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Some properties and results involving the zeta and associated functions

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Abstract. In this research-cum-expository article, we aim at presenting a systematic account of some recent developments involving the Riemann Zeta function $\zeta(s)$, the Hurwitz (or generalized) Zeta function $\zeta(s,a)$, and the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ as well as its various interesting extensions and generalizations. In particular, we begin by looking into the problems associated with the evaluations and representations of $\zeta(s)$ when $s \in \mathbb{N} \setminus \{1\}$, $\mathbb{N}$ being the set of natural numbers, emphasizing upon various potentially useful and computationally friendly classes of rapidly convergent series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ which have been developed in recent years. We then turn toward some other investigations involving certain general classes of Goldbach-Euler type sums. Finally, we present a systematic investigation of various properties and results involving several families of generating functions and their partial sums which are associated with the aforementioned general classes of the extended Hurwitz-Lerch Zeta functions. References to some of latest developments in the theory and applications of several families of the extended Hurwitz-Lerch zeta functions are also provided for the interested researchers on these and other related topics in Analytic Number Theory, Geometric Function Theory of Complex Analysis, and so on.

1. Introduction, Definitions and Preliminaries

Throughout this article, we use the following standard notations:

$$\mathbb{N} := \{1,2,3,\cdots\}, \quad \mathbb{N}_0 := \{0,1,2,3,\cdots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1,-2,-3,\cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

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Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of *positive* numbers and $\mathbb{C}$ denotes the set of complex numbers.

Some rather important and potentially useful functions in *Analytic Number Theory* include (for example) the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s,a)$, which are defined (for $\Re(s) > 1$) by

$$\zeta(s) := \begin{cases} \sum\limits_{n=1}^{\infty} \dfrac{1}{n^s} = \dfrac{1}{1-2^{-s}} \sum\limits_{n=1}^{\infty} \dfrac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\[2ex] \dfrac{1}{1-2^{1-s}} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big) \end{cases} \tag{1.1}$$

and

$$\begin{aligned} \zeta(s,a) &:= \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \qquad \big(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\big) \\ &= \zeta(s, n+a) + \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} \qquad (n \in \mathbb{N}), \end{aligned} \tag{1.2}$$

and $\big($for $\Re(s) \leqq 1;\ s \neq 1\big)$ by their *meromorphic* continuations (see, for details, the excellent works by Titchmarsh [104] and Apostol [6] *as well as* the monumental treatise by Whittaker and Watson [107]; see also [1, Chapter 23] and [83, Chapter 2]), so that (obviously)

$$\zeta(s,1) = \zeta(s) = (2^s-1)^{-1}\ \zeta\left(s, \frac{1}{2}\right) \qquad \text{and} \qquad \zeta(s,2) = \zeta(s) - 1. \tag{1.3}$$

Indeed, in many different ways, both of the Zeta functions $\zeta(s)$ and $\zeta(s,a)$ can be continued *meromorphically* to the whole complex s-plane except for a simple pole at $s = 1$ with their respective residues 1.

The following simple relationships between the Zeta functions $\zeta(s)$ and $\zeta(s,a)$ are worthy of note:

$$\zeta(s) = \frac{1}{m^s - 1} \sum_{j=1}^{m-1} \zeta\left(s, \frac{j}{m}\right) \qquad (m \in \mathbb{N} \setminus \{1\}) \tag{1.4}$$

and

$$\zeta(s, ma) = \frac{1}{m^s} \sum_{j=0}^{m-1} \zeta\left(s, a + \frac{j}{m}\right) \qquad (m \in \mathbb{N}). \tag{1.5}$$

A classical about three-century-old theorem of Christian Goldbach (1690–1764) was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700–1782). **Goldbach's Theorem** was revived and revisited recently in many publications as the following problem:

$$\sum_{\omega \in \mathcal{S}} \frac{1}{\omega - 1} = 1, \tag{1.6}$$

where $\mathcal{S}$ denotes the set of all nontrivial integer kth powers, that is,

$$\mathcal{S} := \left\{ n^k : n,\ k \in \mathbb{N} \setminus \{1\} = \{2,\ 3,\ 4, \cdots\} \right\}.$$

In fact, in terms of the Riemann Zeta function $\zeta(s)$ defined by (1.1), Goldbach's theorem (1.6) can easily be seen to assume the following elegant form:

$$\sum_{\omega\in\mathcal{S}} \frac{1}{\omega-1} = \sum_{k=2}^{\infty} [\zeta(k)-1] = 1. \tag{1.7}$$

Since $\zeta(s)$ is a decreasing function of its argument s for $s \geqq 2$, we have

$$1 < \zeta(n) \leqq \zeta(2) = \frac{\pi^2}{6} < 2, \tag{1.8}$$

the above alternative form (1.7) of Goldbach's Theorem (1.6) can also be rewritten as follows:

$$\sum_{k=2}^{\infty} \mathfrak{f}\big(\zeta(k)\big) = 1,$$

where

$$\mathfrak{f}(x) := x - [x] = \text{The } \textit{fractional} \text{ part of } x \in \mathbb{R}.$$

As a matter of fact, it is fairly straightforward to show also that

$$\sum_{k=2}^{\infty} (-1)^k \, \mathfrak{f}\big(\zeta(k)\big) = \frac{1}{2}, \qquad \sum_{k=1}^{\infty} \mathfrak{f}\big(\zeta(2k)\big) = \frac{3}{4} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \mathfrak{f}\big(\zeta(2k+1)\big) = \frac{1}{4}.$$

It may be of interest to remark in passing that the name of Christian Goldbach (1690–1764) is usually associated with a relatively more popular conjecture dated 1742 (known as **Goldbach's Conjecture**) that *every positive integer greater than* 2 *is the sum of two prime numbers*:

$$4 = 2+2 = 1+3; \quad 6 = 3+3 = 1+5; \quad 8 = 1+7 = 3+5; \quad \textit{et cetera}.$$

Just as the celebrated **Riemann Hypothesis** dated 1859 that *all nontrivial zeros of* $\zeta(s)$ *lie on the critical line*:

$$\Re(s) = \frac{1}{2},$$

Goldbach's conjecture has not been proven as yet. Interestingly, not too long ago in the year 2001, on the occasion of the publication of the following ("very funny, tender, charming, and irresistible") novel:

Uncle Pedros and Goldbach's Conjecture: ***A Novel of Mathematical Obsession*** (by Apostolos Doxiadis), Faber and Faber, London, 2001,

the British publisher (Faber and Faber) had offered a reward of one million U.K. Pounds to anyone who can prove Goldbach's Conjecture.

Another result that has attracted fascinatingly and tantalizingly large number of seemingly independent solutions is the so-called **Basler Problem**:

$$\zeta(2) = \frac{\pi^2}{6}, \tag{1.9}$$

which was used above in (1.8). It was of vital importance to Leonhard Euler (1707-1783) and the Bernoulli brothers [Jakob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748)]. All these remarkably many essentially independent solutions of the **Basler Problem** (1.9) have appeared in the mathematical literature ever since Euler *first* solved this problem in the year 1736. In this context, one other remarkable *classical* result involving Riemann's Zeta function $\zeta(s)$ is the following elegant series representation for $\zeta(3)$:

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)\cdot 2^{2k}}, \tag{1.10}$$

which was actually contained in Euler's 1772 paper entitled "*Exercitationes Analyticae*" (*cf.*, *e.g.*, Ayoub [7, pp. 1084–1085]). Remarkably, this 1772 result (1.10) of Euler was rediscovered (among others) by Ramaswami [64] (see also a paper by Srivastava [71, p. 7, Equation (2.23)]) and (more recently) by Ewell [26]. Moreover, just as pointed out by (for example) Chen and Srivastava [14, pp. 180–181], another series representation:

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}, \tag{1.11}$$

which played a key rôle in the *celebrated* proof (see, for details, [5]) of the *irrationality* of $\zeta(3)$ by Roger Apéry (1916-1994), was derived *independently* by (among others) Hjortnaes [40], Gosper [34], and Apéry [5] himself.

It is easily observed that Euler's series in (1.10) converges faster than the defining series for $\zeta(3)$, but obviously not as fast as the series in (1.11). Evaluations of such Zeta values as $\zeta(3)$, $\zeta(5)$, *et cetera* are known to arise naturally in a wide variety of applications such as those in Elastostatics, Quantum Field Theory, *et cetera* (see, for example, Tricomi [105], Witten [109], and Nash and O'Connor [57], [58]). On the other hand, in the case of *even* integer arguments, we already have the following computationally useful relationship:

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2\cdot(2n)!} B_{2n} \qquad (n \in \mathbb{N}_0) \tag{1.12}$$

with the *well-tabulated* Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi), \tag{1.13}$$

as well as the following familiar recursion formula:

$$\zeta(2n) = \left(n + \frac{1}{2}\right)^{-1} \sum_{k=1}^{n-1} \zeta(2k)\ \zeta(2n-2k) \qquad (n \in \mathbb{N} \setminus \{1\}), \tag{1.14}$$

which, in terms of the Bernoulli numbers B_n, can be rewritten at once as follows:

$$B_{2n} = -\frac{1}{2n+1} \sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k} \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{1.15}$$

This research-cum-expository article is motivated largely by several recent works by Srivastava *et al.* (see, for example, [19], [80] and [81]). It consists of four major parts. In the first part, a genuine need (for computational purposes) for expressing $\zeta(2n+1)$ as a rapidly converging series for all $n \in \mathbb{N}$ has been shown to lead naturally to a rather systematic investigation of the various interesting families of rapidly convergent series representations for the Riemann $\zeta\,(2n+1)$ $(n \in \mathbb{N})$. Relevant connections of the results presented here with many other known series representations for $\zeta\,(2n+1)$ $(n \in \mathbb{N})$ are also briefly indicated. In fact, for two of the many computationally useful special cases considered here, it has been observed that $\zeta\,(3)$ can be represented by means of series which converge much more rapidly than that in Euler's celebrated formula (1.10) as well as that in the series (1.11) which (just as we indicated above) was used earlier by Apéry [5] in his celebrated proof of the *irrationality* of $\zeta\,(3)$. Symbolic and numerical computations using *Mathematica* (Version 4.0) for Linux have shown, among other things, that only 50 terms of one of these series are capable of producing an accuracy of seven decimal places. In the second part of this article, we consider a variety of series and integrals associated with the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ as well as its various interesting extensions and generalizations (see Section 6). In our next two sections (Section 7 and Section 8), we present a systematic account of some recent developments involving certain general classes of Goldbach-Euler type sums as well as various properties and results involving several families of generating functions and their partial sums which are associated with the aforementioned general classes of the extended Hurwitz-Lerch Zeta functions which we introduced in Section 6. Finally, in our last section (Section 9), we choose to present several further closely-related remarks and observations about the developments considered in Section 6 and Section 8 in addition (for example) to the *Open Problem* mentioned in Section 7.

2. Rapidly Converging Series for $\zeta\,(2n+1)$ $(n \in \mathbb{N})$

Suppose, as usual, that $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

which is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\[2mm] \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [72] and [83]). In terms of the above-defined Pochhammer symbol $(\lambda)_\nu$, we begin this section by recalling the following simple consequence of the binomial theorem *and* the definition (1.1):

$$\sum_{k=0}^{\infty} \frac{(s)_k}{k!}\ \zeta\,(s+k,a)\,t^k = \zeta\,(s,a-t) \qquad (|t| < |a|)\,, \tag{2.1}$$

which, for $a = 1$ and $t = \pm 1/m$, yields a useful series identity given by

$$\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!}\ \frac{\zeta\,(s+2k)}{m^{2k}}$$

$$= \begin{cases} (2^s-1)\ \zeta\,(s) - 2^{s-1} & (m = 2) \\[2mm] \dfrac{1}{2}\left[(m^s-1)\ \zeta\,(s) - m^s - \displaystyle\sum_{j=2}^{m-2} \zeta\left(s, \frac{j}{m}\right)\right] & (m \in \mathbb{N} \setminus \{1,2\})\,. \end{cases} \tag{2.2}$$

By making use of the familiar harmonic numbers H_n given by

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \qquad (n \in \mathbb{N}), \tag{2.3}$$

the following set of series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ were proven by Srivastava [75] by appealing appropriately to the series identity (2.2) in its special cases when $m = 2, 3, 4$, and 6, and also to many other properties and characteristics of the Riemann Zeta function such as the familiar functional equation:

$$\zeta(s) = 2 \cdot (2\pi)^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s)\, \zeta(1-s) \tag{2.4}$$

or, equivalently,

$$\zeta(1-s) = 2 \cdot (2\pi)^{-s} \cos\left(\frac{1}{2}\pi s\right) \Gamma(s)\, \zeta(s), \tag{2.5}$$

the familiar derivative formula:

$$\begin{aligned} \zeta'(-2n) &= \lim_{\varepsilon \to 0} \left\{ \frac{\zeta(-2n+\varepsilon)}{\varepsilon} \right\} \\ &= \frac{(-1)^n}{2 \cdot (2\pi)^{2n}} (2n)!\, \zeta(2n+1) \qquad (n \in \mathbb{N}) \end{aligned} \tag{2.6}$$

with, of course,

$$\zeta(0) = -\frac{1}{2}; \qquad \zeta(-2n) = 0 \qquad (n \in \mathbb{N}); \qquad \zeta'(0) = -\frac{1}{2} \log(2\pi), \tag{2.7}$$

and each of the following limit relationships:

$$\lim_{s \to -2n} \left\{ \frac{\sin\left(\frac{1}{2}\pi s\right)}{s+2n} \right\} = (-1)^n \frac{\pi}{2} \qquad (n \in \mathbb{N}) \tag{2.8}$$

and

$$\begin{aligned} &\lim_{s \to -2n} \left\{ \frac{\zeta(s+2k)}{s+2n} \right\} = \frac{(-1)^{n-k}}{2 \cdot (2\pi)^{2(n-k)}} (2n-2k)!\, \zeta(2n-2k+1) \\ &(k = 1, \ldots, n-1;\ n \in \mathbb{N} \setminus \{1\}). \end{aligned} \tag{2.9}$$

Series Representation of the First Kind:

$$\begin{aligned} \zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2^{2n+1}-1} &\left[\frac{H_{2n} - \log \pi}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right. \\ &\left. + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \right] \qquad (n \in \mathbb{N}). \end{aligned} \tag{2.10}$$

Series Representation of the Second Kind:

$$\begin{aligned} \zeta(2n+1) = (-1)^{n-1} \frac{2 \cdot (2\pi)^{2n}}{3^{2n+1}-1} &\left[\frac{H_{2n} - \log\left(\frac{2}{3}\pi\right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}} \right. \\ &\left. + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{3^{2k}} \right] \qquad (n \in \mathbb{N}). \end{aligned} \tag{2.11}$$

Series Representation of the Third Kind:

$$\zeta(2n+1) = (-1)^{n-1} \frac{2 \cdot (2\pi)^{2n}}{2^{4n+1} + 2^{2n} - 1} \left[\frac{H_{2n} - \log\left(\frac{1}{2}\pi\right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} + 2\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{4^{2k}} \right] \qquad (n \in \mathbb{N}). \tag{2.12}$$

Series Representation of the Fourth Kind:

$$\zeta(2n+1) = (-1)^{n-1} \frac{2 \cdot (2\pi)^{2n}}{3^{2n}(2^{2n}+1) + 2^{2n} - 1} \left[\frac{H_{2n} - \log\left(\frac{1}{3}\pi\right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{\left(\frac{1}{3}\pi\right)^{2k}} + 2\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \qquad (n \in \mathbb{N}). \tag{2.13}$$

Here, *as well as elsewhere in this presentation*, an empty sum is understood (as usual) to be zero.

The *first* series representation (2.10) is markedly different from each of the series representations for $\zeta(2n+1)$, which were given earlier by Zhang and Williams [112, p. 1590, Equation (3.13)] and (subsequently) by Cvijović and Klinowski [21, p. 1265, Theorem A] (see also [113] and [114]). Since $\zeta(2k) \to 1$ as $k \to \infty$, the general term in the series representation (2.10) has the following *order estimate*:

$$O\left(2^{-2k} \cdot k^{-2n-1}\right) \qquad (k \to \infty;\ n \in \mathbb{N}),$$

whereas the general term in each of the aforecited *earlier* series representations has the order estimate given below:

$$O\left(2^{-2k} \cdot k^{-2n}\right) \qquad (k \to \infty;\ n \in \mathbb{N}).$$

By suitably combining (2.10) and (2.12), we easily arrive at the following series representation:

$$\zeta(2n+1) = (-1)^{n-1} \frac{2 \cdot (2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} \left[\frac{\log 2}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k \left(2^{2k}-1\right)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} - 2\sum_{k=1}^{\infty} \frac{(2k-1)!\left(2^{2k}-1\right)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \qquad (n \in \mathbb{N}). \tag{2.14}$$

Furthermore, in terms of the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ defined by the generating functions (1.9) and

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \qquad (|z| < \pi), \tag{2.15}$$

respectively, it is known that (*cf.*, *e.g.*, [54, p. 29])

$$E_n(0) = (-1)^n E_n(1) = \frac{2\left(1-2^{n+1}\right)}{n+1} B_{n+1} \qquad (n \in \mathbb{N}). \tag{2.16}$$

Thus, by combining (2.16) with the identity (1.12), we find that

$$E_{2n-1}(0) = \frac{4 \cdot (-1)^n}{(2\pi)^{2n}} (2n-1)!\left(2^{2n}-1\right) \zeta(2n) \qquad (n \in \mathbb{N}). \tag{2.17}$$

If we apply the relationship (2.17), the series representation (2.14) can immediately be put in the following *alternative* form:

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1} \frac{2\cdot(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} &\left[\frac{\log 2}{(2n)!}\right. \\ &+\sum_{k=1}^{n-1} \frac{(-1)^k\left(2^{2k}-1\right)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ &\left.+\frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2n+2k)!} \left(\frac{\pi}{2}\right)^{2k} E_{2k-1}(0)\right] \qquad (n\in\mathbb{N}), \end{aligned} \tag{2.18}$$

which is a slightly *modified* and *corrected* version of a result proven, using a significantly different technique, by Tsumura [106, p. 383, Theorem B].

