$ if and only if $\text{Coker}(T + K)$ is finite-dimensional for all $K \in K(X, Y)$.

Also the following observation is useful:

Remark 4.3. In the previous result we can replace “compact operators” by “nuclear operators with arbitrarily small norm”.

4.2. The strictly singular and the strictly cosingular operators

**Definition 4.4.** An operator $T \in L(X,Y)$ is said to be strictly singular if for every subspace $Z$ of $X$, $TZ$ is an isomorphic embedding only if $Z$ is finite-dimensional.

$T$ is said to be strictly cosingular if for every subspace $Z$ of $Y$, $Q_ZT$ is surjective only if $Z$ is finite-codimensional.

**Proposition 4.5.** The classes $SS$ and $SC$ have the following properties:

(i) $SS$ and $SC$ are closed operator ideals;
(ii) $SS$ is injective but not surjective, and $SC$ is surjective but not injective;
(iii) $SS \subseteq P\Phi_+$ and $SC \subseteq P\Phi_-$.

In the previous proposition, the results for $SS$ are due to Kato [30] and the results for $SC$ are due to Pełczyński [37] and Vladimirskii [47].

**Proposition 4.6.** The following statements hold:

(i) $K \not\subseteq SS \cap SC$;
(ii) $SC \setminus SS \neq \emptyset$;
(iii) $SS \setminus SC \neq \emptyset$.

For part (ii), the natural inclusion operator $U: L_2[0,1] \to L_1[0,1]$ belongs to $SC \setminus SS$, and for part (iii), the inclusion operator $V: L_\infty[0,1] \to L_2[0,1]$ belongs to $SS \setminus SC$. Thus, part (i) is proved by the natural inclusion operator $W := UV: L_\infty[0,1] \to L_1[0,1]$, which belongs to $SS \cap SC \setminus K$.

Indeed, the restriction of $U$ to the subspace generated by the Rademacher functions is an isomorphic injection, so $U \notin SS$. On the other hand, if $N$ is a subspace of $L_1[0,1]$ such that $Q_NU$ is surjective, then $Q_N$ is a surjection onto a reflexive subspace, hence $Q_N$ is weakly compact. Since $L_1[0,1]$ has the Dunford-Pettis property, $Q_N$ is completely continuous, so $Q_NU$ is compact, hence the quotient $L_1[0,1]/N$ is finite-dimensional and therefore, $U \in SC$.

Now, $V = U^*$, so $U \notin SS$ implies $V \notin SC$. Moreover, since $L_\infty[0,1]$ has the Dunford-Pettis property, any restriction $V|_R$ with $R$ reflexive is completely continuous, so it cannot be an isomorphic embedding, which proves $V \in SS$. Finally, $W$ is not compact because the sequence $(f_n)$, where $f_n(t) := \sin 2\pi nt$, is bounded in $L_\infty[0,1]$ but does not have any convergent subsequence in $L_1[0,1]$. For additional details, see [4].

For a while, it was thought that the admissible classes $SS$ and $SC$ might be identified with the perturbation classes $P\Phi_+$ and $P\Phi_-$, but the discovery of the hereditarily indecomposable spaces denied this possibility.

**Definition 4.7.** A Banach space $X$ is said to be indecomposable if for each decomposition $X = X_1 \oplus X_2$ with $X_1$ and $X_2$ closed subspaces, $X_1$ or $X_2$ is finite-dimensional.

The space $X$ is said to be hereditarily indecomposable if all its closed subspaces are indecomposable, and it is said to be quotient indecomposable if all its quotients are indecomposable.

Hereditarily indecomposable spaces, unlike Hilbert spaces, support very few operators. In fact, Weis proved [49] that for every space $Y$ we have $L(X,Y) = \Phi_+(X,Y) \cup SS(X,Y)$ if and only if $X$ is hereditarily indecomposable; and similarly, for every space $Y$ we have $L(Y,X) = \Phi_-(Y,X) \cup SC(Y,X)$ if and only if $X$ is quotient indecomposable.

