To Marigold Maisie Duggal On Her Birthday
Functional Analysis,
Approximation and
Computation
7 (2) (2015), 159–167

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Generalized commutativity theorems for Hilbert space operators

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Abstract. Given Hilbert space operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$, defne $\delta_{A,B}$ and $\triangle_{A,B} \in B(B(\mathcal{K},\mathcal{H}))$ by $\delta_{A,B}(X) = AX - XB$ and $\triangle_{A,B}(X) = AXB - X$. This paper considers the equivalence $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0) \Longleftrightarrow \triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$ for various classes of Hilbert space operator A and B^* satisfying what are essentially very reasonable hypotheses.

1. Introduction

Given Hilbert space operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$, let $\delta_{A,B}$ and $\triangle_{A,B}$ denote, respectively, the generalized derivation $\delta_{A,B} \in B(B(\mathcal{K},\mathcal{H}))$, $\delta_{A,B}(X) = AX - XB$, and the elementary operator $\triangle_{A,B} \in B(B(\mathcal{K},\mathcal{H}))$, $\triangle_{A,B}(X) = AXB - X$. The classical (Putnam-Fuglede) commutativity theorem says that if A, B are normal, then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$. A similar result holds for $\triangle_{A,B}$: if A, B are normal, then $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$ [15]. This symmetric version of the (Putnam-Fuglede) commutativity theorem fails to extend to classes of Hilbert space operators more general than the class of normal operators: Indeed, if A and B are subnormal, then $\delta_{A,B}(X) = 0$ for some $X \in B(\mathcal{H})$ does not always imply $\delta_{A^*,B^*}(X) = 0$. An asymmetric version of the commutativity theorem, wherein one replaces operators A and B by operators A and B^*, is known to hold for operators A and B^* belonging to a number of classes of operators which properly contain the class of normal operators. For example, [1], if A is dominant and B^* is M-hyponormal, then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$. Does this inclusion extend to $\triangle_{A,B}$? In this note we answer this question in the affirmative for dominant A and hyponormal B^* to prove that if $\overline{\mathrm{ran}(X)}$ is invariant for A and $\ker^{\perp}(X)$ is invariant for B^*, then $\triangle_{A,B}(X) = 0$ implies $\triangle_{A^*,B^*}(X) = 0$. Indeed we prove more. Recall from [5] that for operators $A \in B(\mathcal{H})$ and isometries $B^* \in B(\mathcal{K})$, $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$ if and only if $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$. Does this two way implication extend to classes of operators (decidedly) not as general as every Hilbert space operator A, but more general than the class of isometries B^*?

Let $\mathcal{P}(\mathcal{H})$ denote the class of operators in $B(\mathcal{H})$ which are *translation, restriction and invertible invariant*, and which satisfy the property that their normal subspaces are reducing. (All these terms are explained

2010 *Mathematics Subject Classification*. Primary: 47B20, 47B47.
Keywords. Hilbert space; generalized derivation; elementary operator; commutativity theorem; hyponormal operator; class $\mathcal{Y}$ and $\mathcal{A}(s,t)$l operators.
Received: 21 January 2015; Accepted: 7 March 2015
Communicated by Dragan S. Djordjević
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in the following section.) Let $(\mathcal{P}(\mathcal{H}),\mathcal{P}(\mathcal{K}))\in\delta$ denote $\delta_{A,B}^{-1}(0)\subseteq\delta_{A^*,B^*}^{-1}(0)$ for all $A\in\mathcal{P}(\mathcal{H})$ and $B^*\in\mathcal{P}(\mathcal{K})$; let $(\mathcal{P}(\mathcal{H}),\mathcal{P}(\mathcal{K}))\in\triangle$ (resp., $(\mathcal{P}(\mathcal{H}),\mathcal{P}(\mathcal{K}))\in\triangle^\dagger$) denote $\triangle_{A,B}^{-1}(0)\subseteq\triangle_{A^*,B^*}^{-1}(0)$ for all $A\in\mathcal{P}(\mathcal{H})$ and $B^*\in\mathcal{P}(\mathcal{K})$ (resp., for all $A\in\mathcal{P}(\mathcal{H})$ and $B^*\in\mathcal{P}(\mathcal{K})$ for which if $\triangle_{A,B}(X)=0$ for an $X\in B(\mathcal{K},\mathcal{H})$, then $\overline{\text{ran(X)}}$ is invariant for A and $\ker^\perp(X)$ is invariant for B^*). Then $(\mathcal{P}(\mathcal{H}),\mathcal{P}(\mathcal{K}))\in\triangle\Longrightarrow(\mathcal{P}(\mathcal{H}),\mathcal{P}(\mathcal{K}))\in\delta$. Letting $(\mathcal{H}\mathcal{Y})$ denote the class of hyponormal operators in $B(\mathcal{K})$, it is seen that $(\mathcal{P}(\mathcal{H}),(\mathcal{H}\mathcal{Y}))\in\delta\Longleftrightarrow(\mathcal{P}(\mathcal{H}),(\mathcal{H}\mathcal{Y}))\in\triangle^\dagger$. Let $\mathcal{P}_A(\mathcal{H})$ denote those operators $A\in\mathcal{P}(\mathcal{H})$ for which "given a closed subset $S\subset\mathbb{C}$, if there exists a bounded function $f:\mathbb{C}\backslash S\to\mathcal{H}$ such that $(T-\lambda)f(\lambda)=x$ for some $(0\neq)x\in\mathcal{H}$, then there exists an analytic function $g:\mathbb{C}\backslash S\to\mathcal{H}$ such that $(T-\lambda)g(\lambda)=x$". Let class $\mathcal{Y}$ operators be defined as in [20]. Then $(\mathcal{P}_A(\mathcal{H}),B(\mathcal{K})\cap\mathcal{Y})\in\triangle\Longrightarrow(\mathcal{P}_A(\mathcal{H}),B(\mathcal{K})\cap\mathcal{Y})\in\delta$. Conversely, if $(\mathcal{P}_A(\mathcal{H}),B(\mathcal{K})\cap\mathcal{Y})\in\delta$, then, for every $A\in\mathcal{P}_A(\mathcal{H})$, $B^*\in B(\mathcal{K})\cap\mathcal{Y}$ and $X\in B(\mathcal{K},\mathcal{H})$ such that $\overline{\text{ran(X)}}$ is invariant for A, $\ker^\perp(X)$ is invariant for B^* and $A|_{\overline{\text{ran(X)}}}$ is invertible, $\triangle_{A,B}(X)=0$ implies $\triangle_{A^*,B^*}(X)=0$. As particular examples of operators satisfying these implications we (improve upon a vast majority of similar extant results to) prove in the following that if an $A\in B(\mathcal{H})$ is dominant, or w-hyponormal with $A^{-1}(0)\subseteq A^{*-1}(0)$, and $B^*\in B(\mathcal{K})\cap\mathcal{Y}$, or $B^*\in B(\mathcal{K})$ is w-hyponormal with $B^{*-1}(0)\subseteq B^{-1}(0)$, then $d_{A,B}^{-1}(0)\subseteq d_{A^*,B^*}^{-1}(0)$, where $d_{A,B}=\delta_{A,B}$ or $\triangle_{A,B}^\dagger$. Again, let class $\mathcal{A}(s,t)$ operators be defined as in [8]. Then $d_{A,B}^{-1}(0)\subseteq d_{A^*,B^*}^{-1}(0)$ for all $A\in B(\mathcal{H})\cap\mathcal{A}(s,t)$ such that $A^{-1}(0)\subseteq A^{*-1}(0)$ and $B^*\in B(\mathcal{K})\cap\mathcal{A}(s,t)$ such that $B^{*-1}(0)\subseteq B^{-1}(0)$, $\frac{1}{2}<s,t\leq 1$. The results below not only generalize similar results from [1–6, 10, 13–15, 17, 18, 20], but in many a case also provide a different perspective.