One other interesting combination of the series representations (2.10) and (2.12) leads us to the following variant of Tsumura's result (2.14) or (2.18):

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1} \frac{\pi^{2n}}{2^{2n+1}-1} &\left[\frac{H_{2n}-\log\left(\frac{1}{4}\pi\right)}{(2n)!}\right. \\ &+\sum_{k=1}^{n-1} \frac{(-1)^k\left(2^{2k+1}-1\right)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ &\left.-4\sum_{k=1}^{\infty} \frac{(2k-1)!\left(2^{2k-1}-1\right)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}}\right] \qquad (n\in\mathbb{N}), \end{aligned} \tag{2.19}$$

which is essentially the same as the *determinantal* expression for $\zeta(2n+1)$ derived by Ewell [27, p. 1010, Corollary 3] by employing an entirely different technique from ours.

Numerous other similar combinations of the series representations (2.10) to (2.13) would yield some interesting companions of Ewell's result (2.19).

Next, by setting $t = 1/m$ and differentiating both sides with respect to s, we find from the following obvious consequence of the series identity (2.1):

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!} \zeta(s+2k+1,a)\; t^{2k+1} \\ &\qquad = \frac{1}{2}\left[\zeta(s,a-t)-\zeta(s,a+t)\right] \qquad (|t|<|a|)\end{aligned} \tag{2.20}$$

that

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!\, m^{2k}} \left[\zeta'(s+2k+1,a)+\zeta(s+2k+1,a)\sum_{j=0}^{2k}\frac{1}{s+j}\right] \\ &\qquad = \frac{m}{2}\frac{\partial}{\partial s}\left\{\zeta\left(s,a-\frac{1}{m}\right)-\zeta\left(s,a+\frac{1}{m}\right)\right\} \qquad (m\in\mathbb{N}\setminus\{1\}).\end{aligned} \tag{2.21}$$

In the particular case when $m = 2$, (2.21) immediately yields

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!\, 2^{2k}} \left[\zeta'(s+2k+1,a)+\zeta(s+2k+1,a)\sum_{j=0}^{2k}\frac{1}{s+j}\right] \\ &\qquad = -\left(a-\frac{1}{2}\right)^{-s} \log\left(a-\frac{1}{2}\right).\end{aligned} \tag{2.22}$$

Upon letting $s \to -2n-1$ $(n \in \mathbb{N})$ in the *further* special of this last identity (2.22) when $a = 1$, Wilton [83, p. 92] deduced the following series representation for $\zeta(2n+1)$ (see also [39, p. 357, Entry (54.6.9)]):

$$\zeta(2n+1) = (-1)^{n-1} \ \pi^{2n} \left[\frac{H_{2n+1} - \log \pi}{(2n+1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \ \frac{\zeta(2k+1)}{\pi^{2k}} \right.$$
$$\left. +2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \ \frac{\zeta(2k)}{2^{2k}} \right] \qquad (n \in \mathbb{N}), \tag{2.23}$$

which, in light of the elementary identity:

$$\frac{(2k)!}{(2n+2k)!} = \frac{(2k-1)!}{(2n+2k-1)!} - 2n \, \frac{(2k-1)!}{(2n+2k)!} \qquad (n \in \mathbb{N}), \tag{2.24}$$

would combine with the result (2.10) to yield the following series representation:

$$\zeta(2n+1) = (-1)^n \, \frac{(2\pi)^{2n}}{n\,(2^{2n+1}-1)} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1}\,k}{(2n-2k)!} \ \frac{\zeta(2k+1)}{\pi^{2k}} \right.$$
$$\left. + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \ \frac{\zeta(2k)}{2^{2k}} \right] \qquad (n \in \mathbb{N}). \tag{2.25}$$

This last series representation (2.25) is precisely the aforementioned *main* result of Cvijović and Klinowski [21, p. 1265, Theorem A]. As a matter of fact, in view of a known derivative formula [75, p. 389, Equation (2.8)], the series representation (2.25) is essentially the same as a result given earlier by Zhang and Williams [112, p. 1590, Equation (3.13)] (see also Zhang and Williams [112, p. 1591, Equation (3.16)] where an obviously more complicated (*asymptotic*) version of (2.25) was proven similarly).

By making use of another elementary identity:

$$\frac{(2k)!}{(2n+2k+1)!} = \frac{(2k-1)!}{(2n+2k)!} - (2n+1) \, \frac{(2k-1)!}{(2n+2k+1)!} \qquad (n, k \in \mathbb{N}), \tag{2.26}$$

we can obtain the following yet another series representation for $\zeta(2n+1)$ by applying (2.10) and (2.23):

$$\zeta(2n+1) = (-1)^n \, \frac{2 \cdot (2\pi)^{2n}}{(2n-1)\,2^{2n}+1} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1}\,k}{(2n-2k+1)!} \ \frac{\zeta(2k+1)}{\pi^{2k}} \right.$$
$$\left. + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+1)!} \ \frac{\zeta(2k)}{2^{2k}} \right] \qquad (n \in \mathbb{N}), \tag{2.27}$$

which provides a *significantly* simpler (and *much more* rapidly convergent) version of the following other *main* result of Cvijović and Klinowski [21, p. 1265, Theorem B]:

$$\zeta(2n+1) = (-1)^n \, \frac{2 \cdot (2\pi)^{2n}}{(2n)!} \sum_{k=0}^{\infty} \Omega_{n,k} \ \frac{\zeta(2k)}{2^{2k}} \qquad (n \in \mathbb{N}), \tag{2.28}$$

where the coefficients $\Omega_{n,k}$ $(n \in \mathbb{N};\ k \in \mathbb{N}_0)$ are given explicitly as a finite sum of Bernoulli numbers [21, p. 1265, Theorem B(i)] (see, for details, Srivastava [75, pp. 393-394]):

$$\Omega_{n,k} := \sum_{j=0}^{2n} \binom{2n}{j} \frac{B_{2n-j}}{(j+2k+1)\,(j+1)\,2^j} \qquad (n \in \mathbb{N};\ k \in \mathbb{N}_0). \tag{2.29}$$

3. Further Classes of Rapidly Convergent Series for $\zeta(2n+1)$ $(n \in \mathbb{N})$

We begin this section once again from the identity (2.1) with (of course) $a = 1$, $t = \pm 1/m$, and s replaced by $s+1$. Thus, by applying (2.2), we find yet another class of series identities including, for example,

$$\sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \, \frac{\zeta(s+2k)}{2^{2k}} = (2^s - 2) \, \zeta(s) \tag{3.1}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \, \frac{\zeta(s+2k)}{m^{2k}} &= \frac{1}{2m} \left[m\,(m^s - 3)\,\zeta(s) + \left(m^{s+1} - 1\right)\zeta(s+1) - 2\zeta\left(s+1, \frac{1}{m}\right) \right. \\ &\qquad \left. - \sum_{j=2}^{m-2} \left\{ m\zeta\left(s, \frac{j}{m}\right) + \zeta\left(s+1, \frac{j}{m}\right) \right\} \right] \qquad (m \in \mathbb{N} \setminus \{1, 2\}) \,.\end{aligned} \tag{3.2}$$

In fact, it is the series identity (3.1) which was first applied by Zhang and Williams [112] (and, subsequently, by Cvijović and Klinowski [21]) with a view to proving two (only seemingly different) versions of the series representation (2.25). Indeed, if we appeal to (3.2) *with* $m = 4$, we can derive the following much more rapidly convergent series representation for $\zeta(2n+1)$ (see [74, p. 9, Equation (41)]):

$$\begin{aligned}\zeta(2n+1) &= (-1)^n \, \frac{2 \cdot (2\pi)^{2n}}{n\,(2^{4n+1} + 2^{2n} - 1)} \left[\frac{4^{n-1} - 1}{(2n)!} \, B_{2n} \log 2 \right. \\ &\quad - \frac{2^{2n-1} - 1}{2\,(2n-1)!} \, \zeta'(1-2n) - \frac{4^{2n-1}}{(2n-1)!} \, \zeta'\left(1-2n, \frac{1}{4}\right) \\ &\quad \left. + \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k)!} \, \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \, \frac{\zeta(2k)}{4^{2k}} \right] \qquad (n \in \mathbb{N}) \,,\end{aligned} \tag{3.3}$$

where (*and in what follows*) a prime denotes the derivative of $\zeta(s)$ or $\zeta(s,a)$ with respect to s.

By means of the identities (2.24) and (2.26), the results (2.12) and (3.3) would lead us eventually to the following *additional* series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ (see [74, p. 10, Equations (42) and (43)]):

$$\begin{aligned}\zeta(2n+1) &= (-1)^{n-1} \left(\frac{\pi}{2}\right)^{2n} \left[\frac{H_{2n+1} - \log\left(\frac{1}{2}\pi\right)}{(2n+1)!} + \frac{2\,(4^n - 1)}{(2n+2)!} B_{2n+2} \log 2 \right. \\ &\quad - \frac{2^{2n+1} - 1}{(2n+1)!} \, \zeta'(-2n-1) - \frac{2^{4n+3}}{(2n+1)!} \, \zeta'\left(-2n-1, \frac{1}{4}\right) \\ &\quad \left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \, \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \, \frac{\zeta(2k)}{4^{2k}} \right] \qquad (n \in \mathbb{N})\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\zeta(2n+1) = (-1)^n \frac{4\cdot(2\pi)^{2n}}{n\cdot 4^{2n+1}-2^{2n}+1}\Bigg[&\frac{2^{2n+1}-1}{2\cdot(2n)!}\,\zeta'(-2n-1)\\ &+\frac{4^{2n+1}}{(2n)!}\,\zeta'\left(-2n-1,\frac{1}{4}\right)-\frac{(2n+1)(4^n-1)}{(2n+2)!}\,B_{2n+2}\log 2\\ &+\sum_{k=1}^{n-1}\frac{(-1)^{k-1}k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}}+\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{4^{2k}}\Bigg] \qquad (n\in\mathbb{N}).\end{aligned} \tag{3.5}$$

Explicit expressions for the derivatives $\zeta'(-2n\pm 1)$ and $\zeta'\left(-2n\pm 1,\frac{1}{4}\right)$, which are involved in the series representations (3.3), (3.4) and (3.5), can be found and substituted into these results in order to represent $\zeta(2n+1)$ in terms of Bernoulli numbers and polynomials and various rapidly convergent series of the ζ-functions (see, for details, the work by Srivastava [74, Section 3]).

Out of the four seemingly analogous results (2.12), (3.3), (3.4) and (3.5), the infinite series in (3.4) would obviously converge most rapidly, with its general term having the order estimate:

$$O\left(k^{-2n-2}\cdot 4^{-2k}\right) \qquad (k\to\infty;\ n\in\mathbb{N}).$$

Now, from the work by Srivastava and Tsumura [100], we recall the following three *new* members of the class of the series representations (2.12) and (3.4):

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\left(\frac{2\pi}{3}\right)^{2n}\Bigg[&\frac{H_{2n+1}-\log\left(\frac{2}{3}\pi\right)}{(2n+1)!}+\frac{\left(3^{2n+2}-1\right)\pi}{2\sqrt{3}\,(2n+2)!}\,B_{2n+2}\\ &+\frac{(-1)^{n-1}}{\sqrt{3}\,(2\pi)^{2n+1}}\,\zeta\left(2n+2,\frac{1}{3}\right)\\ &+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{3^{2k}}\Bigg] \qquad (n\in\mathbb{N}),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\left(\frac{\pi}{2}\right)^{2n}\Bigg[&\frac{H_{2n+1}-\log\left(\frac{1}{2}\pi\right)}{(2n+1)!}+\frac{2^{2n}\left(2^{2n+2}-1\right)\pi}{(2n+2)!}\,B_{2n+2}\\ &+\frac{(-1)^{n-1}}{2\cdot(2\pi)^{2n+1}}\,\zeta\left(2n+2,\frac{1}{4}\right)\\ &+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{4^{2k}}\Bigg] \qquad (n\in\mathbb{N}),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\left(\frac{\pi}{3}\right)^{2n}\Bigg[&\frac{H_{2n+1}-\log\left(\frac{1}{3}\pi\right)}{(2n+1)!}+\frac{2^{2n}\left(3^{2n+2}-1\right)\pi}{\sqrt{3}\,(2n+2)!}\,B_{2n+2}\\ &+\frac{(-1)^{n-1}}{2\sqrt{3}\,(2\pi)^{2n+1}}\left\{\zeta\left(2n+2,\frac{1}{3}\right)+\zeta\left(2n+2,\frac{1}{6}\right)\right\}\\ &+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\left(\frac{1}{3}\pi\right)^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{6^{2k}}\Bigg] \qquad (n\in\mathbb{N}).\end{aligned} \tag{3.8}$$

The general terms of the infinite series occurring in these three members (3.6), (3.7) and (3.8) have the following order estimates:

$$O\left(k^{-2n-2} \cdot m^{-2k}\right) \qquad (k \to \infty;\ n \in \mathbb{N};\ m = 3, 4, 6), \tag{3.9}$$

which exhibit the fact that *each* of these last three series representations (3.6), (3.7), and (3.8) converges more rapidly than Wilton's result (2.23) and two of them [*cf.* Equations (3.7) and (3.8)] at least as rapidly as Srivastava's result (3.4).

We next recall that, in their aforementioned work on the Ray-Singer torsion and topological field theories, Nash and O'Connor ([57] and [58]) obtained a number of remarkable integral expressions for $\zeta(3)$, including (for example) the following result [26, p. 1489 *et seq.*]:

$$\zeta(3) = \frac{2\pi^2}{7} \log 2 - \frac{8}{7} \int_0^{\pi/2} z^2 \cot z \; dz. \tag{3.10}$$

In fact, by virtue of the following series expansion [24, p. 51, Equation 1.20(3)]:

$$z \cot z = -2 \sum_{k=0}^{\infty} \zeta(2k) \left(\frac{z}{\pi}\right)^{2k} \qquad (|z| < \pi), \tag{3.11}$$

the result (3.10) equivalent to the series representation (*cf.* the work by Dąbrowski [23, p. 202]; see also the paper by Chen and Srivastava [14, p. 191, Equation (3.19)]):

$$\zeta(3) = \frac{2\pi^2}{7} \left(\log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1)\, 2^{2k}} \right). \tag{3.12}$$

Moreover, if we choose to integrate by parts, we easily find that

$$\int_0^{\pi/2} z^2 \cot z \; dz = -2 \int_0^{\pi/2} z \log \sin z \; dz, \tag{3.13}$$

so that the result (3.10) is equivalent *also* to the following integral representation:

$$\zeta(3) = \frac{2\pi^2}{7} \log 2 + \frac{16}{7} \int_0^{\pi/2} z \log \sin z \; dz, \tag{3.14}$$

which was proven in the aforementioned 1772 paper by Euler (*cf.*, *e.g.*, [7, p. 1084]).

Furthermore, since

$$i \cot iz = \coth z = \frac{2}{e^{2z} - 1} + 1 \qquad \left(i := \sqrt{-1}\right), \tag{3.15}$$

by replacing z in the known expansion (3.11) by $\frac{1}{2} i\pi z$, it is easily seen that (*cf.*, *e.g.*, [27, p. 25]; see also [24, p. 51, Equation 1.20(1)])

$$\frac{\pi z}{e^{\pi z} - 1} + \frac{\pi z}{2} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \zeta(2k)}{2^{2k-1}} z^{2k} \qquad (|z| < 2). \tag{3.16}$$

Upon setting $z = it$ in (3.16), multiplying both sides by t^{m-1} $(m \in \mathbb{N})$, and then integrating the resulting equation from $t = 0$ to $t = \tau$ $(0 < \tau < 2)$, Srivastava [54] derived the following series representations for $\zeta(2n+1)$ (see also the work by Srivastava *et al.* [91]):

$$\begin{aligned} \zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!\,(2^{2n+1} - 1)} \\ \cdot \left[\log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n}{2j} \frac{(2j)!\,(2^{2j} - 1)}{(2\pi)^{2j}} \zeta(2j+1) + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+n)\, 2^{2k}} \right] \qquad (n \in \mathbb{N}) \end{aligned} \tag{3.17}$$

and

$$\zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n+1)!\,(2^{2n}-1)}$$
$$\cdot \left[\log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n+1}{2j} \frac{(2j)!\,\left(2^{2j}-1\right)}{(2\pi)^{2j}}\, \zeta(2j+1) + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{\left(k+n+\frac{1}{2}\right) 2^{2k}}\right] \qquad (n \in \mathbb{N}). \tag{3.18}$$

Upon setting $n = 1$, (3.18) immediately reduces to the following series representation for $\zeta(3)$:

$$\zeta(3) = \frac{2\pi^2}{9} \left(\log 2 + 2 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+3)\, 2^{2k}}\right), \tag{3.19}$$

which was proven *independently* by (among others) Glasser [33, p. 446, Equation (12)], Zhang and Williams [112, p. 1585, Equation (2.13)], and Dąbrowski [23, p. 206] (see also the work by Chen and Srivastava [14, p. 183, Equation (2.15)]). Furthermore, a special case of (3.17) when $n = 1$ yields (*cf.* Dąbrowski [23, p. 202]; see also Chen and Srivastava [14, 5, p. 191, Equation (3.19)])

$$\zeta(3) = \frac{2\pi^2}{7} \left(\log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1)\, 2^{2k}}\right). \tag{3.20}$$

In fact, in view of the following familiar sum:

$$\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)\, 2^{2k}} = -\frac{1}{2} \log 2, \tag{3.21}$$

Euler's formula (1.6) is indeed a rather *simple* consequence of (3.20).

In passing, we find it worthwhile to remark that an integral representation for $\zeta(2n+1)$, which is easily seen to be equivalent to the series representation (3.17), was given by Dąbrowski [23, p. 203, Equation (16)], who [23, p. 206] mentioned the existence of (but did not fully state) the series representation (3.18) as well. The series representation (3.17) was derived also in a paper by Borwein *et al.* (*cf.* [11, p. 269, Equation (57)]).