**Proposition 4.8.** There exists a separable reflexive space $Z$ for which $SS(Z) \not\subseteq P\Phi_+(Z)$ and $SC(Z^*) \not\subseteq P\Phi_-(Z^*)$. 
There are at least two different constructions of such a space $Z$ [17, 20], and both of them involve the hereditarily indecomposable spaces discovered in [27] and [12]. Let us succinctly describe the construction given in [20]:

Let $X$ be a reflexive, hereditarily indecomposable space and let $Y$ be an infinite-dimensional closed subspace of $X$ such that $\dim X/Y = \infty$. Then the operator

$$A: Y \times X \to Y \times X$$

defined by $A(y, x) = (0, y)$ is inessential, but clearly not strictly singular. Applying the stability under perturbations of the index of semi-Fredholm operators, we can show that the space $Z := X \times Y$ satisfies

$$\Phi_+(Z) = \Phi(Z).$$

Thus

$$P\Phi_+(Z) = P\Phi(Z) = \mathcal{I}n(Z) \neq SS(Z).$$

A duality argument shows that the conjugate operator $A^*: Z^* \to Z^*$ is inessential, but not strictly singular, and that $\Phi_-(Z^*) = \Phi(Z^*)$. Thus

$$P\Phi_-(Z^*) = P\Phi(Z^*) = \mathcal{I}n(Z^*) \neq SC(Z^*).$$

Since $SS$ and $SC$ are closed operator ideals, it is straightforward that so are $SS^d$ and $SC^d$. The following result describes the duality between $SS$ and $SC$.

**Proposition 4.9.** [50] The ideals $SS$ and $SC$ satisfy the following statements:

(i) $SS^d \subset SC$;

(ii) $SC^d \subset SS$;

An example of an operator in $SC \setminus SS^d$ is the natural inclusion $J_{c_0} \in \mathcal{L}(c_0, \ell_\infty)$; and $SS \setminus SC^d$ contains the operator $Q \in \mathcal{L}(\ell_1(\ell_2^n), \ell_2)$ defined by

$$Q((x_i^n)_{i=1}^{\infty}:=(\lim_{n \to \mathcal{U}} x_i^n)_{i \in \mathbb{N}}$$

where $\mathcal{U}$ is a fixed non-trivial ultrafilter on $\mathbb{N}$. In fact, $Q$ is surjective and weakly compact, and as $\ell_1(\ell_2^n)$ has the Schur property (weakly convergent sequences in $\ell_1(\ell_2^n)$ are norm convergent), then $Q$ is strictly singular. But the range of $Q^*$ is complemented, hence $Q^* \notin SC$ [4].

4.3. The improjective operators

As we have seen, the ideals $K$, $SS$ and $SC$ are intrinsically-defined subclasses of $P\Phi_+$, $P\Phi_-$ and $P\Phi$. In turn, these perturbation classes are subclasses of the following class:

**Definition 4.10.** An operator $T \in \mathcal{L}(X, Y)$ is said to be improjective if for every subspace $Z$ of $X$, $TJ_Z$ is an isomorphic embedding and $T(Z)$ is complemented in $Y$ only if $Z$ is finite-dimensional.

**Proposition 4.11.** [44, 45] The class $\text{Imp}$ is an operator quasi-ideal and satisfies the following statements:

(i) $\text{Imp}^d \subset \text{Imp}$;

(ii) $P\Phi \subset \text{Imp}$.

The inclusion $\text{Imp}(X, Y) \subset \text{Imp}^d(X, Y)$ holds if $X$ is reflexive [44]. Unfortunately, the behaviour of $\text{Imp}$ is not as good as that of other classes. Let $X_{GM}$ be the separable, indecomposable space constructed by Gowers and Maurey [28] after their discovery of the hereditarily indecomposable spaces in [27].
Proposition 4.12. [6] The following statements hold:

(i) $\text{Imp}(X_{GM})$ is not a linear subspace of $L(X_{GM})$; hence $\text{Imp}$ is not an operator ideal;

(ii) $\mathcal{I}n(X_{GM}) \not\subseteq \text{Imp}(X_{GM})$.