2. Preliminaries

We start by introducing some terminology and recalling a few complementary results. In the following, $\mathcal{H}$ and $\mathcal{K}$ shall denote infinite dimensional complex Hilbert spaces, $\mathcal{L}(\mathcal{K},\mathcal{H})$ ($\mathcal{L}(\mathcal{H})=\mathcal{L}(\mathcal{H},\mathcal{H})$ and $\mathcal{L}(\mathcal{K})=\mathcal{L}(\mathcal{K},\mathcal{K})$) the category of (not necessarily bounded) linear transformations with domain in $\mathcal{K}$ and range in $\mathcal{H}$, and $B(\mathcal{K},\mathcal{H})$ ($B(\mathcal{H})=B(\mathcal{H},\mathcal{H})$ and $B(\mathcal{K})=B(\mathcal{K},\mathcal{K})$) the algebra of operators (equivalently, bounded linear transformations) from $\mathcal{K}$ into $\mathcal{H}$. Given $A\in\mathcal{L}(\mathcal{H})$ and $B\in\mathcal{L}(\mathcal{K})$, and an operator $X\in B(\mathcal{K},\mathcal{H})$, the relation $XB\subseteq AX$ means $X\text{dom(B)}\subseteq\text{dom(A)}$ and $XBx=AXx$ for all x in the domain dom(B) of B. A densely defined closed linear transformation $T\in\mathcal{L}(\mathcal{H})$ is hyponormal (resp., normal) if $\text{dom(T)}\subseteq\text{dom(T}^*)$ and $\|T^*x\|\leq\|Tx\|$ (resp., $\text{dom(T)}=\text{dom(T}^*)$ and $\|T^*x\|=\|Tx\|$) for all $x\in\text{dom(T)}$. If an operator $T\in B(\mathcal{H})$ is injective and has a dense range, then T^{-1} exists as a densely defined closed linear transformation. Let $\sigma(T)$, $\sigma_p(T)$ and $\rho(T)$ denote, respectively, the spectrum, the point spectrum and the resolvent set of T. The following lemma is known ([11, Lemma 2.1], [7, Lemma 1]).

Lemma 2.1. *If $A\in\mathcal{L}(\mathcal{H})$ and $B\in\mathcal{L}(\mathcal{K})$ are densely defined closed linear transformations such that $XB\subseteq AX$ for some $X\in B(\mathcal{K},\mathcal{H})$, and if $\lambda\notin\sigma(A)\cup\sigma(B)$, then $X(B-\lambda)^{-1}=(A-\lambda)^{-1}X$.*

Recall that $X\in B(\mathcal{K},\mathcal{H})$ is a quasi-affinity if X is injective and has dense range. The following proposition is proved in [11, Theorem 3.3].

Proposition 2.2. *Let $X\in B(\mathcal{K},\mathcal{H})$ be a quasi-affinity such that $XB\subseteq AX$ for some densely defined closed linear transformations $A\in\mathcal{L}(\mathcal{H})$ and $B\in\mathcal{L}(\mathcal{K})$. If A is hyponormal, then $\sigma(A)\subseteq\sigma(B)$.*

For operators $A\in B(\mathcal{H})$ and $B^*\in B(\mathcal{K})$, let $d_{A,B}\in B(B(\mathcal{K},\mathcal{H}))$ denote either of $\delta_{A,B}\in B(B(\mathcal{K},\mathcal{H}))$ and $\triangle_{A,B}\in B(B(\mathcal{K},\mathcal{H}))$. If an $X\in B(\mathcal{K},\mathcal{H})$ satisfies $\delta_{A,B}(X)=0$, then the closure of the range of X, $\overline{\text{ran(X)}}$, is invariant for A and the orthogonal complement of the kernel of X, $\ker^\perp X$, is invariant for B^*. If we let

$$A_1=A|_{\overline{\text{ran(X)}}},\quad B_1^*=B^*|_{\ker^\perp X}$$

and define the quasi-affinity $X_1\in B(\ker^\perp X,\overline{\text{ran(X)}})$ by setting

$$X_1x=Xx\quad\text{for each}\quad x\in\ker^\perp X,$$

then $\delta_{A_1,B_1}(X_1)=0$.

Suppose that $B^{*-1}(0)$ is a normal subspace of B^* (equivalently, 0 is a normal eigenvalue of B^*, so that $B^*|_{B^{*-1}(0)}$ is normal and $B^{*-1}(0)$ reduces B^*), and that $\triangle_{A,B}(X) = 0$ for some quasi-affinity X. Then B is a quasi-affinity, and $B^{-1} \in \mathcal{L}(\mathcal{H})$ is a densely defined closed linear transformation which satisfies $XB^{-1} \subseteq AX$. The following proposition follows from [19, Theorem 4.2] (also see [11] and [7]).

Proposition 2.3. *If $A \in B(\mathcal{H})$ is hyponormal , $B^{*-1} \in \mathcal{L}(\mathcal{K})$ is a densely defined closed subnormal transformation and $XB^{-1} \subseteq AX$ for some quasi-affinity $X \in B(\mathcal{K}, \mathcal{H})$, then A and B^{-1} are unitarily equivalent normal operators.*

Corollary 2.4. *If $A \in B(\mathcal{H})$ and $B^* \in B(\mathcal{K})$ are subnormal operators, and $\triangle_{A,B}(X) = 0$ for some quasi-affinity $X \in B(\mathcal{K}, \mathcal{H})$, then $\triangle_{A^*,B^*}(X) = 0$.*

Proof. If an $X \in B(\mathcal{K}, \mathcal{H})$ satisfies $\triangle_{A,B}(X) = 0$ for some quasi-affinity X, then B^{*-1} is a densely defined closed subnormal transformation such that $XB^{-1} \subseteq AX$. Hence, Proposition 2.3, A and B^{-1} are unitarily equivalent normal operators which satisfy $AX = XB^{-1}$. But then $A^*X = XB^{*-1} \Longleftrightarrow A^*XB^* = X$. □

Corollary 2.4 is an extension of a result of Shul'man [15, Theorem 5], case $n = 2$, to subnormal operators. The following lemma is well known.

Lemma 2.5. *If $A \in B(\mathcal{H})$, $B^* \in B(\mathcal{K})$, $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$ and $X \in B(\mathcal{K}, \mathcal{H})$ satisfies $\delta_{A,B}(X) = 0$, then $\overline{\mathrm{ranX}}$ reduces A, $\ker^{\perp} X$ reduces B^* and A_1, B_1 (defined as above) are unitarily equivalent normal operators.*

The corresponding result for $\triangle_{A,B}$ is the following.