By suitably combining the series occurring in (3.12), (3.19) and (3.21), it is not difficult to deduce several other series representations for $\zeta(3)$, which are analogous to Euler's formula (1.6). More generally, since

$$\frac{\lambda k^2 + \mu k + \nu}{(2k+2n-1)\,(2k+2n)\,(2k+2n+1)} = \frac{\mathcal{A}}{2k+2n-1} + \frac{\mathcal{B}}{2k+2n} + \frac{\mathcal{C}}{2k+2n+1}, \tag{3.22}$$

where, for convenience,

$$\mathcal{A} = \mathcal{A}_n(\lambda, \mu, \nu) := \frac{1}{2}\left[\lambda n^2 - (\lambda + \mu)\, n + \frac{1}{4}(\lambda + 2\mu + 4\nu)\right], \tag{3.23}$$

$$\mathcal{B} = \mathcal{B}_n(\lambda, \mu, \nu) := -\left(\lambda n^2 - \mu n + \nu\right) \tag{3.24}$$

and

$$\mathcal{C} = \mathcal{C}_n(\lambda, \mu, \nu) := \frac{1}{2}\left[\lambda n^2 + (\lambda - \mu)\, n + \frac{1}{4}(\lambda - 2\mu + 4\nu)\right]. \tag{3.25}$$

By applying (3.17), (3.18), and another result (proven by Srivastava [76, p. 341, Equation (3.17)]):

$$\sum_{j=1}^{n} (-1)^{j-1} \binom{2n+1}{2j} \frac{(2j)!\left(2^{2j}-1\right)}{(2\pi)^{2j}}\, \zeta(2j+1)$$
$$= \log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{\left(k+n+\frac{1}{2}\right) 2^{2k}} \qquad (n \in \mathbb{N}_0), \tag{3.26}$$

with n replaced by $n-1$, Srivastava [76] derived the following unification of a large number of known (or new) series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$, including (for example) Euler's formula (1.6):

$$\zeta(2n+1) = \frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)!\left\{\left(2^{2n+1}-1\right)\mathcal{B} + (2n+1)\left(2^{2n}-1\right)\mathcal{C}\right\}}$$
$$\cdot \left[\frac{1}{4}\lambda \log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-2} \left\{2j(2j-1)\mathcal{A} + [\lambda(4n-1) - 2\mu]\, nj + \lambda n \left(n+\frac{1}{2}\right)\right\}\right.$$
$$\cdot \frac{(2j-2)!\left(2^{2j}-1\right)}{(2\pi)^{2j}}\, \zeta(2j+1)$$
$$\left. + \sum_{k=0}^{\infty} \frac{\left(\lambda k^2 + \mu k + \nu\right)\, \zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)\, 2^{2k}}\right] \qquad (n \in \mathbb{N};\ \lambda, \mu, \nu \in \mathbb{C}), \tag{3.27}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are given by (3.23), (3.24) and (3.25), respectively.

Numerous other interesting series representations for $\zeta(2n+1)$, which are analogous to (3.17) and (3.18), were also given by Srivastava *et al.* [91].

4. Illustrative Examples and Computationally Useful Deductions

In this section, we suitably specialize the parameter λ, μ, and ν in (3.27) and then apply a rather elaborate scheme. We thus eventually arrive at the following remarkably rapidly convergent series representation for $\zeta(2n+1)$ $(n \in \mathbb{N})$, which was derived by Srivastava [76, pp. 348–349, Equation (3.50)]):

$$\zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!\Delta_n} \left[\sum_{j=1}^{n-1} (-1)^j\right.$$
$$\cdot \left(\left\{(2n-3)\, 2^{2n+2} - 2n\right\} \left\{\binom{2n-1}{2j} - \binom{2n+2}{2j} + 6n\binom{2n-1}{2j-2}\right\} - \left(2^{2n+3}-1\right)\right.$$
$$\left. \cdot \left\{\binom{2n}{2j} - \binom{2n+3}{2j} + 3\binom{2n+1}{2j-1}\right\}\right) \frac{(2j)!\left(2^{2j}-1\right)}{(2\pi)^{2j}}\, \zeta(2j+1)\, 4.1$$
$$\left. + 12 \sum_{k=0}^{\infty} \frac{(\xi_n k + \eta_n)\, \zeta(2k)}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)\, 2^{2k}}\right] \qquad (n \in \mathbb{N}), \tag{4.1}$$

where, for convenience,

$$\Delta_n := \left(2^{2n+3}-1\right)\left\{\frac{1}{3}(2n+1)\left(2n^2-4n+3\right)\left(2^{2n}-1\right) - 2^{2n+1} + 1\right\}$$
$$- \left\{(2n-3)\, 2^{2n+2} - 2n\right\}\left\{2^{2n+2} + n(2n-3)\left(2^{2n}-1\right) - 1\right\}, \tag{4.2}$$

$$\xi_n := 2\left\{(2n-5)\,2^{2n+2} - 2n + 1\right\} \tag{4.3}$$

and

$$\eta_n := \left(4n^2 - 4n - 7\right)2^{2n+2} - (2n+1)^2. \tag{4.4}$$

In its special case when $n = 1$, the result (4.1) yields the following (*rather curious*) series representation:

$$\zeta(3) = -\frac{6\pi^2}{23}\sum_{k=0}^{\infty}\frac{(98k+121)\ \zeta(2k)}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)\,2^{2k}}, \tag{4.5}$$

where the series obviously converges much more rapidly than that in *each* of the *celebrated* results (1.6) and (1.7).

An interesting companion of (4.5) in the following form:

$$\zeta(3) = -\frac{120}{1573}\pi^2\sum_{k=0}^{\infty}\frac{8576k^2+24286k+17283}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)(2k+6)(2k+7)}\,\frac{\zeta(2k)}{2^{2k}}. \tag{4.6}$$

was deduced by Srivastava and Tsumura [103], who indeed presented an *inductive* construction of several general series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ (see also [101]).

5. Symbolic Computations and Numerical Verification: Use of *Mathematica* (Version 4.0)

Here, in this section, we first summarize the results of numerical verification and symbolic computations with the series in (4.5) by using *Mathematica* (Version 4.0) for Linux:

$$\text{In}[1] := (98k+121)\ \text{Zeta}[2k] \,/\, \big((2k+1)(2k+2)(2k+3)(2k+4)(2k+5)\,2^{\urcorner}\,(2k)\big)$$

$$\text{Out}[1] = \frac{(121+98k)\,\text{Zeta}\,[2k]}{2^{2k}\,(1+2k)(2+2k)(3+2k)(4+2k)(5+2k)}$$

$$\text{In}[2] := \text{Sum}[\%,\ \{k, 1,\ \text{Infinity}\}]\ //\ \text{Simplify}$$

$$\text{Out}[2] = \frac{121}{240} - \frac{23\ \text{Zeta}[3]}{6\text{Pi}^2}$$

$$\text{In}[3] := \text{N}[\%]$$

$$\text{Out}[3] = 0.0372903$$

$$\text{In}[4] := \text{Sum}[\text{N}\,[\%1]\ //\ \text{Evaluate},\ \{k, 1, 50\}]$$

$$\text{Out}[4] = 0.0372903$$

$$\text{In}[5] := \text{N Sum}[\%1\ //\ \text{Evaluate},\ \{k, 1,\ \text{Infinity}\}]$$

$$\text{Out}[5] = 0.0372903$$

Since

$$\zeta(0) = -\frac{1}{2},$$

Out[2] evidently validates the series representation (4.5) *symbolically.* Furthermore, our *numerical* computations in Out[3], Out[4], and Out[5], together, exhibit the fact that only 50 terms ($k = 1$ to $k = 50$) of the series in (4.5) can produce an accuracy of as many as *seven* decimal places.

Our symbolic computations and numerical verifications with the series in (4.6) using *Mathematica* (Version 4.0) for Linux lead us to the following table:

Number of Terms	Precision of Computation
4	6
10	11
20	18
50	38
98	69

As a matter of fact, since the general term of the series in (4.6) has the following order estimate:

$$O\left(2^{-2k} \cdot k^{-5}\right) \qquad (k \longrightarrow \infty),$$

for getting p *exact* digits, we must have

$$2^{-2k} \cdot k^{-5} < 10^{-p}.$$

Upon solving this inequality *symbolically*, we find that

$$k \cong \frac{5}{\log 4} \text{ProductLog}\left(\frac{10^{p/5} \log 4}{5}\right),$$

where the function ProductLog (also known as *Lambert's function*) is the solution of the equation:

$$xe^x = a.$$

Some relevant details about the symbolic computations and numerical verification with the series in (4.6) using *Mathematica* (Version 4.0) for Linux are being summarized below.

$$\text{In [1]:= expr} = \left(8576k\urcorner 2 + 24286k + 17283\right) \text{ Zeta}[2k] \,/$$

$$\left((2k+1)\,(2k+2)\,(2k+3)\,(2k+4)\,(2k+5)\,(2k+6)\,(2k+7)\,2\urcorner\,(2k)\right)$$

$$\text{Out [1]} = \frac{\left(17283 + 24286k + 8576k^2\right) \text{ Zeta}\,[2k]}{2^{2k}\,(1+2k)\,(2+2k)\,(3+2k)\,(4+2k)\,(5+2k)\,(6+2k)\,(7+2k)}$$

$$\text{In [2] := Sum[expr}, \{k, 0, \text{infinity}\}] \text{ // Simplify}$$

$$\text{Out [2]} = -\frac{1573}{120\text{Pi}^2} \text{ Zeta[3]}$$

$$\text{In [3] :=N}\Big[-1573/\left(120\text{Pi}\urcorner 2\right) \text{ Zeta[3], 50}\Big]$$

$$-\text{Sum[expr}, \{k, 0, 50\}]$$

$$\text{Out [3]} = 4.00751120011 \cdot 10^{-38}$$

In [4] :=N$\Big[$−1573/ $\big($120Pi^2$\big)$ Zeta[3], 100$\Big]$

−Sum [expr, $\{k, 0, 50\}$]

Out [4] = 4.0075112001 <skip> 3481 $\cdot\, 10^{-38}$

Thus, clearly, the result does not change appreciably when we increase the precision of computation of the symbolic result from 50 to 100. This is expected, because of the following numerical computation of the last term for $k = 50$:

In [5] := N [expr, $k \to 50, 50$]

Out [5] = 1.36085303749223768614438874545515142335757702860179 $\cdot\, 10^{-37}$

5. The Hurwitz-Lerch Zeta Function $\Phi(z, s, a)$: Extensions, Generalizations and Other Closely-Related Functions

Several potentially useful and computationally friendly foregoing developments (which we have attempted to present here in a rather concise form) have essentially motivated a large number of further investigations on the subject, not only involving the Riemann Zeta function $\zeta(s)$, the Hurwitz (or generalized) Zeta function $\zeta(s, a)$ (and their such relatives as the multiple Zeta functions and the multiple Gamma functions), as well as the Dirichlet L-functions $L(2n, \chi)$ and $L(2n+1, \chi)$ (see, for details, [102]), but indeed also the substantially general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (*cf.*, *e.g.*, [24, p. 27. Eq. 1.11 (1)]; see also [83, p. 121, *et seq.*])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{6.1}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1\right).$$

Just as in the cases of the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s, a)$, the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be continued *meromorphically* to the whole complex s-plane, except for a simple pole at $s = 1$ with its residue 1. It is also known that [24, p. 27, Equation 1.11 (3)]

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\, \mathrm{e}^{-at}}{1 - z\mathrm{e}^{-t}}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\, \mathrm{e}^{-(a-1)t}}{\mathrm{e}^t - z}\, dt \tag{6.2}$$

$$\big(\Re(a) > 0;\ \Re(s) > 0 \ \text{ when }\ |z| \leqq 1\ (z \neq 1);\ \Re(s) > 1 \ \text{ when }\ z = 1\big).$$

The Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (6.1) contains, as its *special* cases, not only the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s, a)$ [*cf.* Equations (1.1) and (1.2)]:

$$\zeta(s) = \Phi(1, s, 1) \qquad \text{and} \qquad \zeta(s, a) = \Phi(1, s, a) \tag{6.3}$$

and the Lerch Zeta function $\ell_s(\xi)$ defined by (see, for details, [24, Chapter I] and [83, Chapter 2])

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\ \Phi\left(e^{2\pi i\xi}, s, 1\right) \tag{6.4}$$

$$(\xi \in \mathbb{R};\ \Re(s) > 1),$$

but also such other important functions of *Analytic Function Theory* as the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\ \Phi(z, s, 1) \tag{6.5}$$

$$\big(s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\big)$$

and the Lipschitz-Lerch Zeta function (*cf.* [83, p. 122, Eq. 2.5 (11)]):

$$\phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi}, s, a\right) =: L(\xi, s, a) \tag{6.6}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}\big),$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions. For details, the interested reader should be referred, in connection with some of these developments, to the recent works including (among others) [4], [15] to [20], [31], [45], [46] and [53].

Yen *et al.* [111, p. 100, Theorem] derived the following sum-integral representation for the Hurwitz (or generalized) Zeta function $\zeta(s, a)$ defined by (1.2):

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \sum_{j=0}^{k-1} \int_0^\infty \frac{t^{s-1}\ \mathrm{e}^{-(a+j)t}}{1 - \mathrm{e}^{-kt}}\, dt \tag{6.7}$$

$$\big(k \in \mathbb{N};\ \Re(s) > 1;\ \Re(a) > 0\big),$$

which, for $k = 2$, was given earlier by Nishimoto *et al.* [60, p. 94, Theorem 4]. A straightforward generalization of the sum-integral representation (6.7) was given subsequently by Lin and Srivastava [52, p. 727, Eq. (7)] in the form:

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \sum_{j=0}^{k-1} z^j \int_0^\infty \frac{t^{s-1}\ \mathrm{e}^{-(a+j)t}}{1 - z^k \mathrm{e}^{-kt}}\, dt \tag{6.8}$$

$$\big(k \in \mathbb{N};\ \Re(a) > 0;\ \Re(s) > 0 \ \text{ when } \ |z| \leqq 1\ (z \neq 1);\ \Re(s) > 1 \ \text{ when } \ z = 1\big).$$

Motivated essentially by the sum-integral representations (6.7) and (6.8), a generalization of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ was introduced and investigated by Lin and Srivastava [52] in the following form [52, p. 727, Eq. (8)]:

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s} \tag{6.9}$$

$$\big(\mu \in \mathbb{C};\ a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \rho, \sigma \in \mathbb{R}^+;\ \rho < \sigma \ \text{ when } \ s, z \in \mathbb{C};$$

$$\rho = \sigma \ \text{ and } \ s \in \mathbb{C} \ \text{ when } \ |z| < \delta := \rho^{-\rho}\ \sigma^\sigma;\ \rho = \sigma \ \text{ and } \ \Re(s - \mu + \nu) > 1 \ \text{ when } \ |z| = \delta\big),$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined in conjunction with (2.1) and (2.2). Clearly, we find from the definition (6.9) that

$$\Phi_{\nu,\nu}^{(\sigma,\sigma)}(z, s, a) = \Phi_{\mu,\nu}^{(0,0)}(z, s, a) = \Phi(z, s, a) \tag{6.10}$$

and

$$\Phi_{\mu,1}^{(1,1)}(z,s,a)=\Phi_{\mu}^{*}(z,s,a):=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(n+a)^s} \tag{6.11}$$

$$\big(\mu\in\mathbb{C};\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C}\ \text{ when }\ |z|<1;\ \Re(s-\mu)>1\ \text{ when }\ |z|=1\big),$$

where, as already noted by Lin and Srivastava [52], $\Phi_{\mu}^{*}(z,s,a)$ is a generalization of the Hurwitz-Lerch Zeta function considered by Goyal and Laddha [35, p. 100, Equation (1.5)]. For further results involving these classes of generalized Hurwitz-Lerch Zeta functions, see the recent works by Garg *et al.* [31] and Lin *et al.* [53].

A generalization of the above-defined Hurwitz-Lerch Zeta functions $\Phi(z,s,a)$ and $\Phi_{\mu}^{*}(z,s,a)$ was studied by Garg *et al.* [30] in the following form [30, p. 313, Eq. (1.7)]:

$$\Phi_{\lambda,\mu;\nu}(z,s,a):=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n\cdot n!}\frac{z^n}{(n+a)^s} \tag{6.12}$$

$$\big(\lambda,\mu\in\mathbb{C};\ \nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C}\ \text{ when }\ |z|<1;\ \Re(s+\nu-\lambda-\mu)>1\ \text{ when }\ |z|=1\big).$$

By comparing the definitions (6.9) and (6.11), it is easily observed that the function $\Phi_{\lambda,\mu;\nu}(z,s,a)$ studied by Garg *et al.* [30] does *not* provide a generalization of the function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ which was introduced earlier by Lin and Srivastava [52]. Indeed, for $\lambda=1$, the function $\Phi_{\lambda,\mu;\nu}(z,s,a)$ coincides with a *special* case of the function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ when $\rho=\sigma=1$.

For the *Riemann-Liouville fractional derivative operator* $\mathcal{D}_z^{\mu}$ defined by (see, for example, [25, p. 181], [65] and [50, p. 70 *et seq.*])

$$\mathcal{D}_z^{\mu}\left\{f(z)\right\}:=\begin{cases}\dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^z (z-t)^{-\mu-1}\ f(t)\,dt & \big(\Re(\mu)<0\big)\\[2ex] \dfrac{d^m}{dz^m}\Big\{\mathcal{D}_z^{\mu-m}\left\{f(z)\right\}\Big\} & \big(m-1\leqq\Re(\mu)<m\ \ (m\in\mathbb{N})\big),\end{cases} \tag{6.13}$$

it is known that

$$\mathcal{D}_z^{\mu}\left\{z^{\lambda}\right\}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\ z^{\lambda-\mu}\qquad\big(\Re(\lambda)>-1\big), \tag{6.14}$$

which, in view of the definition (6.9), yields the following fractional derivative formula for the generalized Hurwitz-Lerch Zeta function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ *with* $\rho=\sigma$ [52, p. 730, Eq. (24)]:

$$\mathcal{D}_z^{\mu-\nu}\left\{z^{\mu-1}\ \Phi(z^{\sigma},s,a)\right\}=\frac{\Gamma(\mu)}{\Gamma(\nu)}\ z^{\nu-1}\ \Phi_{\mu,\nu}^{(\sigma,\sigma)}(z^{\sigma},s,a) \tag{6.15}$$

$$\big(\Re(\mu)>0;\ \sigma\in\mathbb{R}^+\big).$$

In particular, when $\nu=\sigma=1$, the fractional derivative formula (6.15) would reduce at once to the following form:

$$\Phi_{\mu}^{*}(z,s,a)=\frac{1}{\Gamma(\mu)}\ \mathcal{D}_z^{\mu-1}\left\{z^{\mu-1}\ \Phi(z,s,a)\right\}\qquad\big(\Re(\mu)>0\big), \tag{6.16}$$

which (as already remarked by Lin and Srivastava [52, p. 730]) exhibits the interesting (and useful) fact that $\Phi_{\mu}^{*}(z,s,a)$ is essentially a Riemann-Liouville fractional derivative of the classical

Hurwitz-Lerch function $\Phi(z,s,a)$. Moreover, it is easily deduced from the fractional derivative formula (6.14) that

$$\begin{aligned}\Phi_{\lambda,\mu;\nu}(z,s,a) &= \frac{\Gamma(\nu)}{\Gamma(\lambda)}\, z^{1-\lambda}\, \mathcal{D}_z^{\lambda-\nu}\left\{z^{\lambda-1}\, \Phi_\mu^*(z,s,a)\right\} \\ &= \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\, z^{1-\lambda}\, \mathcal{D}_z^{\lambda-\nu}\left\{z^{\lambda-1}\, \mathcal{D}_z^{\mu-1}\left\{z^{\mu-1}\, \Phi_\mu(z,s,a)\right\}\right\}, \qquad (6.17)\end{aligned}$$

which exhibits the *hitherto unnoticed* fact that the function $\Phi_{\lambda,\mu;\nu}(z,s,a)$ studied by Garg *et al.* [30] is essentially a consequence of the classical Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ when we apply the Riemann-Liouville fractional derivative operator $\mathcal{D}_z^\mu$ *two times* as indicated above (see also [98]). Many other explicit representations for $\Phi_\mu^*(z,s,a)$ and $\Phi_{\mu,\nu}^{(\rho;\sigma)}(z,s,a)$, including a potentially useful Eulerian integral representation of the first kind [52, p. 731, Eq. (28)], were proven by Lin and Srivastava [52].