Using the Continuum Hypothesis, Koszmider constructed a connected compact space $K_0$ such that every operator $T \in L(C(K_0))$ is of the form $T = gI + S$ with $g \in C(K_0)$ and $S$ a weakly compact operator on $C(K_0)$ [33]. As a consequence, the space $C(K_0)$ is indecomposable. Another $C(K)$ space with the same characteristics but whose construction does not depend on the Continuum Hypothesis was found by Plebanek [41].

Theorem 4.13. [8] Let $K_0$ be the compact of Koszmider. Then

$$\mathcal{S}\mathcal{S}(C(K_0)) = \mathcal{I}n(C(K_0)) = \not\subseteq \text{Imp}(C(K_0)),$$

It follows from the original proof of Theorem 4.13 that $\text{Imp}(C(K_0))$ is not a linear subspace of $L(C(K_0))$.

The classes of operators that we have considered in connection with the perturbation classes problem for semi-Fredholm operators and the relationships among them are represented in the following diagram, where $A \rightarrow C$ means that class $A$ is properly contained in class $C$.

$$\mathcal{S}\mathcal{C}^d \rightarrow \mathcal{S}\mathcal{S} \rightarrow P\Phi_+ \rightarrow \mathcal{S}\mathcal{C} \rightarrow \mathcal{S}\mathcal{S}^d \rightarrow P\Phi_-$$

5. Positive answers to the perturbation classes problem

Originally, the perturbation classes problem asked if $\mathcal{S}\mathcal{S} = P\Phi_+$ and $\mathcal{S}\mathcal{C} = P\Phi_-$. But once it is known that $\mathcal{S}\mathcal{S} \neq P\Phi_+$ and $\mathcal{S}\mathcal{C} \neq P\Phi_-$, the problem turns into the following, weaker one: find pairs $X, Y$ of Banach spaces for which $\Phi_+(X, Y) \neq \emptyset$ and $\mathcal{S}\mathcal{S}(X, Y) = P\Phi_+(X, Y)$, or $\Phi_-(X, Y) \neq \emptyset$ and $\mathcal{S}\mathcal{C}(X, Y) = P\Phi_-(X, Y)$. Note that a positive answer for a given pair of spaces $X, Y$ provides an intrinsic characterization of the corresponding component of the perturbation classes. For similar reasons we are interested in spaces $X$ for which $\mathcal{I}n(X) = \text{Imp}(X)$. In this section we shall see that there are many examples among the classical Banach spaces for which this is true.

Since $\mathcal{K}$ is the simplest class of admissible perturbations for $\Phi, \Phi_+$, and $\Phi_-$, we start with the following result, which mixes the beginnings of Fredholm perturbation theory with some recent discoveries of exotic Banach spaces.

Proposition 5.1. The identities $P\Phi(X) = P\Phi_+(X) = P\Phi_-(X) = \mathcal{K}(X)$ hold in two very different cases:

(a) $X = \ell_p$ for $1 \leq p < \infty$;

(b) $X = X_{\text{AH}}$, the hereditarily indecomposable space of Argyros and Haydon [13].

Indeed, the symmetrical structure of the $\ell_p$ spaces lets them support many operators but, at the same time, helps to prove the equality easily. Part (b) holds by the opposite reason: the space of operators $L(X_{\text{AH}})$ is very small, up to the point that every $T \in L(X_{\text{AH}})$ is of the form $T = \lambda I + K$ for some $\lambda \in \mathbb{R}$ and some compact operator $K$. In particular, $\Phi(X_{\text{AH}}) = \Phi_+(X_{\text{AH}})$. 

5.1. When $P\Phi_+(X, Y) = SS(X, Y)$ or $P\Phi_-(X, Y) = SC(X, Y)$

The perturbation classes $P\Phi_+(X, Y)$ and $P\Phi_-(X, Y)$ admit intrinsic characterizations when they coincide with $SS(X, Y)$ and $SC(X, Y)$ respectively. Here we describe some pairs of spaces for which one of the equalities holds.

A Banach space $X$ is said to be subprojective if every closed, infinite-dimensional subspace $Y$ of $X$ contains an infinite-dimensional subspace $Z$ complemented in $X$; if moreover we can always find $Z$ with complement isomorphic to $X$, then $X$ is said to be strongly subprojective.