Lemma 2.6. *If $A \in B(\mathcal{H})$, $B^* \in B(\mathcal{K})$, $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$ and $X \in B(\mathcal{K}, \mathcal{H})$ satisfies $\triangle_{A,B}(X) = 0$, then $\overline{\mathrm{ranX}}$ reduces A, $\ker^{\perp}(X)$ reduces B^* and $A_1 = A|\overline{\mathrm{ranX}}$, $B_1^{-1} = (B^*|_{\ker^{\perp}(X)})^{*-1}$ are unitarily equivalent normal operators.*

Proof. The hypotheses imply that $\triangle_{A,B}(X) = \triangle_{A^*,B^*}(X) = 0$ (implies $A|X^*|^2 - |X^*|^2A = 0 = B|X|^2 - |X|^2B$); hence $\overline{\mathrm{ranX}}$ reduces A and $\ker^{\perp}(X)$ reduces B^*. Furthermore, since $AX \in \triangle^{-1}_{A,B}(0)$, $\triangle^{-1}_{A^*,B^*}(AX) = 0$. Hence $(A^*A - AA^*)XB^* = 0$, which implies that A_1 is normal. Similarly, since $XB \in \triangle^{-1}_{A,B}(0)$ implies $XB \in \triangle^{-1}_{A^*,B^*}(0)$, $(BB^* - B^*B)X^*A = 0 \Longrightarrow B_1^*$ is normal. Consider now the equation $\triangle_{A_1,B_1}(X_1) = 0$ (where X_1 is the quasi-affinity defined above). Evidently B_1 is a quasiaffinity, B_1^{-1} is a densely defined closed normal transformation and $X_1B_1^{-1} \subseteq A_1X_1$. Proposition 2.3 applies, and we conclude that A_1 and B_1^{-1} are unitarily equivalent normal operators. □

Unlike the case $\delta_{A,B}(X) = 0$, it may happen that $\triangle_{A,B}(X) = 0$ but $\overline{\mathrm{ran}(X)}$ is invariant for A and $\ker^{\perp}(X)$ is invariant for B^* does not hold [4]. It is however clear from Lemma 2.6 that we do not then have the implication $\triangle_{A,B}(X) = 0 \Longrightarrow \triangle_{A^*,B^*}(X) = 0$. *Since our interest in the following is in exploring conditions under which the two way implication $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0) \Longleftrightarrow \triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$ holds, in proving $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$ for some given operators A and B, we shall restrict ourselves to only those X satisfying $\triangle_{A,B}(X) = 0$ for which $\overline{\mathrm{ran}(X)}$ is invariant for A and $\ker^{\perp}(X)$ is invariant for B^*.* Unless otherwise stated, henceforth A shall denote an operator in $B(\mathcal{H})$, B^* an operator in $B(\mathcal{K})$, X an operator in $B(\mathcal{K}, \mathcal{H})$, and (if either $\delta_{A,B}(X) = 0$ or $\triangle_{A,B}(X) = 0$, then) the restrictions A_1, B_1 and the quasiaffinity X_1 shall be defined as above. $\delta_{A,B}(X) = 0$ neither implies nor is implied by $\triangle_{A,B}(X) = 0$: however, if an $X \in \delta^{-1}_{A,B}(0) \cap \triangle^{-1}_{A,B}(0)$, and either $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$ or $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$, then A_1 and B_1 have a particularly simple form.

Proposition 2.7. *If $X \in \delta^{-1}_{A,B}(0) \cap \triangle^{-1}_{A,B}(0)$, and either $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$ or $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$, then $\overline{ran(X)}$ reduces A, $\ker^{\perp} X$ reduces B^*, $A_1 = A|_{\overline{ran(X)}}$ and $B_1 = B|_{\ker^{\perp} X}$ are normal operators which satisfy $A_1^2 = B_1^2 = I$.*

Proof. If $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$ and $X \in \delta^{-1}_{A,B}(0) \cap \triangle^{-1}_{A,B}(0)$, then $\overline{ran(X)}$ reduces A, $\ker^{\perp} X$ reduces B^*, and A_1 and B_1 are unitarily equivalent normal operators which satisfy

$$X_1 = A_1X_1B_1 = X_1B_1^2 = A_1^2X_1.$$

In view of the fact that X_1 is a quasi-affinity, this implies $A_1^2 = B_1^2 = I$. The proof for the other case is similar. □

3. Results: The implication $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0) \Longleftrightarrow \triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$

Lemmas 2.5 and 2.6 are an early indicator of the fact that the implications $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0) \Longrightarrow \triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$ and $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0) \Longrightarrow \delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$ fail to hold in general. For example, if either of A and B^* fails to have an invertible part (*a part of an operator is its restriction to an invariant subspace*), then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$ does not imply $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$. It is however well known that the above implications hold for certain classes of Hilbert space operators (for example, normal operators and operators A, B^* such that B^* is an isometry [5]). In the following we prove that the two way equivalence above holds for classes of operators substantially larger than the class of normal operators. Following the lead of Lemmas 2.5 and 2.6, we start by defining a subclass of the class of Hilbert space operators. In the following we say that a set $\mathcal{S}(\mathcal{H}) \subset B(\mathcal{H})$ is:

translation invariant if $T \in \mathcal{S}(\mathcal{H}) \Longrightarrow T - \lambda \in \mathcal{S}(\mathcal{H})$ for every complex λ,
restriction invariant if $T \in \mathcal{S}(\mathcal{H}) \Longrightarrow T|_M \oplus 0 \in \mathcal{S}(\mathcal{H})$ $(= \mathcal{S}(M \oplus M^\perp))$ for every invariant subspace M of T,
and *invertible invariant* if $T \in \mathcal{S}(\mathcal{H}) \Longrightarrow (T|_M)^{-1} \oplus 0 \in \mathcal{S}(\mathcal{H})$ whenever $(T|_M)^{-1}$ exists as a bounded linear transformation.

Definition 3.1. We say that a set $\mathcal{S}(\mathcal{H}) \subset B(\mathcal{H})$ is a $\mathcal{P}(\mathcal{H})$ class of operators if $\mathcal{S}(\mathcal{H})$ is translation, restriction and invertible invariant, and if the normal subspaces of operators $T \in \mathcal{S}(\mathcal{H})$ (i.e., closed invariant subspaces M such that $T|_M$ is normal) are reducing.

Examples of $\mathcal{P}(\mathcal{H})$ classes occur quite naturally: thus, the class consisting of hyponormal operators (as also the classes consisting of M-hyponormal and dominant operators – see definitions below) is a $\mathcal{P}(\mathcal{H})$ class.

Definition 3.2. Let $\mathcal{Q}(\mathcal{H}) \subset B(\mathcal{H})$ and $\mathcal{Q}(\mathcal{K}) \subset B(\mathcal{K})$ be two classes of operators. We say that the pair $(\mathcal{Q}(\mathcal{H}), \mathcal{Q}(\mathcal{K})) \in \delta$ (resp., $(\mathcal{Q}(\mathcal{H}), \mathcal{Q}(\mathcal{K})) \in \triangle$) if, for every $A \in \mathcal{Q}(\mathcal{H})$ and $B^* \in \mathcal{Q}(\mathcal{K})$, $\delta_{A,B}(X) = 0$ for an $X \in B(\mathcal{K}, \mathcal{H})$ implies $\delta_{A^*,B^*}(X) = 0$ (resp., $\triangle_{A,B}(X) = 0$ for an $X \in B(\mathcal{K}, \mathcal{H})$, such that $\overline{\mathrm{ran(X)}}$ is invariant for A and $\ker(X)^\perp$ is invariant for B^*, implies $\triangle_{A^*,B^*}(X) = 0$).