A multiple (or, simply, n-dimentional) Hurwitz-Lerch Zeta function $\Phi_n(z,s,a)$ was studied recently by Choi *et al.* [16, p. 66, Eq. (6)]. Răducanu and Srivastava (see [62] and the references cited therein), on the other hand, made use of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ in defining a certain linear convolution operator in their systematic investigation of various analytic function classes in *Geometric Function Theory* in *Complex Analysis*. Furthermore, Gupta *et al.* [37] revisited the study of the familiar Hurwitz-Lerch Zeta distribution by investigating its structural properties, reliability properties and statistical inference. These investigations by Gupta *et al.* [37] and others (see, for example, [77], [83], [89] and [90]), fruitfully using the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ and some of its above-mentioned generalizations, motivated Srivastava *et al.* [98] to present a further generalization and analogous investigation of a new family of Hurwitz-Lerch Zeta functions defined in the following form [98, p. 491, Equation (1.20)]:

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}\cdot n!}\, \frac{z^n}{(n+a)^s} \qquad (6.18)$$

$$\begin{gathered}\big(\lambda,\mu\in\mathbb{C};\ a,\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \rho,\sigma,\kappa\in\mathbb{R}^+;\ \kappa-\rho-\sigma>-1\ \text{ when }\ s,z\in\mathbb{C}; \\ \kappa-\rho-\sigma=-1\ \text{ and }\ s\in\mathbb{C}\ \text{ when }\ |z|<\delta^*:=\rho^{-\rho}\,\sigma^{-\sigma}\,\kappa^\kappa; \\ \kappa-\rho-\sigma=-1\ \text{ and }\ \Re(s+\nu-\lambda-\mu)>1\ \text{ when }\ |z|=\delta^*\big).\end{gathered}$$

For the above-defined function in (6.18), Srivastava *et al.* [98] established various integral representations, relationships with the $\overline{H}$-function which is defined by means of a Mellin-Barnes type contour integral (see, for example, [95] and [98]), fractional derivative and analytic continuation formulas, as well as an extension of the generalized Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z,s,a)$ in (6.18). This natural further extension and generalization of the function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z,s,a)$ was indeed accomplished by introducing essentially arbitrary numbers of numerator and denominator parameters in the definition (6.18). For this purpose, in addition to the symbol ∇^* defined by

$$\nabla^* := \left(\prod_{j=1}^{p}\rho_j^{-\rho_j}\right)\cdot\left(\prod_{j=1}^{q}\sigma_j^{\sigma_j}\right), \qquad (6.19)$$

the following notations will be employed:

$$\Delta := \sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j \qquad \text{and} \qquad \Xi := s + \sum_{j=1}^{q}\mu_j - \sum_{j=1}^{p}\lambda_j + \frac{p-q}{2}. \qquad (6.20)$$

Then the extended Hurwitz-Lerch Zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)$$

is defined by [98, p. 503, Equation (6.2)] (see also [78])

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \frac{z^n}{(n+a)^s} \tag{6.21}$$

$$\Bigg(p, q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j = 1, \cdots, p);\ a, \mu_j \in \mathbb{C}\backslash Z_0^-\ (j = 1, \cdots, q);\ \rho_j, \sigma_k \in \mathbb{R}^+\ (j = 1, \cdots, p;\ k = 1, \cdots, q);$$
$$\Delta > -1 \text{ when } s, z \in \mathbb{C};\ \Delta = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \nabla^*;\ \Delta = -1 \text{ and } \Re(\Xi) > \frac{1}{2} \text{ when } |z| = \nabla^*\Bigg).$$

The special case of the definition (6.21) when $p - 1 = q = 1$ would obviously correspond to the above-investigated generalized Hurwitz-Lerch Zeta function $\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu;\nu}(z,s,a)$ defined by (6.18).

Remark 1. If we set

$$s = 0, \quad p \mapsto p + 1 \qquad (\rho_1 = \cdots = \rho_p = 1;\ \lambda_{p+1} = \rho_{p+1} = 1)$$

and

$$q \mapsto q + 1 \qquad (\sigma_1 = \cdots = \sigma_q = 1;\ \mu_{q+1} = \beta;\ \sigma_{q+1} = \alpha),$$

then (6.21) reduces to the following generalized M-series which was recently introduced by Sharma and Jain [70] (see also [66] as well as an earlier paper by Sharma [69] for the *special* case when $\beta = 1$):

$$\begin{aligned} {}_p\overset{\alpha,\beta}{M}_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) &:= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\alpha k + \beta)} \\ &= \frac{1}{\Gamma(\beta)}\ {}_{p+1}\Psi^*_{q+1}\left[\begin{array}{c} (a_1, 1), \cdots, (a_p, 1), (1, 1); \\ \\ (b_1, 1), \cdots, (b_q, 1), (\beta, \alpha); \end{array} z\right]. \end{aligned} \tag{6.22}$$

The last relationship in (6.22) exhibits the fact that the so-called generalized M-series is, in fact, an *obvious* variant of the Fox-Wright function ${}_p\Psi_q$ or ${}_p\Psi^*_q$ $(p, q \in \mathbb{N}_0)$, which is a generalization of the familiar generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ such that

$$a_j \in \mathbb{C}\ (j = 1, \cdots, p) \qquad \text{and} \qquad b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ (j = 1, \cdots, q),$$

defined by (see, for details, [24, p. 183] and [94, p. 21]; see also [50, p. 56], [55, p. 30] and [92, p. 19])

$$\begin{aligned} &{}_p\Psi^*_q\left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{array} z\right] \\ &\qquad := \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} \cdots (a_p)_{A_p n}}{(b_1)_{B_1 n} \cdots (b_q)_{B_q n}} \frac{z^n}{n!} \\ &\qquad = \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)}\ {}_p\Psi_q\left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{array} z\right] \end{aligned} \tag{6.23}$$

$$\left(A_j > 0 \quad (j = 1, \cdots, p);\ B_j > 0 \quad (j = 1, \cdots, q);\ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0\right),$$

where the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j}\right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j}\right).$$

In the particular case when

$$A_j = B_k = 1 \qquad (j = 1, \cdots, p;\ \ k = 1, \cdots, q),$$

we have the following relationship (see, for details, [94, p. 21]):

$$ {}_p\Psi_q^* \left[\begin{array}{c} (a_1, 1), \cdots, (a_p, 1); \\ \\ (b_1, 1), \cdots, (b_q, 1); \end{array} z \right] = {}_pF_q \left[\begin{array}{c} a_1, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{array} z \right] $$

$$ = \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)}\ {}_p\Psi_q \left[\begin{array}{c} (a_1, 1), \cdots, (a_p, 1); \\ \\ (b_1, 1), \cdots, (b_q, 1); \end{array} z \right], \tag{6.24}$$

in terms of the generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$. Similarly, for the generalized Mittag-Leffler function considered by Kilbas *et al.* [49], we have the following relationship:

$$E_\rho\left[(\beta_1, \eta_1), \cdots, (\beta_q, \eta_q); z\right] := \sum_{k=0}^{\infty} \frac{(\rho)_k}{\prod\limits_{j=1}^{q} \Gamma(\eta_j k + \beta_j)}$$

$$= \frac{1}{\Gamma(\rho)}\ {}_1\Psi_q \left[\begin{array}{c} (\rho, 1); \\ \\ (\beta_1, \eta_1), \cdots, (\beta_q, \eta_q); \end{array} z \right], \tag{6.25}$$

Next, in an attempt to derive Feynman integrals in two different ways, which arise in perturbation calculations of the equilibrium properties of a magnetic mode of phase transitions, led naturally to the following generalization of Fox's H-function [42, p. 4126] (see also [12] and [41]):

$$\overline{H}(z) = \overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n}\left[z \left| \begin{array}{c} (a_j, A_j; \alpha_j)_{j=1}^{n}, (a_j, A_j)_{j=n+1}^{p} \\ \\ (b_j, B_j)_{j=1}^{m}, (b_j, B_j; \beta_j)_{j=m+1}^{q} \end{array} \right. \right] := \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \chi(s) z^s \,\mathrm{d}s \tag{6.26}$$

$$\left(z \neq 0;\ \mathrm{i} = \sqrt{-1};\ \chi(s) := \frac{\prod\limits_{j=1}^{m} \Gamma(b_j - B_j s) \cdot \prod\limits_{j=1}^{n} \left\{\Gamma(1 - a_j + A_j s)\right\}^{\alpha_j}}{\prod\limits_{j=n+1}^{p} \Gamma(a_j - A_j s) \cdot \prod\limits_{j=m+1}^{q} \left\{\Gamma(1 - b_j + B_j s)\right\}^{\beta_j}} \right),$$

which contains *fractional* powers of some of the Gamma functions involved. Here, and in what follows, the parameters

$$A_j > 0 \quad (j = 1, \cdots, p) \qquad \text{and} \qquad B_j > 0 \ \ (j = 1, \cdots, q),$$

the exponents

$$\alpha_j \quad (j = 1, \cdots, n) \qquad \text{and} \qquad \beta_j \quad (j = m+1, \cdots, q)$$

can take on noninteger values, and $\mathfrak{L} = \mathfrak{L}_{(\mathrm{i}\tau;\infty)}$ is a Mellin-Barnes type contour starting at the point $\tau - \mathrm{i}\infty$ and terminating at the point $\tau + \mathrm{i}\infty$ $(\tau \in \mathbb{R})$ with the usual indentations to separate one set of poles from the other set of poles. The sufficient condition for the absolute convergence of the contour integral in (2.18) was established as follows by Buschman and Srivastava [12, p. 4708]:

$$\Lambda := \sum_{j=1}^{m} B_j + \sum_{j=1}^{n} |\alpha_j| A_j - \sum_{j=m+1}^{q} |\beta_j| B_j - \sum_{j=n+1}^{p} A_j > 0, \tag{6.27}$$

which provides exponential decay of the integrand in (6.26) and the region of absolute convergence of the contour integral in (6.26) is given by

$$|\arg(z)| < \frac{1}{2}\, \pi\Lambda,$$

where Λ is defined by (6.27).

Each of the following results involving the extended Hurwitz-Lerch Zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)$$

can be proven by applying the definition (6.21) in precisely the same manner as for the corresponding result involving the general Hurwitz-Lerch Zeta function $\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu;\nu}(z,s,a)$ (see, for details, [98, Section 6]).

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{e}^{-at}\; {}_p\Psi_q^* \left[\begin{array}{c} (\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); \\ \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); \end{array} z\mathrm{e}^{-t} \right] dt \tag{6.28}$$

$$\big(\min\{\Re(a),\Re(s)\} > 0\big),$$

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) = \frac{\prod_{j=1}^{q}\Gamma(\mu_j)}{\prod_{j=1}^{p}\Gamma(\lambda_j)} \cdot \frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{\Gamma(-\xi)\,\{\Gamma(\xi+a)\}^s \prod_{j=1}^{p}\Gamma(\lambda_j+\rho_j\xi)}{\{\Gamma(\xi+a+1)\}^s \prod_{j=1}^{q}\Gamma(\mu_j+\sigma_j\xi)} (-z)^\xi\, d\xi \tag{6.29}$$

$$\big(|\arg(-z)| < \pi\big)$$

or, equivalently,

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) = \frac{\prod_{j=1}^{q}\Gamma(\mu_j)}{\prod_{j=1}^{p}\Gamma(\lambda_j)}$$

$$\cdot\, \overline{H}^{1,p+1}_{p+1,q+2}\left[-z \left| \begin{array}{c} (1-\lambda_1,\rho_1;1),\cdots,(1-\lambda_p,\rho_p;1),(1-a,1;s) \\ \\ (0,1),(1-\mu_1,\sigma_1;1),\cdots,(1-\mu_q,\sigma_q;1),(-a,1;s) \end{array} \right. \right], \tag{6.30}$$

provided that both sides of the assertions (6.28), (6.29) and (6.30) exist, the path of integration $\mathfrak{L}$ in (6.30) being a Mellin-Barnes type contour in the complex ξ-plane, which, as in the definition (6.26), starts at the point $-i\infty$ and terminates at the point $i\infty$ with indentations, if necessary, in such a manner as to separate the poles of $\Gamma(-\xi)$ from the poles of $\Gamma(\lambda_j + \rho_j \xi)$ $(j = 1, \cdots, p)$.

The $\overline{H}$-function representation given by (6.27) can be applied in order to derive various properties of the extended Hurwitz-Lerch Zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)$$

from those of the $\overline{H}$-function (see, for details, Section 8). Thus, for example, by making use of the following fractional-calculus result due to Srivastava *et al.* [95, p. 97, Eq. (2.4)]:

$$\begin{aligned}
&\mathcal{D}_z^{\nu}\left\{z^{\lambda-1}\,\overline{H}^{m,n}_{p,q}(\omega z^{\kappa})\right\}\\
&\quad= z^{\lambda-\nu-1}\,\overline{H}^{m,n+1}_{p+1,q+1}\left[\omega z^{\kappa}\,\middle|\,\begin{matrix}(1-\lambda,\kappa;1),(a_j,A_j;\alpha_j)_{j=1}^{n},(a_j,A_j)_{j=n+1}^{p}\\[2mm](b_j,B_j)_{j=1}^{m},(b_j,B_j;\beta_j)_{j=m+1}^{q},(1-\lambda+\nu,\kappa;1)\end{matrix}\right] \qquad (6.31)\\
&\qquad\qquad\big(\Re(\lambda)>0;\ \kappa>0\big),
\end{aligned}$$

we readily obtain an extension of such fractional derivative formulas as (for example) (6.15) given by

$$\begin{aligned}
&\mathcal{D}_z^{\nu-\tau}\left\{z^{\nu-1}\,\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z^{\kappa},s,a)\right\} = \frac{\prod\limits_{j=1}^{q}\Gamma(\mu_j)}{\prod\limits_{j=1}^{p}\Gamma(\lambda_j)}\,z^{\tau-1}\\
&\qquad\cdot\,\overline{H}^{1,p+2}_{p+2,q+3}\left[-z^{\kappa}\,\middle|\,\begin{matrix}(1-\lambda_1,\rho_1;1),\cdots,(1-\lambda_p,\rho_p;1),(1-\nu,\kappa;1),(1-a,1;s)\\[2mm](0,1),(1-\mu_1,\sigma_1;1),\cdots,(1-\mu_q,\sigma_q;1),(1-\tau,\kappa;1),(-a,1;s)\end{matrix}\right]\\
&\quad=\frac{\Gamma(\nu)}{\Gamma(\tau)}\,z^{\tau-1}\,\Phi^{(\rho_1,\cdots,\rho_p,\kappa,\sigma_1,\cdots,\sigma_q,\kappa)}_{\lambda_1,\cdots,\lambda_p,\nu;\mu_1,\cdots,\mu_q,\tau}(z^{\kappa},s,a)\qquad\big(\Re(\nu)>0;\ \kappa>0\big). \qquad (6.32)
\end{aligned}$$

Finally, we present the following extension of a known result [98, p. 496, Theorem 3] (see also [98, p. 505, Theorem 9].

Theorem 1. *Let* $(\alpha_n)_{n\in\mathbb{N}_0}$ *be a positive sequence such that the following infinite series*:

$$\sum_{n=0}^{\infty} e^{-\alpha_n t}$$

converges for any $t \in \mathbb{R}^+$. *Then*

$$\begin{aligned}
\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) &= \frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1}\,e^{-(a-\alpha_0+\alpha_n)t}\left(1-e^{-(\alpha_{n+1}-\alpha_n)t}\right)\\
&\quad\cdot\,{}_p\Psi_q^{*}\left[\begin{matrix}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p);\\[2mm](\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q);\end{matrix}\ ze^{-t}\right]dt\qquad\big(\min\{\Re(a),\Re(s)\}>0\big), \qquad (6.33)
\end{aligned}$$

provided that each member of (6.30) *exists.*

It would seem to be interesting and worthwhile to be able to extend the results presented in Sections 2 to 5 of this article to hold true for the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ and for some of its generalizations given by (for example) the Lin-Srivastava Zeta function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z, s, a)$ and the extended Hurwitz-Lerch Zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z, s, a)$$

defined by (6.21) for special values of the varous parameters involved in the definitions (6.9) and (6.21). Section 8 of this article will be devoted to a systematic investigation of various properties and results involving several families of generating functions and their partial sums which are associated with the aforementioned general classes of the extended Hurwitz-Lerch Zeta functions.