Theorem 5.2. Suppose $\Phi_+(X, Y) \neq \emptyset$. Then the identity $P\Phi_+(X, Y) = SS(X, Y)$ holds in the following cases:

(a) $Y$ is subprojective;
(b) $X$ is hereditarily indecomposable [7];
(c) $X$ is separable and $Y$ contains a complemented copy of $C[0,1]$ [11];
(d) $X = L_p(0,1)$ for $1 < p < 2$ and $Y$ satisfies the Orlicz property [26];
(e) $X = L(1,0,1)$ and $Y$ is weakly sequentially complete [26];
(f) $X$ is strongly subprojective [22, 43];
(g) $X$ is a Lorentz space $\Lambda_{p,q}(0,1)$ with finite cotype for $1 < p < 2$, or $L_{p,q}(0,1)$ or $L_{p,q}(0,\infty)$ for $1 < p \leq 2$ and $1 \leq q \leq 2$, and $SS(\ell_2, Y) = K(\ell_2,Y)$ [25];
(h) $X$ is an Orlicz function space $L_\varphi(0,1)$ with $E^\infty_\varphi \equiv \{ t^p \}$ for $1 < p < 2$ and $SS(\ell_2, Y) = K(\ell_2,Y)$ [25].

Part (a) in Theorem 5.2 follows from the fact that if $Y$ is subprojective, then $SS(X, Y) = Imp(X,Y)$ [7].

A Banach space $X$ is said to be superprojective if every closed, infinite-codimensional subspace $Y$ is contained in a complemented, infinite-codimensional subspace $Z$ of $X$; if moreover $Z$ can always be obtained isomorphic to $X$ then $X$ is said to be strongly superprojective.

Theorem 5.3. Suppose $\Phi_-(X, Y) \neq \emptyset$. Then the identity $P\Phi_-(X, Y) = SC(X, Y)$ holds in the following cases:

(a') $X$ is superprojective;
(b') $Y$ is quotient indecomposable [7];
(c') $X$ contains a complemented copy of $\ell_1$ and $Y$ is separable [11];
(d') $Y = L_p(0,1)$ for $2 < p < \infty$ and $X^*$ satisfies the Orlicz property [26];
(e') $Y = L_p(0,1)$ for $1 \leq p \leq 2$ [26];
(f') $Y$ is strongly superprojective [22, 43].
(g') $Y$ is a Lorentz function space $\Lambda_{p,q}(0,1)$ with finite type for $2 < p < \infty$, or $L_{p,q}(0,1)$ or $L_{p,q}(0,\infty)$ for $2 \leq p, q < \infty$ and $SC(X, \ell_2) = K(X,\ell_2)$ [25];
(h') $Y$ is an Orlicz function space $L_\varphi(0,1)$ with $E^\infty_\varphi \equiv \{ t^p \}$ for $2 < p < \infty$ and $SC(X, \ell_2) = K(X,\ell_2)$ [25].

Part (a') in Theorem 5.3 is a consequence of the fact that if $X$ is superprojective, then $SC(X, Y) = Imp(X,Y)$ [7].

It is worth noting that Theorems 5.2 and 5.3 hold in two very different cases: on the one hand, for hereditarily indecomposable spaces and quotient indecomposable spaces, which only admit trivial projections; on the other hand, for (strongly) subprojective spaces and (strongly) superprojective spaces, which admit many projections.

There are many classical spaces that are strongly subprojective or strongly superprojective:
Proposition 5.4. The following Banach spaces are strongly subprojective:

(1) The sequence spaces \( \ell_p \) for \( 1 \leq p < \infty \), and \( c_0 \).

(2) The James space \( J \).

(3) The Lorentz sequence spaces \( d(w,p) \) for \( 1 \leq p < \infty \) and \( w = (w_n) \) a non-increasing null sequence with \( \sum_{n=1}^{\infty} w_n \) divergent. This applies to \( \ell_{p,q} \) for \( 1 \leq p,q < \infty \).

(4) The Baernstein spaces \( B_p \) for \( 1 < p < \infty \).

(5) The Tsirelson space \( T \).

(6) The function spaces \( L_p(0,1) \) for \( 2 \leq p < \infty \).

(7) The function spaces \( L_p(0,\infty) \cap L_2(0,\infty) \) for \( 1 \leq p \leq 2 \).