The implication $(\mathcal{P}(\mathcal{H}), \mathcal{P}(\mathcal{K})) \in \triangle \Longrightarrow (\mathcal{P}(\mathcal{H}), \mathcal{P}(\mathcal{K})) \in \delta$ is fairly straightforward to prove.

Proposition 3.3. $(\mathcal{P}(\mathcal{H}), \mathcal{P}(\mathcal{K})) \in \triangle$ *implies* $(\mathcal{P}(\mathcal{H}), \mathcal{P}(\mathcal{K})) \in \delta$.

Proof. Let $A \in \mathcal{P}(\mathcal{H})$ and $B^* \in \mathcal{P}(\mathcal{K})$. For an $X \in \delta^{-1}_{A,B}(0)$, define $A_1 \in B(\overline{\mathrm{ran} X})$, $B_1^* \in B(\ker^\perp X)$ and $X_1 \in B(\ker^\perp X, \overline{\mathrm{ran} X})$ as before. Then $\delta_{A_1,B_1}(X_1) = 0$. Choose a $\overline{\lambda} \in \rho(B_1^*)$, the resolvent set of B_1^*, and define $C \in B(\mathcal{H})$, $D^* \in B(\mathcal{K})$ and $Y \in B(\mathcal{K}, \mathcal{H})$ by $C = (A_1 - \lambda) \oplus 0$, $D^* = (B_1 - \lambda)^{*-1} \oplus 0$ and $Y = X_1 \oplus 0$. Then

$$\delta_{A_1-\lambda, B_1-\lambda}(X_1) = 0 \Longrightarrow \triangle_{C,D}(Y) = 0,$$

where $C \in \mathcal{P}(\mathcal{H})$ and $D^* \in \mathcal{P}(\mathcal{K})$. It being evident that $\overline{\mathrm{ran(Y)}}$ reduces C and $\ker(Y)^\perp$ reduces D^*, it follows from the hypotheses that $\triangle_{C^*,D^*}(Y) = 0$. Hence $A_1 - \lambda$ and $B_1 - \lambda$ are unitarily equivalent normal operators. But then A_1 and B_1 are unitarily equivalent normal operators, leading us thereby to conclude that $\delta_{A_1^*,B_1^*}(X_1) = 0$. Since normal subspaces of A and B^* are reducing, $\delta_{A^*,B^*}(X) = 0$. □

The reverse implication $(\mathcal{P}(\mathcal{H}), \mathcal{P}(\mathcal{K})) \in \delta \Longrightarrow (\mathcal{P}(\mathcal{H}), \mathcal{P}(\mathcal{K})) \in \triangle$ is not as straightforward. Evidently, for an $A \in B(\mathcal{H})$ and an invertible $B^* \in \mathcal{K}$, $(A, B^*) \in \triangle \Longrightarrow (A, B^{*-1}) \in \delta$. Observe that if $X \in \triangle^{-1}_{A,B}(0)$, $\overline{\mathrm{ran(X)}}$ is invariant for A and $\ker^\perp(X)$ is invariant for B^*, then (upon defining A_1, B_1 and X_1 as before) B_1^{-1} is well defined (as a densely defined closed linear transformation) which satisfies $X_1 B_1^{-1} \subseteq A_1 X_1$. Here $\rho(B^{-1}) \cap \rho(A_1)$ may be the empty set; however, if $\rho(B^{-1}) \cap \rho(A_1) \neq \emptyset$, then there exists a complex number λ such that $(A_1 - \lambda)$ and $B_1^{-1} - \lambda$ are boundedly invertible operators which satisfy $X_1(B_1^{-1} - \lambda) \subseteq (A_1 - \lambda)X_1$. Lemma 2.1 applies and we conclude that $(A_1 - \lambda)^{-1} X_1 = X_1 (B_1^{-1} - \lambda)^{-1}$. This, if $(A, B^*) \in \delta$ entails $(A_1 - \lambda)^{*-1} X_1 = X_1 (B_1^{-1} - \lambda)^{*-1}$, then implies that A_1 and B_1^{-1} are unitarily equivalent normal operators. Consequently, $\triangle_{A_1,B_1}(X_1)$, and hence

$\triangle_{A^*,B^*}(X)$, = 0. A case in hand where this argument holds is that in which $B^* \in B(\mathcal{K})$ is a hyponormal operator (i.e., B^* satisfies $BB^* \leq B^*B$). Recall from [16, Lemma 3] that a hyponormal quasi-affinity has a densely defined closed hyponormal inverse. If $\triangle_{A,B}(X) = 0$ for an operator $A \in B(\mathcal{H})$, a hyponormal operator $B^* \in B(\mathcal{K})$ and an operator $X \in B(\mathcal{K},\mathcal{H})$ such that $\overline{\mathrm{ran(X)}}$ is invariant for A and $\ker^{\perp}(\mathrm{X})$ is invariant for B^*, then $X_1^*A_1^* \subseteq B_1^{*-1}X_1^*$ (where B_1^{*-1} is a densely defined closed hyponormal transformation and X_1^* is a quasi-affinity). Proposition 2.2 applies, and we conclude that $\sigma(B_1^{-1}) \subseteq \sigma(A_1)$.

Proposition 3.4. *Let $(\mathcal{HY})$ denote the class of hyponormal operators in $B(\mathcal{K})$. Then*

$$(\mathcal{P}(\mathcal{H}),(\mathcal{HY})) \in \delta \Longleftrightarrow (\mathcal{P}(\mathcal{H}),(\mathcal{HY})) \in \triangle.$$

Proof. Since hyponormal operators in $B(\mathcal{K})$ constitute a $\mathcal{P}(\mathcal{K})$ class, the backwards implication follows from Proposition 3.3. For the forwards implication, choose an $A \in \mathcal{P}(\mathcal{H})$, a hyponormal operator $B^* \in B(\mathcal{K})$ and an $X \in B(\mathcal{K},\mathcal{H})$ such that $\triangle_{A,B}(X) = 0$. Then it follows from the above that $(A_1-\lambda)^{-1}$ and $(B_1^{-1}-\lambda)^{-1}$ are bounded operators which satisfy $(A_1-\lambda)^{-1}X_1 = X_1(B_1^{-1}-\lambda)^{-1}$ for every $\lambda \in \rho(A_1)$. Evidently the operator $E = (A_1-\lambda)^{-1} \oplus 0 \in \mathcal{P}(\mathcal{H})$ and the operator $F^* = {(B_1^{-1}-\lambda)^*}^{-1} \oplus 0 \in B(\mathcal{K})$ is a hyponormal operator. Since $\delta_{E,F}(Y) = 0$, where $Y = X_1 \oplus 0 \in B(\mathcal{K},\mathcal{H})$, and since $(\mathcal{P}(\mathcal{H}),(\mathcal{HY})) \in \delta$, $\delta_{E^*,F^*}(Y) = 0$. Consequently E, F, and hence also A_1, B_1^{-1}, are unitarily equivalent normal operators. This implies that A and B have direct sum representations $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$ such that A_1, B_1 are normal, and $\triangle_{A,B}(X) = 0 \Longrightarrow \triangle_{A_1,B_1}(X_1) = 0 \Longrightarrow \triangle_{A_1^*,B_1^*}(X_1) = 0 \Longrightarrow \triangle_{A^*,B^*}(X) = 0.$ □

An operator $T \in B(\mathcal{H})$ is M-hyponormal, $T \in M-(\mathcal{HY})$, if there exists a number $M > 0$ such that $|(T-\lambda)^*|^2 \leq M|T-\lambda|^2$ for all complex λ. (M-hyponormal operators define a $\mathcal{P}(\mathcal{H})$ class.) We do not know if Proposition 2.2 extends to M-hyponormal operators A, and hence whether Proposition 3.4 extends to $A \in \mathcal{P}(\mathcal{H})$ and M-hyponormal B^*. However an alternative approach bears fruit.