6. General Families of the Goldbach-Euler Series

The following general family of the so-called Goldbach-Euler series has been widely investigated and recorded in the form (see [59, p. 59, Eq. (9)]; see also [36, p. 894, Entry 8.363 (7)] and [56, p. 88, Eq. (5)]),

$$\sum_{k=2}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(pn+r)^k-1}=\frac{1}{p}\left[\psi\left(\frac{r}{p}\right)-\psi\left(\frac{r-1}{p}\right)\right] \tag{7.1}$$

$$(p\in\mathbb{N};\ r=p\neq 1 \qquad \text{or} \qquad r=p+1),$$

where, in terms of the familiar (Euler's) Gamma function $\Gamma(z)$, the *Psi* (or *Digamma*) *function* $\psi(z)$ is defined (*as usual*) by

$$\psi(z):=\frac{d}{dz}\{\log\Gamma(z)\}=\frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\Gamma(z)=\int_1^z\psi(t)\,dt. \tag{7.2}$$

By recalling a familiar series representation of the Psi (or Digamma) function $\psi(z)$ defined by (7.2) as follows (see [83, p. 14, Eq. 1.2 (3)]):

$$\psi(z)=-\gamma-\frac{1}{z}+\sum_{n=1}^{\infty}\frac{z}{n(n+z)}=-\gamma-\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right), \tag{7.3}$$

where γ denotes the *Euler-Mascheroni constant* defined by (see, for details, [28, Section 1.5])

$$\begin{aligned}\gamma &:= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)=-\psi(1)\\ &\cong 0.57721\,56649\,01532\,8606\,06512\,09008\,24024\,31042\cdots.\end{aligned}$$

we can rewrite the cases

$$r=p\neq 1 \qquad \text{and} \qquad r=p+1 \qquad (p\in\mathbb{N})$$

of the generalized Goldbach-Euler series (7.1) in the following *separate* forms:

$$\sum_{k=2}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(pn)^k-1}=\frac{1}{p}\left[\psi(1)-\psi\left(1-\frac{1}{p}\right)\right] \qquad (p\in\mathbb{N}\setminus\{1\}) \tag{7.4}$$

and

$$\sum_{k=2}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(pn+1)^k-1}=1+\frac{1}{p}\left[\psi\left(\frac{1}{p}\right)-\psi\left(1\right)\right]\qquad (p\in\mathbb{N}), \tag{7.5}$$

where we have used the following well-known identity:

$$\psi(z+1)=\psi(z)+\frac{1}{z}. \tag{7.6}$$

Recently, Choi and Srivastava [19] made use of *Mathematica* (Version 6) to show that the special case of (for example) the generalized Goldbach-Euler series (7.5) when $p=1$ is recorded *erroneously* in [56, p. 88] (see also [59, p. 59, Eq. (10)]). First of all, it would be helpful to state the *simple* corrected form of (7.4) (see also Theorem 4 for other relevant details) as follows:

$$\begin{aligned}\sum_{k=2}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(pn)^k}=\sum_{n=1}^{\infty}\frac{1}{pn(pn-1)}&=\frac{1}{p^2}\sum_{n=1}^{\infty}\frac{1}{n\left(n-\frac{1}{p}\right)}\\&=\frac{1}{p}\left[\psi(1)-\psi\left(1-\frac{1}{p}\right)\right]\qquad (p\in\mathbb{N}\setminus\{1\};\ p\ \text{fixed}),\end{aligned} \tag{7.7}$$

where we have used the following known summation identity:

$$\sum_{n=1}^{\infty}\frac{1}{(n+\lambda)(n+\mu)}=\frac{\psi(\mu+1)-\psi(\lambda+1)}{\mu-\lambda} \tag{7.8}$$

for

$$\lambda=0\qquad\text{and}\qquad \mu=-\frac{1}{p}.$$

The duly-corrected forms of the generalized Goldbach-Euler series (7.4) *as well as* (7.5) are asserted by Theorem 2 below (see, for details, [19]).

Theorem 2. *Each of the following results holds true*:

$$\sum_{\omega\in\mathcal{S}_{p,0}}\frac{1}{\omega-1}=\frac{1}{p}\left[\psi\left(1\right)-\psi\left(1-\frac{1}{p}\right)\right]\qquad (p\in\mathbb{N}\setminus\{1\}) \tag{7.9}$$

and

$$\sum_{\omega\in\mathcal{S}_{p,1}}\frac{1}{\omega-1}=1+\frac{1}{p}\left[\psi\left(\frac{1}{p}\right)-\psi\left(1\right)\right]\qquad (p\in\mathbb{N}), \tag{7.10}$$

where the set $\mathcal{S}_{p,0}$ *is defined* (*for fixed* $p\in\mathbb{N}\setminus\{1\}$) *by*

$$\mathcal{S}_{p,0}:=\left\{(pn)^k : n\in\mathbb{N}\qquad \textit{and}\qquad k\in\mathbb{N}\setminus\{1\}\right\} \tag{7.11}$$

and the set $\mathcal{S}_{p,1}$ *is defined* (*for fixed* $p\in\mathbb{N}$) *by*

$$\mathcal{S}_{p,1}:=\left\{(pn+1)^k : n\in\mathbb{N}\qquad \textit{and}\qquad k\in\mathbb{N}\setminus\{1\}\right\}. \tag{7.12}$$

On the remarkably widely and extensively investigated subject of *closed-form* evaluation of series involving the Zeta functions, we may recall here the following sum:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(2k+1)\cdot 2^{2k}}=\log 2-\gamma, \tag{7.13}$$

which, as noted by Srivastava [71], is contained in a memoir of 1781 by Leonhard Euler (1707–1783) (see also [32, p. 28, Eq. (8)]; it was rederived by Wilton [108, p. 92]). A rather extensive collection of closed-form sums of series involving the Zeta functions was presented in [83] and [84].

Just as the Zeta-function series in (1.6), the series in (7.9) and (7.10) can be expressed as series involving the Riemann and Hurwitz (or generalized) Zeta functions. Theorem 3 below is intended to provide these *further* insights into (and the equivalences for) the assertions (7.9) and (7.10) of Theorem 2.

Theorem 3. *Each of the following results holds true*:

$$\sum_{\omega\in\mathcal{S}_{p,0}} \frac{1}{\omega-1} = \sum_{k=2}^{\infty} \frac{\zeta(k)}{p^k} = \frac{1}{p}\left[\psi(1) - \psi\left(1-\frac{1}{p}\right)\right] \qquad (p \in \mathbb{N}\setminus\{1\}) \tag{7.14}$$

and

$$\begin{aligned}\sum_{\omega\in\mathcal{S}_{p,1}} (\omega-1)^{-1} &= \sum_{k=2}^{\infty} \frac{1}{p^k}\,\zeta\left(k, 1+\frac{1}{p}\right) \\ &= 1 + \frac{1}{p}\left[\psi\left(\frac{1}{p}\right) - \psi(1)\right] \qquad (p\in\mathbb{N}),\end{aligned} \tag{7.15}$$

where the sets $\mathcal{S}_{p,0}$ *and* $\mathcal{S}_{p,1}$ *are defined by* (7.11) *and* (7.12), *respectively.*

Proof. For the sake of completeness, we choose the summarize the demonstration of Theorem 3 as follows. Indeed, if we let $\mathcal{T}_p$ denote the set of all p-multiple positive integers that are not in $\mathcal{S}_{p,0}$, then it is easily observed that that

$$\begin{aligned}\sum_{\omega\in\mathcal{S}_{p,0}} \frac{1}{\omega-1} &= \sum_{k=2}^{\infty}\sum_{a\in\mathcal{T}_p} \frac{1}{a^k-1} \\ &= \sum_{k=2}^{\infty}\sum_{a\in\mathcal{T}_p}\sum_{j=1}^{\infty} \frac{1}{a^{kj}} \\ &= \sum_{k=2}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(p\,n)^k} = \sum_{k=2}^{\infty} \frac{\zeta(k)}{p^k},\end{aligned}$$

which obviously proves the equivalence asserted in the result (7.14) of Theorem 3.

By following the same process as in the above-summarized demonstration of (7.14), we are led to the equivalence asserted in the result (7.15) of Theorem 3. □

Remark 2. In view of the identity in (1.2), the special case of (7.15) when $p = 1$ yields the *classical* about three-century-old Goldbach theorem (1.6).

Remark 3. Each of the series involving the Zeta function in (7.14) and (7.15) is an obvious special case of the following formula (see [83, p. 159, Eq. 3.4 (5)] and [84, p. 266, Eq. 3.4 (5)]):

$$\sum_{k=2}^{\infty} \zeta(k,a)\,t^{k-1} = \psi(a) - \psi(a-t) \qquad (|t| < |a|). \tag{7.16}$$

Remark 4. The double series occurring on the left-hand sides of (7.4) and (7.5) can be rewritten as the following series involving the Riemann and Hurwitz (or generalized) Zeta functions, respectively:

$$\begin{aligned}\sum_{k=2}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(pn)^k-1}&=\sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{1}{p^{kj}}\,\zeta(kj)\\&=\sum_{\omega\in\mathcal{S}_{p,0}}\frac{1}{\omega-1}+\sum_{k=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{p^{kj}}\,\zeta(kj)\qquad(p\in\mathbb{N}\setminus\{1\})\end{aligned}\tag{7.17}$$

and

$$\begin{aligned}\sum_{k=2}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(pn+1)^k-1}&=\sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{1}{p^{kj}}\,\zeta\left(kj,\,1+\frac{1}{p}\right)\\&=\sum_{\omega\in\mathcal{S}_{p,1}}\frac{1}{\omega-1}+\sum_{k=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{p^{kj}}\,\zeta\left(kj,\,1+\frac{1}{p}\right)\qquad(p\in\mathbb{N}),\end{aligned}\tag{7.18}$$

where the sets $\mathcal{S}_{p,0}$ and $\mathcal{S}_{p,1}$ are defined by (7.11) and (7.12), respectively.

We conclude this section by posing a natural question as the following *open* problem (see also [19]).

Open Problem. For each of the following double sums:

$$\sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{1}{p^{kj}}\,\zeta(kj)\qquad\text{and}\qquad\sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{1}{p^{kj}}\,\zeta\left(kj,\,1+\frac{1}{p}\right),$$

which occur as the second members of (7.17) and (7.18), respectively, find a closed-form evaluation or expression as in the known formula (7.16).

7. Further Results Involving the Families of the Extended Hurwitz-Lerch Zeta Functions

In this sequel to our presentation in Section 6, we make use of the Pochhammer symbol $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$, which is defined already in Section 2, in order to recall the following well-known companion of the expansion formula (2.1):

$$\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\,\Phi(z,s+n,a)t^n=\Phi(z,s,a-t)\qquad(|t|<|a|).\tag{8.1}$$

More generally, it is not difficult to show similarly that

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,\Phi(z,s+n,a)t^n=\sum_{k=0}^{\infty}\frac{z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}}=:\vartheta_\lambda(z,t;s,a)\qquad(|t|<|a|),\tag{8.2}$$

which would reduce immediately to the expansion formula (8.1) in its special case when $\lambda=s$. Moreover, in the limit case when

$$t\mapsto\frac{t}{\lambda}\qquad\text{and}\qquad|\lambda|\to\infty,$$

this last result (8.2) yields

$$\sum_{n=0}^{\infty} \Phi(z, s+n, a)\frac{t^n}{n!} = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}\ \exp\left(\frac{t}{k+a}\right) =: \varphi(z,t;s,a) \qquad (|t| < \infty). \tag{8.3}$$

Wilton [108] applied the expansion formula (2.1) in order to rederive Burnside's formula [24, p. 48, Equation 1.18 (11)] for the sum of a series involving the Hurwitz (or generalized) Zeta function $\zeta(s, a)$. Srivastava (see, for details, [71] and [72]), on the other hand, made use of such expansion formulas as (2.1) and (8.1) as well as the obvious special case of (2.1) when $a = 1$ for finding the sums of various classes of series involving the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s, a)$ (see also [83, Chapter 3] and [84, Chapter 3]).

Various results for the generating functions $\vartheta_\lambda(z, t; s, a)$ and $\varphi(z, t; s, a)$, which are defined by (8.2) and (8.3), respectively, were given recently by Bin-Saad [10, p. 46, Equations (5.1) to (5.4)] who also considered each of the following *truncated* forms of these generating functions:

$$\vartheta_\lambda^{(0,r)}(z,t;s,a) := \sum_{k=0}^{r} \frac{z^k}{(k+a)^{s-\lambda}(k+a-t)^\lambda} \qquad (r \in \mathbb{N}_0), \tag{8.4}$$

$$\vartheta_\lambda^{(r+1,\infty)}(z,t;s,a) := \sum_{k=r+1}^{\infty} \frac{z^k}{(k+a)^{s-\lambda}(k+a-t)^\lambda} \qquad (r \in \mathbb{N}_0), \tag{8.5}$$

$$\varphi^{(0,r)}(z,t;s,a) := \sum_{k=0}^{r} \frac{z^k}{(k+a)^s}\ \exp\left(\frac{t}{k+a}\right) \qquad (r \in \mathbb{N}_0) \tag{8.6}$$

and

$$\varphi^{(r+1,\infty)}(z,t;s,a) := \sum_{k=r+1}^{\infty} \frac{z^k}{(k+a)^s}\ \exp\left(\frac{t}{k+a}\right) \qquad (r \in \mathbb{N}_0), \tag{8.7}$$

so that, obviously, we find that

$$\vartheta_\lambda^{(0,r)}(z,t;s,a) + \vartheta_\lambda^{(r+1,\infty)}(z,t;s,a) = \vartheta_\lambda(z,t;s,a) \tag{8.8}$$

and

$$\varphi^{(0,r)}(z,t;s,a) + \varphi^{(r+1,\infty)}(z,t;s,a) = \varphi(z,t;s,a). \tag{8.9}$$

In the case of the Riemann Zeta function $\zeta(s)$, the special case of each of the generating functions $\vartheta_\lambda(z, t; s, a)$ and $\varphi(z, t; s, a)$ in (8.2) and (8.3) when $z = a = 1$ was investigated by Katsurada [48]. Subsequently, various results involving the generating functions $\vartheta_\lambda(z, t; s, a)$ and $\varphi(z, t; s, a)$ defined by (8.2) and (8.3), respectively, together with their such partial sums as those given by (8.4) to (8.7), were derived by Bin-Saad [10] (see also the more recent sequels to [10] and [48] by Gupta and Kumari [38] *and* by Saxena *et al.* [67]).

The main objective in this section is to investigate, in a rather systematic manner, much more general families of generating functions and their partial sums than those associated with the generating functions $\vartheta_\lambda(z, t; s, a)$ and $\varphi(z, t; s, a)$ defined by (8.2) and (8.3), respectively. We also observe the fact that the so-called τ-generalized Riemann Zeta function, which happens to be the main subject of investigation by Gupta and Kumari [38] and by Saxena *et al.* [67], is simply a seemingly trivial notational variation of the familiar general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (6.1).

We now introduce the following generating functions and their partial sums involving the extended Hurwitz-Lerch Zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s, a)$$

defined by (6.21). Indeed, as a generalization of the generating functions (2.1) and (8.1), we have (see also [81] and [82])

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s, a-t) = \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s+n, a)t^n \qquad (|t| < |a|), \tag{8.10}$$

which can easily be put in the following considerably more general form:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s+n, a)t^n = \sum_{k=0}^{\infty} \frac{\Xi_k\, z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}} =: \Omega_\lambda(z, t; s, a) \qquad (|t| < |a|), \tag{8.11}$$

where (*and in what follows*) the sequence $\{\Xi_n\}_{n\in\mathbb{N}_0}$ of the coefficients is given by

$$\Xi_n := \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \qquad (n \in \mathbb{N}_0). \tag{8.12}$$

The generating function (8.11) reduces immediately to the expansion formula (8.10) in its special case when $\lambda = s$. Moreover, in its limit case when

$$t \mapsto \frac{t}{\lambda} \qquad \text{and} \qquad |\lambda| \to \infty,$$

the generating function (8.11) yields

$$\sum_{n=0}^{\infty} \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s+n, a) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \frac{\Xi_k\, z^k}{(k+a)^s} \exp\left(\frac{t}{k+a}\right) =: \Theta(z, t; s, a) \qquad (|t| < \infty), \tag{8.13}$$

where the sequence $\{\Xi_n\}_{n\in\mathbb{N}_0}$ of the coefficients is given, as before, by (8.12).

We shall also consider each of the following *truncated* forms of the generating functions $\Omega_\lambda(z, t; s, a)$ and $\Theta(z, t; s, a)$ in (8.11) and (8.13), respectively:

$$\Omega^{(0,r)}_\lambda(z, t; s, a) := \sum_{k=0}^{r} \frac{\Xi_k\, z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}} \qquad (r \in \mathbb{N}_0), \tag{8.14}$$

$$\Omega_{\lambda}^{(r+1,\infty)}(z,t;s,a) := \sum_{k=r+1}^{\infty} \frac{\Xi_k \; z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}} \qquad (r \in \mathbb{N}_0), \tag{8.15}$$

$$\Theta^{(0,r)}(z,t;s,a) := \sum_{k=0}^{r} \frac{\Xi_k \; z^k}{(k+a)^s} \; \exp\left(\frac{t}{k+a}\right) \qquad (r \in \mathbb{N}_0) \tag{8.16}$$

and

$$\Theta^{(r+1,\infty)}(z,t;s,a) := \sum_{k=r+1}^{\infty} \frac{\Xi_k \; z^k}{(k+a)^s} \; \exp\left(\frac{t}{k+a}\right) \qquad (r \in \mathbb{N}_0), \tag{8.17}$$

which do obviously satisfy the following decomposition formulas:

$$\Omega_{\lambda}^{(0,r)}(z,t;s,a) + \Omega_{\lambda}^{(r+1,\infty)}(z,t;s,a) = \Omega_{\lambda}(z,t;s,a) \tag{8.18}$$

and

$$\Theta^{(0,r)}(z,t;s,a) + \Theta^{(r+1,\infty)}(z,t;s,a) = \Theta(z,t;s,a). \tag{8.19}$$

The first set of integral representations for the above-defined generating functions is contained in Theorem 4 below.