(8) The Lorentz spaces \( \Lambda_{W,p}(0,1) \), \( L_{p,q}(0,\infty) \) and \( L_{p,q}(0,1) \) for \( 2 < p < \infty \) and \( 1 \leq q < \infty \).

(9) The spaces of continuous functions \( C(K) \), with \( K \) a scattered compact.

(10) Closed subspaces of the previous examples.

We refer to a recent paper [36] for additional examples of (strongly) subprojective Banach spaces.

Recall that a compact space \( K \) is said to be scattered (or dispersed) if every non-empty subset of \( K \) has an isolated point. Examples of scattered compact spaces are the ordinal intervals \( [0, \kappa] = \{ \alpha \text{ ordinal} : 0 \leq \alpha \leq \kappa \} \), endowed with the order topology, and the one-point compactification \( \Gamma_{\infty} \) of any set \( \Gamma \) endowed with the discrete topology. Note that \( C(\Gamma_{\infty}) \) is isomorphic to \( c_0(\Gamma) \).

Proposition 5.5. The following Banach spaces are strongly superprojective:

(1') The sequence spaces \( \ell_p \) for \( 1 < p < \infty \), and \( c_0 \).

(2') The dual \( J^* \) of James space.

(3') The dual spaces \( d(w,p)^* \) of \( d(w,p) \) for \( 1 < p < \infty \) and \( w = (w_n) \) a non-increasing null sequence with \( \sum_{n=1}^{\infty} w_n \) divergent. This applies to \( \ell_{p,q}^* \) for \( 1 < p,q < \infty \).

(4') The dual spaces \( B_p^* \) of Baernstein's spaces for \( 1 < p < \infty \).

(5') The dual \( T^* \) of Tsirelson space.

(6') The function spaces \( L_p(0,1) \) for \( 1 < p \leq 2 \).

(7') The function spaces \( L_p(0,\infty) + L_2(0,\infty) \) for \( 2 \leq p < \infty \).

(8') The dual spaces \( \Lambda_{W,p}(0,1)^* \), \( L_{p,q}(0,\infty)^* \) and \( L_{p,q}(0,1)^* \) for \( 2 < p < \infty \) and \( 1 < q < \infty \).

(9') The spaces of continuous functions \( C(K) \), with \( K \) a scattered compact.

(10') Quotients of the previous examples.

Propositions 5.4 and 5.5 were proved in [22]. It is unknown if there are subprojective spaces which are not strongly subprojective, or superprojective spaces which are not strongly superprojective.
5.2. When $P \Phi(X, Y) = \mathcal{I}n(X, Y)$ admits an intrinsic characterization

Here we describe some pairs of spaces for which $\mathcal{I}n(X, Y)$ coincides with $\mathcal{I}mp(X, Y)$, $SS(X, Y)$ or $SC(X, Y)$.

**Proposition 5.6.** [5, 44]

(a) If $X$ or $Y$ is subprojective or superprojective, then $\mathcal{I}n(X, Y) = \mathcal{I}mp(X, Y)$.
(b) If $Y$ is subprojective, then $SS(X, Y) = \mathcal{I}n(X, Y) = \mathcal{I}mp(X, Y)$.
(c) If $X$ is superprojective, then $SC(X, Y) = \mathcal{I}n(X, Y) = \mathcal{I}mp(X, Y)$.

After Theorem 4.13, it is natural to ask for which compacts $K$ the set $\mathcal{I}mp(C(K))$ coincides with $SS(C(K))$ or $\mathcal{I}n(C(K))$.

**Proposition 5.7.** [8] The identity $\mathcal{I}n(C(K)) = \mathcal{I}mp(C(K))$ holds in the following cases:

(i) $K$ is scattered,
(ii) every separable subspace of $C(K)$ is contained in a separable complemented subspace,
(iii) every non-weakly compact operator acting on $C(K)$ is an isomorphism on some subspace isomorphic to $\ell_1$.