Putnam [12] proved that if an operator $T \in B(\mathcal{H})$ satisfies the *condition* $(T;D)$,

$$(T-\lambda)(T-\lambda)^* \geq D^2, \quad \text{for some } D \geq 0 \text{ and all complex } \lambda,$$

then for every vector x in the range of D there exists a bounded function $f : \mathbb{C} \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$. Hyponormal operators T^* satisfy condition $(T;|TT^*-T^*T|)$ and M-hyponormal operators T^* satisfy condition $(T;K|TT^*-T^*T|)$ for some $K > 0$ [13]. More generally, let T^* be a class $\mathcal{Y}$ operator, i.e., let $T^* \in \bigcup_{\alpha\geq 1}\mathcal{Y}_\alpha$, where $\mathcal{Y}_\alpha$ is the class of (Hilbert space) operators E^* for which there exists a number $K_\alpha > 0$ such that $|EE^*-E^*E|^\alpha \leq K_\alpha^2(E-\lambda)(E-\lambda)^*$ for all $\lambda \in \mathbb{C}$ [20]. (Thus class $\mathcal{Y}$ operators T^* satisfy condition $(T;K_\alpha^2|TT^*-T^*T|^\alpha)$ for some $K_\alpha > 0$.) Recall here that $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$ for all $1 \leq \alpha < \beta$, $E^* \in \mathcal{Y}_1$ implies E^* is M-hyponormal, E^* M-hyponormal implies $E^* \in \mathcal{Y}_2$, and $T^* \in \mathcal{Y}$ implies the existence of an integer $n > 1$ such that $|TT^*-T^*T|^{2^n} \leq K_{2^n}^2(E-\lambda)(E-\lambda)^*$ for all $\lambda \in \mathbb{C}$ [20]. The class of dominant operators T, i.e. operators $T \in B(\mathcal{H})$ for which to every complex λ there corresponds a number $M_\lambda > 0$ such that $|(T-\lambda)^*|^2 \leq M_\lambda|T-\lambda|^2$ for all $\lambda \in \mathbb{C}$, is independent of the class $\mathcal{Y}$. (Thus there exists a dominant operator which is not class $\mathcal{Y}$ [10].) Dominant operators satisfy the following (*local spectral*) property

(A): *Given a closed subset $S \subset \mathbb{C}$, if there exists a bounded function $f : \mathbb{C}\backslash S \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) = x$ for some $(0 \neq)x \in \mathcal{H}$, then there exists an analytic function $g : \mathbb{C}\backslash S \to \mathcal{H}$ such that $(T-\lambda)g(\lambda) = x$.*

Recall that a *part of an operator* is its restriction to a closed invariant subspace. Clearly, a part of a dominant operator is a dominant operator, and an invertible dominant operator is again dominant. *Let $\mathcal{P}_A(\mathcal{H})$ denote those operators in $\mathcal{P}(\mathcal{H})$ for which every part of the operator satisfies property* (A). For operators $A \in B(\mathcal{H})$ satisfying property (A) and operators $B^* \in B(\mathcal{K}) \cap \mathcal{Y}$ such that $\delta_{A,B}(X) = 0$ we have:

Theorem 3.5. *Let $A \in B(\mathcal{H})$ and $B^* \in B(\mathcal{K})$ be quasi-affinities.*
(i) If A satisfies property (A), $B^* \in \mathcal{Y}$ and there exists a quasi-affinity $X \in B(\mathcal{K},\mathcal{H})$ such that $\delta_{A,B}(X) = 0$, then B is normal and $\sigma(A) = \sigma(B)$.

(ii) If $A \in \mathcal{P}_A(\mathcal{H})$, $B^* \in (\mathcal{HY})$ and there exists a quasi-affinity $X \in B(\mathcal{K},\mathcal{H})$ such $\triangle_{A,B}(X) = 0$, then B is normal and $\sigma(A) = \sigma(B^{-1})$.
(iii) If A^{-1} exists and satisfies property (A), $B^* \in \mathcal{Y}$ and there exists a quasi-affinity $X \in B(\mathcal{K},\mathcal{H})$ such $\triangle_{A,B}(X) = 0$, then B is normal and $\sigma(A) = \sigma(B^{-1})$.

Proof. The argument below that we use to prove the theorem has a long history, beginning with the work of Putnam [12] and continuing with the work of Stampfli and Wadhwa [17, 18], Radjabalipour [13, 14], and more recently the work by Uchiyama *et al* [10, 20].

(i) The hypothesis $B^* \in \mathcal{Y}$ implies the existence of a positive integer α such that for every $x \in |BB^* - B^*B|^{\frac{\alpha}{2}}\mathcal{K}$ there exists a bounded function $f : \mathbb{C} \to \mathcal{K}$ such that $(B - \lambda)f(\lambda) = x$ for all complex λ. If $\delta_{A,B}(X) = 0$, then $Xx = X(B - \lambda)f(\lambda) = (A - \lambda)Xf(\lambda)$. We claim that $x = 0$. For if not, then A satisfies property (A) implies the existence of an entire function g such that $(A - \lambda)Xg(\lambda) = Xx$. But then $Xg(\lambda) = (A - \lambda)^{-1}Xx \longrightarrow 0$ as $\lambda \longrightarrow \infty$, i.e., $g(\lambda) \equiv 0$. Hence $Xx = 0 \Longrightarrow x = 0$ (since X is a quasi-affinity) – a contradiction. Thus we must have had all along that $x = 0$. Consequently, $|BB^* - B^*B|^{\frac{\alpha}{2}}\mathcal{K} = 0$, and hence $BB^* - B^*B = 0$, i.e., B is normal. As a normal operator B satisfies both *Bishop's property* (β) and *the decomposition property* (δ). (A Banach space operator $T \in B(\mathcal{X})$ satisfies property (β) if for every open subset $\mathcal{U}$ of $\mathbb{C}$ and every sequence of functions $f_n : \mathcal{U} \to \mathcal{X}$ with the property that $(T - \lambda)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of $\mathcal{U}$, $f_n(\lambda) \to 0$ locally uniformally on $\mathcal{U}$; T satisfies property (δ) if T^* satisfies property (β) [9, Pages 11, 32 and Theorem 2.5.18].) Since $AX = XB$ with X a quasi-affinity, a straightforward argument (using no more than the definition above) shows that B^* satisfies property (β) implies A^* satisfies property (β). Now apply [9, Corollary 3.5.16] to $B^*X^* = X^*A^*$ to conclude $\sigma(A) = \sigma(B)$.