Theorem 4. *Each of the following integral representation formulas holds true*:

$$\Omega_{\lambda}(z,\omega;s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \, \mathrm{e}^{-at} \; {}_p\Psi_q^* \left[\begin{array}{c} (\lambda_1,\rho_1), \cdots, (\lambda_p,\rho_p); \\ \\ (\mu_1,\sigma_1), \cdots, (\mu_q,\sigma_q); \end{array} z\mathrm{e}^{-t} \right]$$

$$\cdot \; {}_1F_1\,(\lambda; s; \omega t)\,\mathrm{d}t \qquad \big(\min\{\Re(a), \Re(s)\} > 0\big) \tag{8.20}$$

and

$$\Theta(z,\omega;s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \, \mathrm{e}^{-at} \; {}_p\Psi_q^* \left[\begin{array}{c} (\lambda_1,\rho_1), \cdots, (\lambda_p,\rho_p); \\ \\ (\mu_1,\sigma_1), \cdots, (\mu_q,\sigma_q); \end{array} z\mathrm{e}^{-t} \right]$$

$$\cdot \; {}_0F_1\,(\underline{\quad\;\;}\,; s; \omega t)\,\mathrm{d}t \qquad \big(\min\{\Re(a), \Re(s)\} > 0\big), \tag{8.21}$$

provided that both sides of each of the assertions (8.20) *and* (8.21) *exist.*

Proof. We find it to be convenient to denote by $\mathfrak{S}$ the second member of the assertion (8.20) of Theorem 4. Then, upon expanding the functions ${}_p\Psi_q^*$ and ${}_1F_1$ in series forms, we find that

$$
\begin{aligned}
\mathfrak{S} &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{e}^{-at}\, {}_p\Psi_q^* \left[\begin{array}{c} (\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); \\ \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); \end{array} z\mathrm{e}^{-t} \right] {}_1F_1\,(\lambda;s;\omega t)\,\mathrm{d}t \\
&= \frac{1}{\Gamma(s)} \sum_{m,n=0}^{\infty} \Xi_m\, z^m\, \frac{(\lambda)_n}{(s)_n}\, \frac{\omega^n}{n!} \int_0^\infty t^{s+n-1}\, \mathrm{e}^{-(a+m)t}\,\mathrm{d}t,
\end{aligned}
\tag{8.22}
$$

where the inversion of the order of integration and double summation can easily be justified by absolute convergence under the conditions stated with (8.20), Ξ_n being defined by (8.12). If we now evaluate the innermost integral in (8.22) by appealing to the following well-known result:

$$
\int_0^\infty t^{\mu-1}\, \mathrm{e}^{-\kappa t}\,\mathrm{d}t = \frac{\Gamma(\mu)}{\kappa^\mu} \qquad \big(\min\{\Re(\kappa),\Re(\mu)\} > 0\big), \tag{8.23}
$$

we find that

$$
\mathfrak{S} = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\sum_{m=0}^{\infty} \Xi_m\, \frac{z^m}{(m+a)^{s+n}} \right) \omega^n \qquad \big(\min\{\Re(a),\Re(s)\} > 0\big), \tag{8.24}
$$

which, in light of the definitions in (6.21) and (8.11), yields the left-hand side of the first assertion (8.20) of Theorem 4.

The second assertion (8.21) of Theorem 4 can be proven in a similar manner. We choose to skip the details involved. $\square$

Remark 5. For $\omega = 0$, each of the assertions (8.20) and (8.21) of Theorem 4 yields a known integral representation formula due to Srivastava *et al.* [99, p. 504, Equation (6.4)]. Moreover, in their special case when

$$
\omega = 0 \qquad \text{and} \qquad \Xi_n = 1 \qquad (n \in \mathbb{N}_0),
$$

the assertions (8.20) and (8.21) of Theorem 4 would reduce immediately to the classical integral representation (6.2) for the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$.

The proof of Theorem 5 below would run parallel to that of Theorem 4, which we already have detailed above fairly adequately. It is based essentially upon the Hankel type contour integral in the following form [24, p. 14, Equation 1.6 (4)]:

$$
2\mathrm{i}\,\sin(\pi\nu)\Gamma(\nu) = -\int_\infty^{(0+)} (-t)^{\nu-1}\,\mathrm{e}^{-t}\,\mathrm{d}t \qquad \big(|\arg(-t)| \leqq \pi\big) \tag{8.25}
$$

or, equivalently,

$$
\frac{1}{\Gamma(1-\nu)} = -\frac{1}{2\pi\mathrm{i}} \int_\infty^{(0+)} (-t)^{\nu-1}\,\mathrm{e}^{-t}\,\mathrm{d}t \qquad \big(|\arg(-t)| \leqq \pi\big). \tag{8.26}
$$

Theorem 5. *Each of the following Hankel type contour integral representation formulas holds true*:

$$\Omega_\lambda(z,\omega;s,a) = -\frac{\Gamma(1-s)}{2\pi \mathrm{i}} \int_\infty^{(0+)} (-t)^{s-1}\, \mathrm{e}^{-at}\; {}_p\Psi_q^* \left[\begin{array}{c} (\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); \\ \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); \end{array} z\mathrm{e}^{-t} \right]$$

$$\cdot\, {}_1F_1\,(\lambda; s; \omega t)\, \mathrm{d}t \qquad \big(\Re(a) > 0;\ |\arg(-t)| \leqq \pi\big) \tag{8.27}$$

and

$$\Theta(z,\omega;s,a) = -\frac{\Gamma(1-s)}{2\pi \mathrm{i}} \int_\infty^{(0+)} (-t)^{s-1}\, \mathrm{e}^{-at}\; {}_p\Psi_q^* \left[\begin{array}{c} (\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); \\ \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); \end{array} z\mathrm{e}^{-t} \right]$$

$$\cdot\, {}_0F_1\,(\text{——}; s; \omega t)\, \mathrm{d}t \qquad \big(\Re(a) > 0;\ |\arg(-t)| \leqq \pi\big), \tag{8.28}$$

provided that both sides of each of the assertions (8.27) *and* (8.28) *exist.*

Remark 6. For $\omega = 0$, each of the assertions (8.27) and (8.28) of Theorem 5 yields the following (*presumably new*) integral representation formula (see also [81]):

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) = -\frac{\Gamma(1-s)}{2\pi \mathrm{i}} \int_\infty^{(0+)} (-t)^{s-1}\, \mathrm{e}^{-at}$$
$$\cdot\, {}_p\Psi_q^* \left[\begin{array}{c} (\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); \\ \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); \end{array} z\mathrm{e}^{-t} \right] \mathrm{d}t \qquad \big(\Re(a) > 0;\ |\arg(-t)| \leqq \pi\big). \tag{8.29}$$

Furthermore, in their special case when

$$\omega = 0 \qquad \text{and} \qquad \Xi_n = 1 \qquad (n \in \mathbb{N}_0),$$

the assertions (8.27) and (8.28) of Theorem 5 would reduce to the classical Hankel type contour integral representation for the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ (see, for example, [24, p. 28, Equation 1.11 (5)]; see also [84, p. 195, Equation 2.5 (8)]).

Next, by making use of the following known result (see, for example, [97, p. 86, Problem 1]):

$$\int_a^b (t-a)^{\alpha-1}\,(b-t)^{\beta-1}\,\mathrm{d}t = (b-a)^{\alpha+\beta-1}\,B(\alpha,\beta) \qquad \big(b \neq a;\ \min\{\Re(\alpha),\Re(\beta)\} > 0\big), \tag{8.30}$$

we evaluate several Eulerian Beta-function integrals involving the generating functions $\Omega_\lambda(z,t;s,a)$ and $\Theta(z,t;s,a)$ defined by (8.11) and (8.13), respectively, $B(\alpha,\beta)$ being the familiar Beta function.

Theorem 6. *In terms of the sequence* $\{\Xi_n\}_{n\in\mathbb{N}_0}$ *of the coefficients given by the definition* (8.12), *each of the following Eulerian Beta-function integral formulas holds true*:

$$\int_{\xi}^{\eta} (t-\xi)^{\alpha-1}\,(\eta-t)^{\beta-1}\,\Omega_\lambda\left(z,\omega(t-\xi)^{\gamma}\,(\eta-t)^{\delta};s,a\right)\mathrm{d}t$$
$$= (\eta-\xi)^{\alpha+\beta-1}\,B(\alpha,\beta)\sum_{n=0}^{\infty}\Xi_n\,\frac{z^n}{(n+a)^s}\;{}_3\Psi_1^*\left[\begin{array}{cc}(\lambda,1),(\alpha,\gamma),(\beta,\delta); & \dfrac{\omega(\eta-\xi)^{\gamma+\delta}}{n+a}\\ (\alpha+\beta,\gamma+\delta); & \end{array}\right] \tag{8.31}$$
$$\left(\eta\neq\xi;\ \min\{\Re(\alpha),\Re(\beta)\}>0;\ \gamma,\delta>0\right)$$

and

$$\int_{\xi}^{\eta} (t-\xi)^{\alpha-1}\,(\eta-t)^{\beta-1}\,\Theta\left(z,\omega(t-\xi)^{\gamma}\,(\eta-t)^{\delta};s,a\right)\mathrm{d}t$$
$$= (\eta-\xi)^{\alpha+\beta-1}\,B(\alpha,\beta)\sum_{n=0}^{\infty}\Xi_n\,\frac{z^n}{(n+a)^s}\;{}_2\Psi_1^*\left[\begin{array}{cc}(\alpha,\gamma),(\beta,\delta); & \dfrac{\omega(\eta-\xi)^{\gamma+\delta}}{n+a}\\ (\alpha+\beta,\gamma+\delta); & \end{array}\right] \tag{8.32}$$
$$\left(\eta\neq\xi;\ \min\{\Re(\alpha),\Re(\beta)\}>0;\ \gamma,\delta> provided that both 0\right),$$

provided that both sides of each of the assertions (8.31) *and* (8.32) *exist, the Fox-Wright function* ${}_3\Psi_1^*$ *in* (8.31) *being tacitly interpreted as an H-function contained in the definition* (6.26).

Proof. Each of the assertions (8.31) and (8.32) of Theorem 6 can be proven fairly easily by appealing to the definitions (8.11) and (8.13), respectively, in conjunction with the Eulerian Beta-function integral (8.30). The details involved are being skipped here. □

Remark 7. In addition to their relatively more familiar cases when $\xi = \eta - 1 = 0$, various interesting limit cases of the integral formulas (8.31) and (8.32) asserted by Theorem 6 can be deduced by letting

$$\lim_{\gamma\downarrow 0} \qquad \text{or} \qquad \lim_{\delta\downarrow 0}.$$

Several such very specialized cases of Theorem 6 as those that are indicated above can be found in the recent works [10], [38] and [67].

The Eulerian Gamma-function integrals involving the generating functions $\Omega_\lambda(z,t;s,a)$ and $\Theta(z,t;s,a)$ defined by (8.11) and (8.13), respectively, which are asserted by Theorem 7 below, can be evaluated by applying the well-known formula (8.23).

Theorem 7. *Let the function* $\Phi_\mu^*(z,s,a)$ *be defined by* (6.11). *Then, in terms of the sequence* $\{\Xi_n\}_{n\in\mathbb{N}_0}$ *of the coefficients given by the definition* (8.12), *each of the following single or double Eulerian Gamma-function integral formulas holds true*:

$$\frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}\,\mathrm{e}^{-\kappa t}\,\Omega_\lambda\left(z,\omega\mathrm{e}^{-\delta t};s,a\right)\mathrm{d}t$$
$$=\delta^{-\mu}\sum_{n=0}^{\infty}\frac{\Xi_n z^n}{(n+a)^s}\,\Phi_\lambda^*\left(\frac{\omega}{n+a},\mu,\frac{\kappa}{\delta}\right)\qquad\left(\min\{\Re(\kappa),\Re(\mu),\Re(\delta)\}>0\right), \tag{8.33}$$

$$\frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}\,\mathrm{e}^{-\kappa t}\,\Theta\left(z,\omega t;s,a\right)\mathrm{d}t=\kappa^{-\mu}\,\Omega_\mu\left(z,\frac{\omega}{\kappa};s,a\right)\qquad\left(\min\{\Re(\kappa),\Re(\mu)\}>0\right) \tag{8.34}$$

and

$$\frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\infty \int_0^\infty u^{\mu-1}\, v^{\nu-1}\, \mathrm{e}^{-\kappa u - \delta v}\, \Theta\left(z, \omega u \mathrm{e}^{-\sigma v}; s, a\right) \mathrm{d}u\, \mathrm{d}v$$
$$= \kappa^{-\mu}\, \sigma^{-\nu} \sum_{n=0}^\infty \frac{\Xi_n z^n}{(n+a)^s}\, \Phi_\mu^* \left(\frac{\omega}{\kappa(n+a)}, \mu, \frac{\delta}{\sigma} \right) \tag{8.35}$$
$$\left(\min\{\Re(\kappa), \Re(\mu), \Re(\nu), \Re(\delta), \Re(\sigma)\} > 0 \right),$$

provided that both sides of each of the assertions (8.33), (8.34) *and* (8.35) *exist.*

Remark 8. Some very specialized cases of Theorem 7 when

$$\Xi_n = 1 \qquad (n \in \mathbb{N}_0)$$

were derived in the recent works [10], [38] and [67].

Remark 9. Two of the claimed integral formulas in Bin-Saad's paper [10, p. 42, Theorem 3.2, Equations (3.10) *and* (3.11)] can easily be shown to be *divergent*, simply because the improper integrals occurring on their left-hand sides obviously violate the required convergence conditions at their *lower* terminal $t = 0$.

We now turn toward the *truncated* forms of the generating functions $\Omega_\lambda(z, t; s, a)$ and $\Theta(z, t; s, a)$ in (8.11) and (8.13), respectively, which are defined by (8.14) to (8.17). Indeed, by appealing appropriately to the definitions in (8.14) to (8.17) in conjunction with the Eulerian Gamma-function integral in (8.23), it is fairly straightforward to derive the integral representation formulas asserted by Theorem 8 below.

Theorem 8. *In terms of the sequence* $\{\Xi_n\}_{n\in\mathbb{N}_0}$ *of the coefficients given by the definition* (8.12), *each of the following Eulerian Gamma-function integral formulas holds true*:

$$\Omega_\lambda^{(0,r)}(z, \omega; s, a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{e}^{-at} \left(\sum_{k=0}^r \Xi_k\, \left(z \mathrm{e}^{-t}\right)^k \right) {}_1F_1(\lambda; s; \omega t) \mathrm{d}t \tag{8.36}$$
$$\left(\min\{\Re(s), \Re(a)\} > 0 \right),$$

$$\Omega_\lambda^{(r+1,\infty)}(z, \omega; s, a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{e}^{-at} \left(\sum_{k=r+1}^\infty \Xi_k\, \left(z \mathrm{e}^{-t}\right)^k \right) {}_1F_1(\lambda; s; \omega t) \mathrm{d}t \tag{8.37}$$
$$\left(\min\{\Re(s), \Re(a)\} > 0 \right),$$

$$\Theta^{(0,r)}(z, \omega; s, a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{e}^{-at} \left(\sum_{k=0}^r \Xi_k\, \left(z \mathrm{e}^{-t}\right)^k \right) {}_0F_1(\text{——}\, ; s; \omega t) \mathrm{d}t \tag{8.38}$$
$$\left(\min\{\Re(s), \Re(a)\} > 0 \right)$$

and

$$\Theta^{(r+1,\infty)}(z,\omega;s,a)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{e}^{-at}\left(\sum_{k=r+1}^{\infty}\Xi_k\,\left(z\mathrm{e}^{-t}\right)^k\right)\,{}_0F_1(\text{——};s;\omega t)\mathrm{d}t \tag{8.39}$$

$$\left(\min\{\Re(s),\Re(a)\}>0\right),$$

provided that both sides of each of the assertions (8.36) *to* (8.39) *exist.*

Remark 10. Several specialized cases of Theorem 8 when

$$\Xi_n=1 \qquad (n\in\mathbb{N}_0)$$

can be found in the recent works [10], [38] and [67].

Remark 11. It is not difficult to derive various other properties and results involving the generating functions $\Omega_\lambda(z,t;s,a)$ and $\Theta(z,t;s,a)$ in (8.11) and (8.13), respectively, as well as their *truncated* forms which are defined by (8.14) to (8.17). For example, by applying the definition (8.11) in conjunction with the definition (6.23), it is easy to derive the following general form of the generating relations asserted by (for example) Bin-Saad [10, p. 44, Theorem 4.2]:

$$\sum_{n=0}^{\infty}\frac{(\alpha_1)_{nu_1}\cdots(\alpha_\ell)_{nu_\ell}}{(\beta_1)_{nv_1}\cdots(\beta_m)_{nv_m}}\,\Omega_\lambda(z,\omega;s+n,a)\,\frac{t^n}{n!}$$
$$=\sum_{k=0}^{\infty}\Xi_k\,\left(1-\frac{\omega}{k+a}\right)^{-\lambda}\,{}_\ell\Psi_m^*\left[\begin{array}{cc}(\alpha_1,u_1),\cdots,(\alpha_\ell,u_\ell)\,; & \\ & \dfrac{t}{k+a}\\ (\beta_1,v_1),\cdots,(\beta_m,v_m)\,; & \end{array}\right]\frac{z^k}{(k+a)^s} \tag{8.40}$$

$$\left(\ell,m\in\mathbb{N}_0;\ \alpha_j\in\mathbb{C},\ u_j\in\mathbb{R}^+\ \ (j=1,\cdots,\ell);\ \beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ v_j\in\mathbb{R}^+\ \ (j=1,\cdots,m);\ \max\{|\omega|,|t|\}<1\right),$$

where the sequence $\{\Xi_n\}_{n\in\mathbb{N}_0}$ of the coefficients is given by the definition (8.12) and it is *tacitly* assumed that each member of the generating relation (8.40) exists. We do, however, choose to leave the details involved in all such derivations as exercises for the interested reader.