An additional condition improves Proposition 5.7. Let us recall that a compact set $K$ is a Valdivia compact if it contains a dense subset $A$ for which there exists a set $\Gamma$ and a homeomorphism $h: K \rightarrow \mathbb{R}^\Gamma$ such that $h(A)$ is the set of elements in $h(K)$ with countable support. Eberlein, Gulko, and Corson compact sets are examples of Valdivia compacts [29].

**Proposition 5.8.** [8] The identity $SS(C(K)) = \mathcal{I}mp(C(K))$ holds when $K$ is a continuous image of a Valdivia compact.

6. Some open problems

All known counterexamples to the perturbation classes problem for semi-Fredholm operators involve hereditarily indecomposable Banach spaces, whose properties are quite exotic.

**Problem 1.** Find examples of classical Banach spaces $X$ and $Y$ such that $P \Phi_+(X, Y) \neq SS(X, Y)$ or $P \Phi_-(X, Y) \neq SC(X, Y)$.

We can consider candidates by lifting some of the restrictions from the results shown in Subsection 5.1.

**Problem 2.** Suppose that $1 < p < 2$ and $Y$ is a Banach space containing a subspace isomorphic to $L_p(0, 1)$. Is it true that $P \Phi_+(L_p(0, 1), Y) \neq SS(L_p(0, 1), Y)$?

We showed in Theorem 5.2 that the answer is positive when $Y$ satisfies the Orlicz property. We can also consider the following dual problem, whose answer is again known true only under some restrictions.

**Problem 3.** Suppose that $2 < p < \infty$ and $X$ is a Banach space admitting a quotient isomorphic to $L_p(0, 1)$. Is it true that $P \Phi_-(X, L_p(0, 1)) \neq SS(X, L_p(0, 1))$?

All the examples of spaces $Z$ for which $P \Phi_+(Z) \neq SS(Z)$ have the form $Z = X \times Y$ with $X$ and $Y$ infinite-dimensional. However, it was proved in [21] that a hereditarily indecomposable space $Z_T$ constructed in [46] satisfies $P \Phi_+(Z_T) \neq SC(Z_T)$.
Problem 4. Does there exist an indecomposable space $Z$ for which $P\Phi_+(Z) \neq SS(Z)$?

Note that $Z$ in the previous problem cannot be hereditarily indecomposable (see (b) in Theorem 5.2).

Finally, we know the answer to the following problem in some special cases [8], but this could be improved.

Problem 5. Characterize those compact spaces $K$ for which $\mathcal{I}(C(K)) = \mathcal{I}(\text{Imp}(C(K)))$.

7. Final remarks

The term semigroup in reference to the classes $\Phi_+, \Phi_-$ and $\Phi$ was first used by Lebow and Schechter [34]. Later, the term operator semigroup was extended by Aiena et al. [9] to any class of operators closed under composition and cartesian product of operators, and containing all bijective isomorphisms.

Weis proved that $P\Phi_+(X, Y) = SS(X, Y)$ when $X = Y = L_p(0,1)$ for $1 \leq p < 2$ [48]. This result is a consequence of a more general result included in Theorem 5.2, and it implies that $P\Phi_-(X, Y) = SC(X, Y)$ when $X = Y = L_p(0,1)$ for $2 < p < \infty$. We refer to [43] for a recent exposition of the properties of $P\Phi_+$ and $P\Phi_-$.

In [16], Friedman introduced a certain condition $(C)$ and showed that

$$T \in SS \Rightarrow T \text{ satisfies (C)} \Rightarrow T \in P\Phi_+.$$  

In [10] it was shown that $T \in P\Phi_+$ does not imply property $(C)$, leaving it open whether the converse of the other implication is valid.

We have considered the perturbation classes problems for bounded operators. The corresponding problems for closed semi-Fredholm and Fredholm operators with dense range were studied by Weis in [49]. We observe that the problems for bounded operators are different from their counterparts for closed operators. Indeed, we showed in Proposition 4.8 that there exists a separable reflexive space $Z$ for which $SS(Z) \neq P\Phi_+(Z)$ and $SC(Z^*) \neq P\Phi_-(Z^*)$. However, it was proved in [49] that, for a separable space, the perturbation classes for closed upper and lower semi-Fredholm operators with dense range coincide respectively with the strictly singular and the strictly cosingular operators.

We refer to [3] for additional information.

References