(ii) Given a quasi-affinity $A \in B(\mathcal{H})$ and a quasi-affinity B^* in $B(\mathcal{K})$ such that $\triangle_{A,B}(X) = 0$ for some quasi-affinity X, B^{-1} exists as a densely defined closed linear transformation which satisfies $XB^{-1} \subseteq AX \Longleftrightarrow X^*A^* \subseteq B^{*-1}X^*$. It is easily seen that if (in the above) B^* is hyponormal, then B^{*-1} is a densely defined closed hyponormal transformation which satisfies $\rho(A) \subseteq \rho(B^{-1})$. Choose a $\lambda \in \rho(A)$. Then $X(B^{-1} - \lambda)^{-1} = X(A - \lambda)^{-1}$, where ${(B^{-1} - \lambda)^*}^{-1}$ is a bounded hyponormal operator and $(A - \lambda)^{-1} \in \mathcal{P}_A(\mathcal{H})$. Arguing as in part (i) of the proof it is seen that $(B^{-1} - \lambda)$ is normal and $\sigma(A - \lambda) = \sigma(B_1^{-1} - \lambda)$. This implies B is normal and $\sigma(A) = \sigma(B^{-1})$.

(iii) If A^{-1} satisfies property (A), then the argument of the proof of part (i) applies to $\delta_{A^{-1},B}(X) = 0$ to prove that B is normal and $\sigma(A^{-1}) = \sigma(B)$. □

The operator B of Theorem 3.5 being normal, $\sigma(B) = \sigma_a(B)$, (where $\sigma_a(.)$ denotes *approximate point spectrum*). A straightforward argument shows that $\sigma(A) = \sigma_a(A) = \sigma(B)$ for the operator A of Theorem 3.5(i). We remark here that the hypothesis on the invertibility of A in Theorem 3.5(iii) is not unreasonable (for the reason that our interest here lies mostly in proving the inclusion $\triangle_{A,B}^{-1}(0) \subseteq \triangle_{A^*,B^*}^{-1}(0)$): Recall from Lemma 2.6 that if $\triangle_{A,B}(X) = 0$ implies $\triangle_{A^*,B^*}(X) = 0$ for a quasi-affinity X, then both A and B are invertible.

Theorem 3.6. *(i) $(\mathcal{P}_A(\mathcal{H}), B(\mathcal{K}) \cap \mathcal{Y}) \in \triangle \Longrightarrow (\mathcal{P}_A(\mathcal{H}), B(\mathcal{K}) \cap \mathcal{Y}) \in \delta$.*
(ii) Suppose that $(\mathcal{P}_A(\mathcal{H}), B(\mathcal{K}) \cap \mathcal{Y}) \in \delta$. Let $A \in \mathcal{P}_A(\mathcal{H})$ and $B^* \in B(\mathcal{K}) \cap \mathcal{Y}$. If an $X \in \triangle_{A,B}^{-1}(0)$, $X \in B(\mathcal{K},\mathcal{H})$, is such that ($\overline{\mathrm{ran}(X)}$ is invariant under A, $\overline{\mathrm{ran}(X^*)}$ is invariant under B^* and) $A_1 = A|_{\overline{\mathrm{ran}(X)}}$ is invertible, then $X \in \triangle_{A^*,B^*}^{-1}(0)$.

Proof. (i) Let $(\mathcal{P}_A(\mathcal{H}), B(\mathcal{K}) \cap \mathcal{Y}) \in \delta$, and assume that $\delta_{A,B}(X) = 0$ for some $A \in \mathcal{P}_A(\mathcal{H})$, $B^* \in B(\mathcal{K}) \cap \mathcal{Y}$ and $X \in B(\mathcal{K},\mathcal{H})$. Define A_1, B_1^* and the quasi-affinity X_1 as before. Since normal subspaces of A (and B^*) reduce A (resp., B^*), either 0 is in the point spectrum of both A_1 and B_1^*, or it is not in the point spectrum of either of them. If $0 \in \sigma_p(A_1)$ and $0 \in \sigma_p(B_1^*)$, then A_1 and B_1^* have a direct sum decomposition $A_1 = 0 \oplus A_{12}$ and (respectively) $B_1^* = 0 \oplus B_{12}^*$, where A_{12} and B_{12}^* are quasi-affinities. Letting X_1 have the corresponding matrix representation $X_1 = [X_{ij}]_{i,j=1}^2$, it is then seen that $A_{12}X_{22} = X_{22}B_{12}$, where X_{22} is a quasi-affinity. Theorem 3.5 applies and we conclude that (B_{12}^*, and so also) B_1^* is normal and ($\sigma(A_{12}) = \sigma(B_{12})$ implies) $\sigma(A_1) = \sigma(B_1)$.

Choose a $(0 \neq)\lambda \in \rho(A_1)$ and define operators E, F^* and Y by $E = (A_1 - \lambda) \oplus -\lambda$, $F^* = (B_1 - \lambda)^{*-1} \oplus -\frac{1}{\lambda}$ and $Y = X_1 \oplus 0$. Then $E \in \mathcal{P}_A(\mathcal{H})$, F^* is normal (therefore, $\in \mathcal{Y}$) and $\triangle_{E,F}(Y) = 0$. Hence $\triangle_{E^*,F^*}(Y) = 0$, which implies that ($E$ and F, hence) A_1 and B_1 are normal. Thus A and B^* have direct sum decompositions $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$ such that

$$\delta_{A,B}(X) = 0 \Longrightarrow \delta_{A_1,B_1}(X_1) = 0 \Longrightarrow \delta_{A_1^*,B_1^*}(X_1) = 0 \Longrightarrow \delta_{A^*,B^*}(X) = 0,$$

and the implication follows.

(ii) The hypotheses imply that $A_1X_1B_1 = X_1 \Longleftrightarrow A_1^{-1}X_1 = X_1B_1$. Defining $C \in B(\mathcal{H})$, $Y \in B(\mathcal{K}, \mathcal{H})$ and $D^* \in B(\mathcal{K})$ by $C = A_1^{-1} \oplus 0$, $Y = X_1 \oplus 0$ and $D^* = B_1^* \oplus 0$, it is seen that $\delta_{C,D}(Y) = 0$, where $C \in \mathcal{P}_A(\mathcal{H})$ and $D^* \in \mathcal{Y}$. The hypothesis $(\mathcal{P}_A(\mathcal{H}), B(\mathcal{K}) \cap \mathcal{Y}) \in \delta$ now implies that $\delta_{C^*,D^*}(Y) = 0$, and hence (C and D, and so) A_1^{-1} and B_1 are unitarily equivalent normal operators. Since normal subspaces of operators in $\mathcal{P}(\mathcal{H})$ and $\mathcal{Y}$ reduce the operator, $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$ (for some operators A_2 and B_2^*). The proof follows, since $\triangle_{A,B}(X) = 0 \Longrightarrow \triangle_{A_1,B_1}(X_1) = 0 \Longrightarrow \triangle_{A_1^*,B_1^*}(X_1) = 0 \Longrightarrow \triangle_{A^*,B^*}(X) = 0$. □

Remark 3.7. (i). Recall that $\delta_{A,B}^{-1}(0) \subseteq \delta_{A^*,B^*}^{-1}(0)$ for dominant A and (M-hyponormal, indeed) class $\mathcal{Y}$ operators B^* [1, 20]. Does $\triangle_{A,B}^{-1}(0) \subseteq \triangle_{A^*,B^*}^{-1}(0)$ hold for dominant A and $B^* \in \mathcal{Y}$ (or, even, M-hyponormal B^*)? Theorem 3.6(ii) says that if $\triangle_{A,B}(X) = 0$ for some dominant operator $A \in B(\mathcal{H})$, $B^* \in B(\mathcal{K}) \cap \mathcal{Y}$ and $X \in B(\mathcal{K}, \mathcal{H})$ such that ($\overline{\text{ran}(X)}$ is invariant under A, $\overline{\text{ran}(X^*)}$ is invariant under B^* and) $A_1 = A|_{\overline{\text{ran}(X)}}$ is invertible, then $\triangle_{A^*,B^*}(X) = 0$. No hypothesis on the invertibility of A_1 is required in the case in which B^* is hyponormal: This follows from Proposition 3.4
(ii). Some of the implications above have been considered by the author in [2]; the current version of the results, along with being a generalization of the results, removes some of the ambiguity from the results (and the argument used to prove them), of [2].