8. Further Remarks and Observations

As observed already by Srivastava [81], Saxena *et al.* [67] considered a so-called τ-generalization of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ in (6.1) in the following form [67, p. 311, Equation (2.1)]:

$$\Phi(\tau;z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(\tau n+a)^s} \qquad (\tau\in\mathbb{R}^+). \tag{9.1}$$

Subsequently, by similarly introducing a parameter $\tau>0$ in the definition (2.5), Gupta and Kumari [38] studied a τ-generalization of the extended Hurwitz-Lerch Zeta function $\Phi_\mu^*(z,s,a)$ in (2.5) as follows:

$$\Phi_\mu^*(\tau;z,s,a):=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\,\frac{z^n}{(\tau n+a)^s} \qquad (\tau\in\mathbb{R}^+), \tag{9.2}$$

which, when compared with the definition (9.1), yields the relationship:

$$\Phi(\tau;z,s,a)=\Phi_1^*(\tau;z,s,a) \qquad (\tau\in\mathbb{R}^+). \tag{9.3}$$

By looking closely at the definitions (9.1) and (9.2), in conjunction with the earlier definitions (6.1) and (6.11), respectively, we immediately get the following rather obvious connections:

$$\Phi(\tau;z,s,a)=\frac{1}{\tau^s}\,\Phi\left(z,s,\frac{a}{\tau}\right)\qquad\text{or}\qquad\Phi\,(z,s,a)=\tau^s\,\Phi(\tau;z,s,a\tau)\qquad(\tau\in\mathbb{R}^+)\tag{9.4}$$

and

$$\Phi_\mu^*(\tau;z,s,a)=\frac{1}{\tau^s}\,\Phi_\mu^*\left(z,s,\frac{a}{\tau}\right)\qquad\text{or}\qquad\Phi_\mu^*\,(z,s,a)=\tau^s\,\Phi_\mu^*(\tau;z,s,a\tau)\qquad(\tau\in\mathbb{R}^+)\tag{9.5}$$

Clearly, therefore, the definitions in (9.1) and (9.2) (*with* $\tau\in\mathbb{R}^+$) are no more general than their corresponding well-known cases when $\tau=1$ given by the definitions in (6.1) and (6.11), respectively. Thus, by *trivially* appealing to the parametric changes exhibited by the connections in (9.4) and (9.5), all of the results involving the so-called τ-generalized functions $\Phi(\tau;z,s,a)$ and $\Phi_\mu^*(\tau;z,s,a)$ can be derived simply from the corresponding (usually known) results involving the familiar functions $\Phi(z,s,a)$ and $\Phi_\mu^*(z,s,a)$, respectively. Just for illustration of the triviality associated with such straightforward derivations, we recall the following sum-integral representation formula due to Lin and Srivastava [52, p. 729, Equation (20)] (see also [99, p. 494, Equation (2.6)] for the special case when $k=1$):

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)=\frac{1}{\Gamma(s)}\sum_{j=0}^{k-1}\frac{(\mu)_{\rho j}}{(\nu)_{\sigma j}}\,z^j\int_0^\infty t^{s-1}\,\mathrm{e}^{-(a+j)t}\;{}_2\Psi_1\left[\begin{array}{c}(\mu+\rho j,\rho k)\,,(1,1);\\ \\ (\nu+\sigma j,\sigma k)\,;\end{array}z^k\,\mathrm{e}^{-kt}\right]\mathrm{d}t\tag{9.6}$$

$$\big(k\in\mathbb{N};\ \min\{\Re(a),\Re(s)\}>0;\ \sigma>\rho>0\quad\text{when}\quad z\in\mathbb{C};\ \sigma\geqq\rho>0\quad\text{when}\quad|z|^{1/k}<\rho^{-\rho}\,\sigma^\sigma\big),$$

it being tacitly assumed that each member of (9.6) exists. Indeed, in the special case when

$$\rho=\sigma=\nu=1,$$

(9.6) yields the following sum-integral representation for the generalized Hurwitz-Lerch Zeta function $\Phi_\mu^*(z,s,a)$ involved in (6.11):

$$\Phi_\mu^*(z,s,a)=\frac{1}{\Gamma(s)}\sum_{j=0}^{k-1}\frac{(\mu)_j}{(\nu)_j}\,z^j\int_0^\infty t^{s-1}\,\mathrm{e}^{-(a+j)t}\;{}_2\Psi_1\left[\begin{array}{c}(\mu+j,k)\,,(1,1);\\ \\ (\nu+j,k)\,;\end{array}z^k\,\mathrm{e}^{-kt}\right]\mathrm{d}t\tag{9.7}$$

$$\big(k\in\mathbb{N};\ \min\{\Re(a),\Re(s)\}>0;\ |z|<1\big)$$

or, equivalently,

$$\Phi_\mu^*(z,s,a)=\frac{1}{\Gamma(s)}\sum_{j=0}^{k-1}\frac{(\mu)_j}{(\nu)_j}\,z^j\int_0^\infty t^{s-1}\,\mathrm{e}^{-(a+j)t}\;{}_{k+1}F_k\left[\begin{array}{c}\Delta^*\,(k;\mu+j)\,,(1,1);\\ \\ \Delta^*\,(k;\nu+j)\,;\end{array}z^k\,\mathrm{e}^{-kt}\right]\mathrm{d}t\tag{9.8}$$

$$\big(k\in\mathbb{N};\ \min\{\Re(a),\Re(s)\}>0;\ |z|<1\big),$$

where, for convenience, $\Delta^*\,(n;\lambda)$ abbreviates the array of n parameters

$$\frac{\lambda}{n},\frac{\lambda+1}{n},\cdots,\frac{\lambda+n-1}{n}\qquad(n\in\mathbb{N}),$$

the array being empty when $n=0$.

Now, with a view to rewriting this last result (9.8) in terms of the τ-generalized Hurwitz-Lerch Zeta function $\Phi_\mu^*(\tau; z, s, a)$ defined by (9.2), we simply make the following *rather trivial* parameter and variable changes:

$$a \mapsto \frac{a}{\tau}, \quad t \mapsto \tau t \qquad \text{and} \qquad \mathrm{d}t \mapsto \tau \mathrm{d}t \qquad (\tau \in \mathbb{R}^+)$$

and multiply the resulting equation by τ^{-s}. By using the connection in (9.5), we are thus led immediately to the following sum-integral representation for the τ-generalized Hurwitz-Lerch Zeta function $\Phi_\mu^*(\tau; z, s, a)$ defined by (9.2) :

$$\Phi_\mu^*(\tau; z, s, a) = \frac{1}{\Gamma(s)} \sum_{j=0}^{k-1} \frac{(\mu)_j}{(\nu)_j} z^j \cdot \int_0^\infty t^{s-1} \mathrm{e}^{-(a+\tau j)t} \; {}_{k+1}F_k \left[\begin{array}{c} \Delta^*(k; \mu + j), (1,1); \\ \\ \Delta^*(k; \nu + j); \end{array} z^k \mathrm{e}^{-k\tau t} \right] \mathrm{d}t \tag{9.9}$$

$$\big(k \in \mathbb{N};\ \min\{\Re(a), \Re(s)\} > 0;\ |z| < 1\big),$$

provided that each member of the assertion (9.9) exists.

In its obvious particular case when $k = 1$, the sum-integral representation (9.9) would simplify at once to the following form [67, p. 311, Equation (2.2)]:

$$\Phi_\mu^*(\tau; z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{e}^{-at} \left(1 - z\mathrm{e}^{-\tau t}\right)^{-\mu} \mathrm{d}t \tag{9.10}$$

$$\big(\Re(a) > 0;\ \Re(s)\} > 0 \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad z = 1\big),$$

which obviously is equivalent to (and certainly not a generalization of) of the $\tau = 1$ case derived earlier by Goyal and Laddha [35, p. 100, Equation (1.6)].

Remark 12. The so-called τ-generalizations ${}_2R_1^\tau$ and ${}_1R_1^\tau$ of the Gauss hypergeometric function ${}_2F_1$ and Kummer's confluent hypergeometric function ${}_1F_1$, respectively, which were used in the aforecited paper by Saxena *et al.* [67, p. 315], are obviously very specialized cases of the well-known and extensively-investigated Fox-Wright function ${}_p\Psi_q$ defined by (6.23). In fact, it is ealsily seen from the definition (6.23) that [67, pp. 315 and 317] (see also [2], [3] and [110])

$$\begin{aligned} {}_2R_1^\tau(a, b; c; z) &:= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{(a)_n \Gamma(b + \tau n)}{\Gamma(c + \tau n)} \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \; {}_2\Psi_1 \left[\begin{array}{c} (a,1), (b,\tau); \\ \\ (c,\tau); \end{array} z \right] \\ &= {}_2\Psi_1^* \left[\begin{array}{c} (a,1), (b,\tau); \\ \\ (c,\tau); \end{array} z \right] \qquad (|z| < 1;\ \tau \in \mathbb{R}^+;\ c \notin \mathbb{Z}_0^-) \end{aligned} \tag{9.11}$$

and

$${}_1R_1^\tau(b; c; z) := \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{\Gamma(b + \tau n)}{\Gamma(c + \tau n)} \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(b)} \; {}_1\Psi_1 \left[\begin{array}{c} (b,\tau); \\ \\ (c,\tau); \end{array} z \right] = {}_1\Psi_1^* \left[\begin{array}{c} (b,\tau); \\ \\ (c,\tau); \end{array} z \right] \tag{9.12}$$

$$(|z| < \infty;\ \tau \in \mathbb{R}^+;\ c \notin \mathbb{Z}_0^-).$$

Similar remarks and observations would apply equally strongly to the other τ-generalizations of well-known and extensively-investigated hypergeometric functions in one, two and more variables.

We conclude our investigation by presenting a generalization of the sum-integral representation formula (9.6) due to Lin and Srivastava [52, p. 729, Equation (20)].

Theorem 9. *The following sum-integral representation formula holds true*:

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)=\frac{1}{\Gamma(s)}\sum_{j=0}^{k-1}\frac{\prod_{\ell=1}^{p}(\lambda_\ell)_{j\rho_\ell}}{\prod_{\ell=1}^{q}(\mu_\ell)_{j\sigma_\ell}}\frac{z^j}{j!}\int_0^\infty t^{s-1}\,\mathrm{e}^{-(a+j)t}$$

$$\cdot\,{}_{p+1}\Psi^*_{q+1}\left[\begin{array}{c}(\lambda_1+j\rho_1,k\rho_1),\cdots,(\lambda_p+j\rho_p,k\rho_p),(1,1);\\ \\ (\mu_1+j\sigma_1,k\sigma_1),\cdots,(\mu_q+j\sigma_q,k\sigma_q),(j+1,k);\end{array}z^k\,\mathrm{e}^{-kt}\right]\mathrm{d}t \tag{9.13}$$

$$\big(k\in\mathbb{N};\ \min\{\Re(a),\Re(s)\}>0\big),$$

provided that each member of the assertion (9.13) *exists.*

Proof. First of all, in light of the following elementary series identity:

$$\sum_{n=0}^{\infty}f(n)=\sum_{j=0}^{k-1}\sum_{n=0}^{\infty}f(kn+j)\qquad(k\in\mathbb{N}),$$

we find from the definition (6.21) that

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)=k^{-s}\sum_{j=0}^{k-1}\frac{\prod_{\ell=1}^{p}(\lambda_\ell)_{j\rho_\ell}}{\prod_{\ell=1}^{q}(\mu_\ell)_{j\sigma_\ell}}\frac{z^j}{j!}$$

$$\cdot\,\Phi^{(k\rho_1,\cdots,k\rho_p,1,k\sigma_1,\cdots,k\sigma_q,k)}_{\lambda_1+j\rho_1,\cdots,\lambda_p+j\rho_p,1;\mu_1+j\sigma_1,\cdots,\mu_q+j\sigma_q,j+1}\left(z^k,s,\frac{a+j}{k}\right)\qquad(k\in\mathbb{N}). \tag{9.14}$$

The assertion (9.13) of Theorem 9 would now emerge readily upon first appealing to the aforementioned known result due to Srivastava *et al.* [99, p. 504, Equation (6.4)] (see also Remark 7 above) given by

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{e}^{-at}\;{}_p\Psi^*_q\left[\begin{array}{c}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p);\\ \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q);\end{array}z\mathrm{e}^{-t}\right]\mathrm{d}t \tag{9.15}$$

$$\big(\min\{\Re(a),\Re(s)\}>0\big)$$

and then setting

$$t\mapsto kt\qquad\text{and}\qquad \mathrm{d}t\mapsto k\,\mathrm{d}t\qquad(k\in\mathbb{N}).$$

□

Furthermore, in its special case when

$$p=2\qquad(\lambda_1=\mu\quad\text{and}\quad\rho_1=\rho;\ \ \lambda_2=1\quad\text{and}\quad\rho_2=1)$$

and

$$q=1\qquad(\mu_1=\nu\quad\text{and}\quad\sigma_1=\sigma),$$

the general result (9.13) asserted by Theorem 9 can be seen to reduce immediately to the known sum-integral representation formula (9.6) due to Lin and Srivastava [52, p. 729, Equation (20)].

Recently, by suitably modifying this last integral representation formula (9.15), Srivastava [79] introduced and systematicall investigated the various properties of a significantly more general class of Hurwitz-Lerch zeta type functions defined by

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda) := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\exp\left(-at-\frac{b}{t^\lambda}\right) \cdot {}_p\Psi_q^*\left[\begin{array}{c} (\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); \end{array} ze^{-t}\right]\mathrm{d}t, \tag{9.16}$$

$$\big(\min\{\Re(a),\Re(s)\}>0;\ \Re(b)\geqq 0;\ \lambda\geqq 0\big),$$

so that, obviously, we have the following relationship:

$$\begin{aligned}\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;0,\lambda) &= \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) \\ &= e^b\,\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,0).\end{aligned} \tag{9.17}$$

In its special case when

$$p-1=q=0 \qquad (\lambda_1=\mu;\ \rho_1=1),$$

the above definition (9.16) would reduce immediately to the following form:

$$\Theta^\lambda_\mu(z,s,a;b) := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\exp\left(-at-\frac{b}{t^\lambda}\right)\,\left(1-ze^{-t}\right)^{-\mu}\,\mathrm{d}t \tag{9.18}$$

$$\big(\min\{\Re(a),\Re(s)\}>0;\ \Re(b)\geqq 0;\ \lambda\geqq 0;\ \mu\in\mathbb{C}\big),$$

where we have assumed *further* that

$$\Re(s)>0 \quad \text{when} \quad b=0 \quad \text{and} \quad |z|\leqq 1 \quad (z\neq 1)$$

or

$$\Re(s-\mu)>0 \quad \text{when} \quad b=0 \quad \text{and} \quad z=1,$$

provided, of course, that the integral in (9.18) exists. The function $\Theta^\lambda_\mu(z,s,a;b)$ was introduced and studied by Raina and Chhajed [63, p. 90, Equation (1.6)] and (more recently) by Srivastava *et al.* [96]. An interesting *further* special case of the function $\Theta^\lambda_\mu(z,s,a;b)$ arises when we set $\lambda=\mu=1$ and $z=1$ in the definition (9.18). We thus find that

$$\Theta^1_1(1,s,a;b)=\zeta_b(s,a) := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\exp\left(-at-\frac{b}{t}\right)\,\left(1-e^{-t}\right)^{-1}\,\mathrm{d}t, \tag{9.19}$$

where $\zeta_b(s,a)$ is the extended Hurwitz zeta function defined in [13, p. 308]. Furthermore, in a series of recent papers, Bayad *et al.* (see [8], [9] and [29]) introduced and studied the so-called *generalized Hurwitz-Lerch zeta function* $\zeta(s,\mu;a,z)$ *of order* μ, which they defined by (*cf.* [9, p. 608, Equation (6)])

$$\zeta(s,\mu;a,z) := \frac{\Gamma(\mu)}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}\,e^{-at}}{(1-ze^{-t})^\mu}\,\mathrm{d}t \tag{9.20}$$

$$\big(\Re(a)>0;\ \Re(s)>0 \quad \text{when} \quad |z|\leqq 1\ \ (z\neq 1);\ \Re(s-\mu)>0 \quad \text{when} \quad z=1\big)$$

or, equivalently, by (*cf.* [9, p. 608, Equation (7)])

$$\zeta(s,\mu;a,z) := \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n)}{n!} \frac{z^n}{(a+n)^s}. \tag{9.21}$$

By comparing the definitions (6.11) and (9.21), it is easily observed that

$$\zeta(s,\mu;a,z) = \Gamma(\mu) \cdot \Phi_{\mu}^{*}(z,s,a) \quad \text{and} \quad \Phi_{\mu}^{*}(z,s,a) = \frac{1}{\Gamma(\mu)}\, \zeta(s,\mu;a,z). \tag{9.22}$$

Clearly, therefore, Equations in (9.22) exhibit the fact that the generalized Hurwitz-Lerch zeta function $\zeta(s,\mu;a,z)$ of order μ, which was considered recently by Bayad *et al.* (see [8], [9] and [29]) is only a constant multiple of the *widely- and extensively-investigated* extended Hurwitz-Lerch Zeta function $\Phi_{\mu}^{*}(z,s,a)$ defined by (6.11).

For a detailed discussion of the various properties and results for, and the potential applications of, the so-called λ-generalized Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)$$

defined by (9.16), and also for their potential applications in Number Theory and Geometric Function Theory in Complex Analysis and for some other statistical applications in probability distribution theory, the interested reader should refer to the works by Srivastava *et al.* (see, for example, [79], [85], [87], [86] and [88]).

Finally, in addition to the *Open Problem* mentioned in Section 7 (see also [19]), many other properties and results involving such important higher transcendental functions of (for example) *Analytic Number Theory* and *Mathematical Physics* as the Riemann Zeta function $\zeta(s)$, the Hurwitz (or generalized) Zeta function $\zeta(s,a)$, and the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ as well as its various interesting extensions and generalizations, than those that we have considered in this article, deserve to be investigated further.

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References

[1] M. Abramowitz and I. A. Stegun (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series **55**, National Bureau of Standards, Washington, D.C., 1964; Reprinted by Dover Publications, New York, 1965 (see also [61]).

[2] A. Al-Zamel, On a generalized gamma-type distribution with τ-confluent hypergeometric function, *Kuwait J. Sci. Engrg.* **28** (2001), 25–36.

[3] I. Ali, S. L. Kalla and H. G. Khajah, A generalized inverse Gaussian distribution with τ-confluent hypergeometric function, *Integral Transforms Spec. Funct.* **12** (2001), 101–114.

[4] H. Alzer, D. Karayannakis and H. M. Srivastava, Series representations for some mathematical constants, *J. Math. Anal. Appl.* **320** (2006), 145–162.

[5] R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, in *Journées Arithmétiques de Luminy* (Colloq. Internat. CNRS, Centre Univ. Luminy, Luminy, 1978), pp. 11–13, *Astérisque* **61**, Soc. Math. France, Paris, 1979.

[6] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, Heidelberg and Berlin, 1976.

[7] R. Ayoub, Euler and the Zeta function, *Amer. Math. Monthly* **81** (1974), 1067–1086.

[8] A. Bayad and J. Chikhi, Reduction and duality of the generalized Hurwitz-Lerch zetas, *Fixed Point Theory Appl.* **2013** (2013), Article ID 82, 1–14.

[9] A. Bayad and S. Gaboury, Generalized Dirichlet L-function of arbitrary order with applications, *Adv. Stud. Contemp. Math.* **23** (2013), 607–619.

[10] M. G. Bin-Saad, Sums and partial sums of double power series associated with the generalized Zeta function and their N-Fractional calculus, *Math. J. Okayama Univ.* **49** (2007), 37–52.

[11] J. M. Borwein, D. M. Bradley and R. E. Crandall, Computational strategies for the Riemann Zeta function, *J. Comput. Appl. Math.* **121** (2000), 247–296.