4. Applications

An operator $T \in B(\mathcal{H})$ is:
p-hyponormal, $0 < p \leq 1$, if $|T^*|^{2p} \leq |T|^{2p}$ (a 1-hypononormal operator is hyponormal);
log-hyponormal if T is invertible and $\log|T^*| \leq \log|T|$;
class $\mathcal{A}(s,t)$ $(0 < s, t \leq 1)$ operator if $|T^*|^{2t} \leq (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{t+s}}$;
(k, p)-quasihyponormal for some integer $k \geq 1$ and $0 < p \leq 1$ if $T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \geq 0$.

It is well known that the *translation invariance property* fails for operators belonging to these classes; hence none of these classes of operators is a $\mathcal{P}(\mathcal{H})$ class. Letting T have the polar decomposition $T = U|T|$, let $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ denote the (first) Aluthge transform of T. The operator T is said to be w-hyponormal if $|\widetilde{T}^*| \leq |T| \leq |\widetilde{T}|$. Recall, [8], that $\mathcal{A}(s,t) \subseteq \mathcal{A}(\alpha, \beta)$ for all $0 < s \leq \alpha$ and $0 < t \leq \beta$, an $\mathcal{A}(\frac{1}{2}, \frac{1}{2})$ operator is w-hyponormal, an $\mathcal{A}(1,1)$ operator is a class $\mathcal{A}$ operator (i.e., an operator such that $|A|^2 \leq |A^2|$), operators $T \in \mathcal{A}(s,t)$ for $0 < s, t \leq \frac{1}{2}$ are w-hyponormal operators and, for operators $T \in \mathcal{A}(s,t)$ for $\frac{1}{2} < s, t \leq 1$, T^2 is w-hyponormal. The following facts about these classes of operators are either well known or easily proved:

(a) The eigenvalues of a p-hyponormal and log-hyponormal operators are normal (i.e., the corresponding eigenspaces are reducing), and the non-zero eigenvalues of $\mathcal{A}(s,t)$ and (k,p)-quasihyponormal operators are normal;

(b) A (k,p)-quasihyponormal operator with dense range is p-hyponormal. The restriction $T|_M$ to an invariant subspace M of an operator T in any one of the above defined classes of operators is again an element of the class.

(c) Let $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, the first Aluthge transform of $T \in B(\mathcal{H})$, have the polar decomposition $\widetilde{T} = V|\widetilde{T}|$. If T is p-hyponormal or log-hyponormal or w-hyponormal or a (k,p)-quasihyponormal operator with dense range, then the second Aluthge transform $\widetilde{\widetilde{T}} = |\widetilde{T}|^{\frac{1}{2}}V|\widetilde{T}|^{\frac{1}{2}}$ of T is a hyponormal operator which satisfies the property that $\widetilde{\widetilde{T}}$ is normal if and only if T is normal.

Assume that $A \in B(\mathcal{H})$ and $B^* \in B(\mathcal{K})$ are such that $A^{-1}(0) \subseteq A^{*-1}(0)$ and $B^{*-1}(0) \subseteq B^{-1}(0)$. If $X \in B(\mathcal{K},\mathcal{H})$ is a quasi-affinity such that $\delta_{A,B}(X) = 0$, then $0 \in \sigma_p(A) \iff 0 \in \sigma_p(B^*)$, and hence if $0 \notin \sigma_p(B^*)$ or $0 \notin \sigma_p(A)$ then A, B^* are quasi-affinities. If, instead, $0 \in \sigma_p(A)$ or $0 \in \sigma_p(B^*)$, then A and B^* have direct sum decompositions $A = 0 \oplus A_2$ and $B^* = 0 \oplus B_2^*$ (such that A_2 and B_2^* are quasi-affinities). Letting X have the corresponding matrix representation $X = [X_{ij}]_{i,j=1}^2$, it is then seen seen that $A_2X_{22} = X_{22}B_2$, where X_{22} is a quasi-affinity. Since $\delta_{A,B}(X) = 0$ implies $\delta_{A^*,B^*}(X) = 0$ if and only if $\delta_{A_2,B_2}(X_{22}) = 0$ implies $\delta_{A_2^*,B_2^*}(X_{22}) = 0$, in proving $\delta_{A,B}(X) = 0 \Longrightarrow \delta_{A^*,B^*}(X) = 0$ we may as well assume that A, B are quasi-affinities. Now if $\tilde{\tilde{A}}$ is hyponormal and $B^* \in \mathcal{Y}$ (resp., $\tilde{\tilde{B^*}}$ is hyponormal), then $\tilde{\tilde{A}}Y = YB$ (resp., $\tilde{\tilde{A}}Y = Y\tilde{\tilde{B^*}}^*$), where Y is the quasi-affinity $Y = |\tilde{A}|^{\frac{1}{2}}|A|^{\frac{1}{2}}X$ (resp., $Y = |\tilde{A}|^{\frac{1}{2}}|A|^{\frac{1}{2}}X|B^*|^{\frac{1}{2}}|\tilde{B^*}|^{\frac{1}{2}}$). Applying Theorem 3.5(i), it follows that B is normal (resp., $\tilde{\tilde{B^*}}^*$ is normal). Again, considering $B^*Y^* = Y^*\tilde{\tilde{A}}^*$ (resp., $\tilde{\tilde{B^*}}Y^* = Y^*\tilde{\tilde{A}}^*$ with B normal and $\tilde{\tilde{A}}$ hyponormal, Theorem 3.5 implies that $\tilde{\tilde{A}}^*$ is normal. Hence $\tilde{\tilde{A}}$ and B (resp., $\tilde{\tilde{A}}$ and $\tilde{\tilde{B^*}}^*$) are unitarily equivalent normal operators. *Let, for convenience, $\mathcal{Q}$ denote the class of (Hilbert space) operators T such that T is either p-hyponormal, or log-hyponormal, or w-hyponormal with $T^{-1}(0) \subseteq T^{*-1}(0)$, or (k, p)-quasihyponormal with $T^{-1}(0) \subseteq T^{*-1}(0)$.*

Lemma 4.1. *If $\delta_{A,B}(X) = 0$ either (i) for an operator $A \in B(\mathcal{H}) \cap \mathcal{Q}$, operator $B^* \in B(\mathcal{K}) \cap \mathcal{Y}$ or $B(\mathcal{K}) \cap \mathcal{Q}$, and a quasi-affinity $X \in B(\mathcal{K},\mathcal{H})$ or (ii) for operators $A \in B(\mathcal{H}) \cap \mathcal{A}(s,t)$ and $B^* \in B(\mathcal{K}) \cap \mathcal{A}(s,t)$, $\frac{1}{2} < s,t \leq 1$, and a quasi-affinity $X \in B(\mathcal{K},\mathcal{H})$, then $\delta_{A^*,B^*}(X) = 0$.*