[12] R. G. Buschman and H. M. Srivastava, The $\overline{H}$-function associated with a certain class of Feynman integrals, *J. Phys. A*: *Math. Gen.* **23** (1990) 4707–4710.

[13] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, Chapman and Hall (CRC Press Company), Boca Raton, London, New York and Washington, D.C., 2001.

[14] M.-P. Chen and H. M. Srivastava, Some families of series representations for the Riemann $\zeta(3)$, *Resultate Math.* **33** (1998), 179–197.

[15] J. Choi, Y. J. Cho and H. M. Srivastava, Series involving the Zeta and multiple Gamma functions, *Appl. Math. Comput.* **159** (2004), 509–537.

[16] J. Choi, D. S. Jang and H. M. Srivastava, A generalization of the Hurwitz-Lerch Zeta function, *Integral Transforms Spec. Funct.* **19** (2008), 65–79.

[17] J. Choi and H. M. Srivastava, Certain families of series associated with the Hurwitz-Lerch Zeta function, *Appl. Math. Comput.* **170** (2005), 399–409.

[18] J. Choi and H. M. Srivastava, Explicit evaluation of Euler and related sums, *Ramanujan J.* **10** (2005), 51–70.

[19] J. Choi and H. M. Srivastava, Series involving the Zeta functions and a family of generalized Goldbach-Euler series, *Amer. Math. Monthly* **121** (2014), 229–236.

[20] J. Choi, H. M. Srivastava and V. S. Adamchik, Multiple Gamma and related functions, *Appl. Math. Comput.* **134** (2003), 515–533.

[21] D. Cvijović and J. Klinowski, New rapidly convergent series representations for $\zeta(2n+1)$, *Proc. Amer. Math. Soc.* **125** (1997), 1263–1271.

[22] D. Cvijović and H. M. Srivastava, Limit representations of Riemann's Zeta function, *Amer. Math. Monthly* **119** (2012), 324–330.

[23] A. Dąbrowski, A note on the values of the Riemann Zeta function at positive odd integers, *Nieuw Arch. Wisk.* (*Ser.* 4) **14** (1996), 199–207.

[24] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[25] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of Integral Transforms*, Vol. **II**, McGraw-Hill Book Company, New York, Toronto and London, 1954.

[26] J. A. Ewell, A new series representation for $\zeta(3)$, *Amer. Math. Monthly* **97** (1990), 219–220.

[27] J. A. Ewell, On the Zeta function values $\zeta(2k+1)$, $k = 1, 2, \cdots$, *Rocky Mountain J. Math.* **25** (1995), 1003–1012.

[28] S. R. Finch, *Mathematical Constants*, Encyclopedia of Mathematics and Its Applications, vol. 94, Cambridge University Press, Cambridge, New York, Port Melbourne, Madrid and Cape Town, 2003.

[29] S. Gaboury and A. Bayad, Series representations at special values of generalized Hurwitz-Lerch zeta function, *Abstr. Appl. Anal.* **2013** (2013), Article ID 975615, 1–8.

[30] M. Garg, K. Jain and S. L. Kalla, A further study of general Hurwitz-Lerch zeta function, *Algebras Groups Geom.* **25** (2008), 311–319.

[31] M. Garg, K. Jain and H. M. Srivastava, Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions, *Integral Transforms Spec. Funct.* **17** (2006), 803–815.

[32] J. W. L. Glaisher, On the history of Euler's constant, *Messenger Math.* **1** (1872) 25–30.

[33] M. L. Glasser, Some integrals of the arctangent function, *Math. Comput.* **22** (1968), 445–447.

[34] R. W. Gosper Jr., A calculus of series rearrangements, in *Algorithms and Complexity*: *New Directions and Recent Results* (J. F. Traub, Editor), pp. 121–151, Academic Press, New York, London and Toronto, 1976.

[35] S. P. Goyal and R. K. Laddha, On the generalized Zeta function and the generalized Lambert function, *Gaṇita Sandesh* **11** (1997), 99–108.

[36] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals*, *Series*, *and Products* (Corrected and Enlarged edition prepared by A. Jeffrey), Academic Press, New York, 1980; sixth edition, 2000.

[37] P. L. Gupta, R. C. Gupta, S.-H. Ong and H. M. Srivastava, A class of Hurwitz-Lerch Zeta distributions and their applications in reliability, *Appl. Math. Comput.* **196** (2008), 521–531.

[38] R. K. Gupta and M. Kumari, Some results on a τ-generalized Riemann Zeta function, *Jñānābha* **41** (2011), 63–68.

[39] E. R. Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.

[40] M. M. Hjortnaes, Overføring av rekken $\sum_{k=1}^{\infty}\left(1/k^3\right)$ til et bestemt integral, *Proceedings of the Twelfth Scandanavian Mathematical Congress* (Lund; August 10-15, 1953), pp. 211–213, Scandanavian Mathematical Society, Lund, 1954.

[41] A. A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynman integrals. I: Transformation and reduction formulae, *J. Phys. A*: *Math. Gen.* **20** (1987), 4109–4117.

[42] A. A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynman integrals. II: A generalization of the H-function, *J. Phys. A*: *Math. Gen.* **20** (1987), 4119–4128.

[43] D. Jankov, T. K. Pogány and R. K. Saxena, An extended general Hurwitz-Lerch Zeta function as a Mathieu $(\boldsymbol{a}, \boldsymbol{\lambda})$-series, *Appl. Math. Lett.* **24** (2011), 1473–1476.

[44] S. Kanemitsu, M. Katsurada and M. Yoshimoto, On the Hurwitz-Lerch Zeta-function, *Aequationes Math.* **59** (2000), 1–19.

[45] S. Kanemitsu, H. Kumagai and M. Yoshimoto, Sums involving the Hurwitz Zeta functions, *Ramanujan J.* **5** (2001), 5–19.

[46] S. Kanemitsu, H. Kumagai, H. M. Srivastava and M. Yoshimoto, Some integral and asymptotic formulas associated with the Hurwitz Zeta function, *Appl. Math. Comput.* **154** (2004), 641–664.
[47] M. Katsurada, On Mellin-Barnes type of integrals and sums associated with the Riemann Zeta-function, *Publ. Inst. Math.* (*Beograd*) (*Nouvelle Ser.*) **62** (**76**) (1997), 13–25.
[48] M. Katsurada, Rapidly convergent series representations for $\zeta(2n+1)$ and their χ-analogue, *Acta Arith.* **90** (1999), 79–89.
[49] A. A. Kilbas, M. Saigo and J. J. Trujillo, On the generalized Wright functions, *Fract. Calc. Appl. Anal.* **5** (2002), 437–466.
[50] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.
[51] N. Koblitz, *p-Adic Numbers*, *p-Adic Analysis*, *and Zeta-Functions*, Graduate Texts in Mathematics, Vol. **58**, Springer-Verlag, New York, Heidelberg and Berlin, 1977.
[52] S.-D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, *Appl. Math. Comput.* **154** (2004), 725–733.
[53] S.-D. Lin, H. M. Srivastava and P.-Y. Wang, Some expansion formulas for a class of generalized Hurwitz-Lerch Zeta functions, *Integral Transforms Spec. Funct.* **17** (2006), 817–827.
[54] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third Enlarged Edition, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtingung der Anwendungsgebiete, Bd. **52**, Springer-Verlag, New York, 1966.
[55] A. M. Mathai, R. K. Saxena and H. J. Haubold, *The H-Function*: *Theory and Applications*, Springer, New York, Dordrecht, Heidelberg and London, 2010.
[56] Z. A. Melzak, *Companion to Concrete Mathematics*, Vol. I: *Mathematical Techniques and Various Applications*, John Wiley and Sons, New York, London, Sydney and Toronto, 1973.
[57] C. Nash and D. O'Connor, Ray-Singer torsion, topological field theories and the Riemann Zeta function at $s = 3$, in *Low-Dimensional Topology and Quantum Field Theory* (Proceedings of a NATO Advanced Research Workshop held at the Isaac Newton Institute at Cambridge, U.K.; September 6-12, 1992) (H. Osborn, Editor), pp. 279–288, Plenum Press, New York and London, 1993.
[58] C. Nash and D. O'Connor, Determinants of Laplacians, the Ray-Singer torsion on lens spaces and the Riemann Zeta function, *J. Math. Phys.* **36** (1995), 1462–1505.
[59] N. Nielsen, *Handbuch der Theorie der Gammafunktion*, Druck und Verlag von B. G. Teubner, Leipzig, 1906; Reprinted by Chelsea Publishing Company, New York, 1965.
[60] K. Nishimoto, C.-E Yen and M.-L. Lin, Some integral forms for a generalized Zeta function, *J. Fract. Calc.* **22** (2002), 91–97.
[61] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Editors), *NIST Handbook of Mathematical Functions* [With 1 CD-ROM (Windows, Macintosh and UNIX)], U. S. Department of Commerce, National Institute of Standards and Technology, Washington, D. C., 2010; Cambridge University Press, Cambridge, London and New York, 2010 (see also [1]).
[62] D. Răducanu and H. M. Srivastava, A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function, *Integral Transforms Spec. Funct.* **18** (2007), 933–943.
[63] R. K. Raina and P. K. Chhajed, Certain results involving a class of functions associated with the Hurwitz zeta function, *Acta Math. Univ. Comenianae* **73** (2004), 89–100.
[64] V. Ramaswami, Notes on Riemann's ζ-function, *J. London Math. Soc.* **9** (1934), 165–169.
[65] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives*: *Theory and Applications*, Translated from the Russian: *Integrals and Derivatives of Fractional Order and Some of Their Applications* ("Nauka i Tekhnika", Minsk, 1987), Gordon and Breach Science Publishers, Reading, Tokyo, Paris, Berlin and Langhorne (Pennsylvania), 1993.
[66] R. K. Saxana, A remark on a paper on M-series [MR2401326 (2009b:26008)]. With an editorial note, *Fract. Calc. Appl. Anal.* **12** (2009), 109–110.
[67] R. K. Saxena, R. K. Gupta and M. Kumari, Integrals and series expansions of the τ-generalized Riemann Zeta function, *J. Indian Acad. Math.* **33** (2011), 309–320.
[68] R. K. Saxena, T. K. Pog'any, R. Saxena and D. Jankov, On generalized Hurwitz-Lerch Zeta distributions occurring in statistical inference, Acta Univ. Sapientiae Math. **3** (2011), 43–59.
[69] M. Sharma, Fractional integration and fractional differentiation of the M-series, *Fract. Calc. Appl. Anal.* **11** (2008), 187–191.
[70] M. Sharma and R. Jain, A note on a generalzed M-series as a special function of fractional calculus, *Fract. Calc. Appl. Anal.* **12** (2009), 449–452.
[71] H. M. Srivastava, A unified presentation of certain classes of series of the Riemann Zeta function, *Riv. Mat. Univ. Parma* (*Ser.* 4) **14** (1988), 1–23.
[72] H. M. Srivastava, Sums of certain series of the Riemann Zeta function, *J. Math. Anal. Appl.* **134** (1988), 129–140.
[73] H. M. Srivastava, Certain families of rapidly convergent series representations for $\zeta\,(2n+1)$, *Math. Sci. Res. Hot-Line* **1** (6) (1997), 1–6 (Research Announcement).

[74] H. M. Srivastava, Further series representations for $\zeta(2n+1)$, *Appl. Math. Comput.* **97** (1998), 1–15.
[75] H. M. Srivastava, Some rapidly converging series for $\zeta(2n+1)$, *Proc. Amer. Math. Soc.* **127** (1999), 385–396.
[76] H. M. Srivastava, Some simple algorithms for the evaluations and representations of the Riemann Zeta function at positive integer arguments, *J. Math. Anal. Appl.* **246** (2000), 331-351.
[77] H. M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), 77–84.
[78] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inform. Sci.* **5** (2011), 390–444.
[79] H. M. Srivastava, A new family of the λ-generalized Hurwitz-Lerch zeta functions with applications, *Appl. Math. Inform. Sci.* **8** (2014), 1485-1500.
[80] H. M. Srivastava, Riemann, Hurwitz and Hurwitz-Lerch Zeta functions and associated series and integrals, in *Essays in Mathematics and Its Applications* (In Honor of Stephen Smale's 80th Birthday) (Panos M. Pardalos and Th. M. Rassias, Editors), Springer-Verlag, Berlin, Heidelberg and New York, 2012, pp. 431-461.
[81] H. M. Srivastava, Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions, *SpringerPlus* **2** (2013), Article ID 2:67, 1–14.
[82] H. M. Srivastava, Some τ-extensions of higher transcendental functions, *J. Indian Acad. Math.* **35** (2013), 165–176.
[83] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
[84] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
[85] H. M. Srivastava and S. Gaboury, New expansion formulas for a family of the lambda-generalized Hurwitz-Lerch zeta functions, *Internat. J. Math. Math. Sci.* **2014** (2014), Article ID 131067, 1–13.
[86] H. M. Srivastava, S. Gaboury and B.-J. Fugère, Further results involving a class of generalized Hurwitz-Lerch Zeta functions, *Russian J. Math. Phys.* **21** (2014), 521-537.
[87] H. M. Srivastava, S. Gaboury and A. Bayad, Expansion formulas for an extended Hurwitz-Lerch zeta function obtained via fracional calculus, *Adv. Difference Equations* **2014** (2014), Article ID 169, 1–17.
[88] H. M. Srivastava, S. Gaboury and R. Tremblay, New relations involving an extended multiparameter Hurwitz-Lerch zeta function with applications, *Internat. J. Anal.* **2014** (2014), Article ID 680850, 1–14.
[89] H. M. Srivastava, M. Garg and S. Choudhary, A new generalization of the Bernoulli and related polynomials, *Russian J. Math. Phys.* **17** (2010), 251–261.
[90] H. M. Srivastava, M. Garg and S. Choudhary, Some new families of generalized Euler and Genocchi polynomials, *Taiwanese J. Math.* **15** (2011), 283–305.
[91] H. M. Srivastava, M. L. Glasser and V. S. Adamchik, Some definite integrals associated with the Riemann Zeta function, *Z. Anal. Anwendungen* **19** (2000), 831–846.
[92] H. M. Srivastava, K. C. Gupta and S. P. Goyal, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.
[93] H. M. Srivastava, D. Jankov, T. K. Pogány and R. K. Saxena, Two-sided inequalities for the extended Hurwitz-Lerch Zeta function, *Comput. Math. Appl.* **62** (2011), 516–522.
[94] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
[95] H. M. Srivastava, S.-D. Lin and P.-Y. Wang, Some fractional-calculus results for the $\overline{H}$-function associated with a class of Feynman integrals, *Russian J. Math. Phys.* **13** (2006), 94–100.
[96] H. M. Srivastava, M.-J. Luo and R. K. Raina, New results involving a class of generalized Hurwitz-Lerch zeta functions and their applications, *Turkish J. Anal. Number Theory* **1** (2013), 26–35.
[97] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
[98] H. M. Srivastava, R. K. Saxena, T. K. Pogány and R. Saxena, Integral and computational representations of the extended Hurwitz–Lerch Zeta function, *Integral Transforms Spec. Funct.* **22** (2011), 487–506.
[99] H. M. Srivastava, R. K. Saxena, T. K. Pogány and R. Saxena, Integral and computational representations of the extended Hurwitz-Lerch Zeta function, *Integral Transforms Spec. Funct.* **22** (2011), 487–506.
[100] H. M. Srivastava and H. Tsumura, A certain class of rapidly convergent series representations for $\zeta(2n+1)$, *J. Comput. Appl. Math.* **118** (2000), 323–335.
[101] H. M. Srivastava and H. Tsumura, New rapidly convergent series representations for $\zeta(2n+1)$, $L(2n,\chi)$ and $L(2n+1,\chi)$, *Math. Sci. Res. Hot-Line* **4**(7) (2000), 17–24 (Research Announcement).
[102] H. M. Srivastava and H. Tsumura, Certain classes of rapidly convergent series representations for $L(2n,\chi)$ and $L(2n+1,\chi)$, *Acta Arith.* **100** (2001), 195–201.
[103] H. M. Srivastava and H. Tsumura, Inductive construction of rapidly convergent series representations for $\zeta(2n+1)$, *Internat. J. Comput. Math.* **80** (2003), 1161–1173.
[104] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University (Clarendon) Press, Oxford and London, 1951; Second Edition (Revised by D. R. Heath-Brown), 1986.
[105] F. G. Tricomi, Sulla somma delle inverse delle terze e quinte potenze dei numeri naturali, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (*Ser.* 8) **47** (1969), 16–18.
[106] H. Tsumura, On evaluation of the Dirichlet series at positive integers by q-calculation, *J. Number Theory* **48**

(1994), 383–391.

[107] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*: *An Introduction to the General Theory of Infinite Processes and of Analytic Functions*; *with an Account of the Principal Transcendental Functions*, Fourth Edition, Cambridge University Press, Cambridge, London and New York, 1927.

[108] J. R. Wilton, A proof of Burnside's formula for $\log \Gamma(x+1)$ and certain allied properties of Riemann's ζ-function, *Messenger Math.* **52** (1922–1923), 90–93.

[109] E. Witten, On quantum gauge theories in two dimensions, *Comm. Math. Phys.* **141** (1991), 153–209.

[110] N. O. Virchenko, S. L. Kalla and A. Al-Zamel, Some results on a generalized hypergeometric function, *Integral Transforms Spec. Funct.* **12** (2001), 89?–100.

[111] C.-E Yen, M.-L. Lin and K. Nishimoto, An integral form for a generalized Zeta function, *J. Fract. Calc.* **22** (2002), 99–102.

[112] N.-Y. Zhang and K. S. Williams, Some series representations of $\zeta(2n+1)$, *Rocky Mountain J. Math.* **23** (1993), 1581–1592.

[113] N.-Y. Zhang and K. S. Williams, Some infinite series involving the Riemann Zeta function, in *Analysis*, *Geometry and Groups*: *A Riemann Legacy Volume*, Parts **I** and **II** (H. M. Srivastava and Th. M. Rassias, Editors), Part **II**, pp. 691–712, Hadronic Press, Palm Harbor, Florida, 1993.

[114] N.-Y. Zhang and K. S. Williams, Values of the Riemann Zeta function and integrals involving $\log(2\sinh\frac{\theta}{2})$ and $\log(2\sin\frac{\theta}{2})$, *Pacific J. Math.* **168** (1995), 271–289.