Proof. If $A \in B(\mathcal{H}) \cap \mathcal{Q}$, then (it follows from the argument above that $\tilde{\tilde{A}}$ and B, or $\tilde{\tilde{A}}$ and $\tilde{\tilde{B^*}}$, are normal, and hence) A and B are normal. This implies $\delta_{A^*,B^*}(X) = 0$. If, instead, $A, B^* \in \mathcal{A}(s,t)$ for $\frac{1}{2} < s,t \leq 1$, then consider $A^2X = XB^2$. The operators A^2 and B^{*2} being w-hyponormal, it is seen that $\tilde{\tilde{A^2}}$ and $\tilde{\tilde{B^{*2}}}^*$ are normal operators. Hence A^2 and B^{*2} are normal. Since every w-hyponormal operator is an $\mathcal{A}(1,1)$ operator, and since an $\mathcal{A}(1,1)$ operator such that A^2 is normal is normal [4, Lemma 2.1], A and B^* are normal. Once again, $\delta_{A^*,B^*}(X) = 0$. □

Remark 4.2. (i). Let $\mathsf{D} \subset \mathcal{P}_A(\mathcal{H})$ denote operators T in $B(\mathcal{H})$ which are dominant. Recall from [17] that $\delta_{A,B}^{-1}(0) \subseteq \delta_{A^*,B^*}^{-1}(0)$ for operators $A \in \mathsf{D}$ and normal operators $B^* \in B(\mathcal{K})$. Hence the hypothesis that $A \in B(\mathcal{H}) \cap \mathcal{Q}$ in Lemma 4.1(i) may be replaced by the hypothesis that $A \in \mathsf{D}$. Furthermore, restricting ourselves to only those $A \in \mathsf{D}$ for which $A^2 \in \mathsf{D}$, the hypothesis $A \in B(\mathcal{H}) \cap \mathcal{A}(s,t)$ may be replaced in Lemma 4.1(ii) by $A \in \mathsf{D}$. We note here from [4, Lemma 2.1] that a dominant operator A such that A^2 is normal is normal.

(ii). The hypothesis $B^* \in B(\mathcal{H}) \cap \mathcal{A}(s,t)$, $\frac{1}{2} < s,t \leq 1$, of Lemma 4.1(ii) may be replaced by the hypothesis that both B^* and $B^{*2} \in B(\mathcal{K},\mathcal{H}) \cap \mathcal{Y}$. Then B^{*2} is normal, and it follows from the argument of the proof of [4, Lemma 2.1] that (since the restriction of a class $\mathcal{Y}$ operator to a closed invariant subspace is again class $\mathcal{Y}$, the operator) B^* is normal.

Let $d_{A,B}$ denote either of $\delta_{A,B}$ and $\triangle_{A,B}$. The following theorem shows that (if in considering the inclusion $\triangle_{A,B}^{-1}(0) \subseteq \triangle_{A^*,B^*}^{-1}(0)$ one restricts oneself to considering only those $X \in \triangle_{A,B}^{-1}(0)$ for which $\overline{\text{ran}(X)}$ is invariant for A and $\ker^{\perp}(X)$ is invariant for B^*, then) the two way implication $\delta_{A,B}^{-1}(0) \subseteq \delta_{A^*,B^*}^{-1}(0) \iff \triangle_{A,B}^{-1}(0) \subseteq \triangle_{A^*,B^*}^{-1}(0)$ is valid for a large number of the commonly considered classes of Hilbert space operators.

Theorem 4.3. *$d_{A,B}^{-1}(0) \subseteq d_{A^*,B^*}^{-1}(0)$ for all: (i)$A \in B(\mathcal{H}) \cap \mathcal{Q} \cup \mathsf{D}$ and $B^* \in B(\mathcal{K}) \cap \mathcal{Q}$; (ii) $A \in B(\mathcal{H}) \cap \mathcal{A}(s,t)$ and $B^* \in B(\mathcal{K}) \cap \mathcal{A}(s,t)$, $\frac{1}{2} < s,t \leq 1$.*

Proof. If $\delta_{A,B}(X) = 0$, then $\delta_{A_1,B_1}(X_1) = 0$, where X_1 is a quasi-affinity, and either $A_1, B_1^* \in \mathcal{Q}$, or $A_1 \in \mathsf{D}$ and $B_1^* \in \mathcal{Q}$, or $A_1, B_1^* \in \mathcal{A}(s,t)$ for $\frac{1}{2} < s,t \leq 1$. Lemma 4.1 and Remark 4.2(i) apply, and we conclude that $\delta_{A_1^*,B_1^*}(X_1) = 0$, and A_1 and B_1 are normal. Since normal parts of operators A and B^* reduce A and (resp.) B^*, $\delta_{A_1^*,B_1^*}(X_1) = 0$ implies $\delta_{A^*,B^*}(X) = 0$. Consider now $\triangle_{A,B}(X) = 0$ (where $A : \overline{\text{ran}}(X) \to \overline{\text{ran}}(X)$ and $B^* : \ker^{\perp}(X) \to \ker^{\perp}(X)$). Then $\triangle_{A_1,B_1}(X_1) = 0$ implies

$$\widetilde{\widetilde{A_1}} Y_1 \widetilde{\widetilde{B_1^*}}^* = Y_1, \quad Y_1 = |\widetilde{A_1}|^{\frac{1}{2}} |A_1|^{\frac{1}{2}} X_1 |B_1^*|^{\frac{1}{2}} |\widetilde{B_1^*}|^{\frac{1}{2}}$$

if $A, B^* \in \mathcal{Q}$,

$$A_1 Y_1 \widetilde{\widetilde{B_1^*}}^* = Y_1, \quad Y_1 = X_1 |B_1^*|^{\frac{1}{2}} |\widetilde{B_1^*}|^{\frac{1}{2}}$$

if $A \in \mathsf{D}$ and $B^* \in \mathcal{Q}$, and

$$\widetilde{\widetilde{A_1^2}} Y_1 (\widetilde{\widetilde{B_1^{*2}}})^* = Y_1, \quad Y_1 = |\widetilde{A_1^2}|^{\frac{1}{2}} |A_1^2|^{\frac{1}{2}} X_1 |B_1^{*2}|^{\frac{1}{2}} |\widetilde{B_1^{*2}}|^{\frac{1}{2}}$$

if $A, B^* \in \mathcal{A}(s,t)$ ($\frac{1}{2} < s, t \leq 1$). Since the operator $\widetilde{\widetilde{A_1}}$ and $\widetilde{\widetilde{B_1^*}}^*$ are hyponormal in the case in which $A_1, B_1^* \in \mathcal{Q}$, the operator A_1 is dominant and the operator $\widetilde{\widetilde{B_1^*}}^*$ is hyponormal in the case in which $A \in \mathsf{D}$ and $B^* \in \mathcal{Q}$, and the operators $\widetilde{\widetilde{A_1^2}}$ and $(\widetilde{\widetilde{B_1^{*2}}})^*$ are hyponormal in the case in which $A, B^* \in \mathcal{A}(s,t)$ with $\frac{1}{2} < s, t \leq 1$, Lemma 4.1, Remark 4.2(i) and Proposition 3.4 imply $\widetilde{\widetilde{A_1}}^* Y_1 \widetilde{\widetilde{B_1^*}} = Y_1$ (resp., $A_1^* Y_1 \widetilde{\widetilde{B_1^*}} = Y_1$, resp. $\widetilde{\widetilde{A_1^2}}^* Y_1 \widetilde{\widetilde{B_1^{*2}}} = Y_1$), and hence that A_1 and B_1^* are normal operators. This, as before, implies $\triangle_{A^*,B^*}(X) = 0$. □

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