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Spectral permanence II

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Abstract. "Spectral permanence" for homomorphisms $T : A \to B$ is extended from the obvious subsemigroups of invertibles and semi-invertibles to more or less arbitrary $H_X \subseteq X$, in particular when there is a "functorial" property $T(H_A) \subseteq H_B$.

1. Invertibility

Suppose A is a semigroup (with identity), more generally [2] an abstract category: then we can identify the *invertible group*

1.1 $$A^{-1} = \{x \in A : 1 \in Ax \cap xA\} .$$

Now if $T : A \to B$ is a (unital) homomorphism of semigroups then there is inclusion

1.2 $$T(A^{-1}) \subseteq B^{-1} \subseteq B ;$$

equivalently

1.3 $$A^{-1} \subseteq T^{-1}(B^{-1}) \subseteq A .$$

If there is equality in (1.3),

1.4 $$T^{-1}B^{-1} \subseteq A^{-1} ,$$

we shall say that the homomorphism T has the *Gelfand property*. In other terminology we may say that T "is a determinant", or alternatively "has spectral permanence". In the inclusion (1.2) the invertible group A^{-1} can be replaced by the *left invertibles*

1.5 $$A^{-1}_{left} = \{x \in A : 1 \in Ax\} ,$$

the *right invertibles*

1.6 $$A^{-1}_{right} = \{x \in A : 1 \in xA\} ,$$

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and the *generalized invertibles*

1.7 $$A^{\cap} = \{x \in A : x \in xAx\}\ .$$

The analogues of equality (1.4) may be described as *left*, *right* and *generalized* permanence. To further extend this idea we might replace the invertible group A^{-1} by some more or less arbitrary semigroup $H_A \subseteq A$; to be relevant we are likely to require inclusion

1.8 $$A^{-1} \subseteq H_A \subseteq A\ .$$

More subtle is to see that $H_B \subseteq B$ is in some sense consistent with $H_A \subseteq A$: we will ask that the passage from X to H_X is *functorial*. Specifically if $T : A \to B$ is a semigroup homomorphism there is to be an induced homomorphism $H_T : H_A \to H_B$ for which

1.9 $$(H)_{ST} = H_S H_T\ ;\ H_I = I\ .$$

What we require is inclusion

1.10 $$T(H_A) \subseteq H_B\ \ :$$

then $H_T = T_H = T : H_A \to H_B$ is the restriction. When $H_X = X^{-1}$ then H_T is a semigroup homomorphism between groups; when $H_X = X^{\cap}$, not itself a semigroup, we find $T(a'a) = (Ta')(Ta)$ whenever $\{a, a', a'a\} \subseteq A^{\cap}$.

2. Exactness

In fact the semigroup assumption for $H_A \subseteq A$ is unnecessarily restrictive: following Vladimir Müller [11],[6] we shall ask that $H_A \subseteq A$ is a *regularity*. Here we specialize to semigroups which are *rings*; more generally [1] *additive categories*. We shall describe the ordered pair $(c, a) \in A^2$ as a *chain* provided

2.1 $$ca = 0 \in A\ ,$$

and as *aplitting exact* [3],[7],[8] (whether or not it is a chain) provided

2.2 $$1 \in Ac + aA \subseteq A\ .$$

Evidently a ring homomorphism $T : A \to B$ sends chains $(c, a) \in A^2$ to chains $(Tc, Ta) \in B^2$, and splitting exact (c, a) to splitting exact (Tc, Ta). Notice that $(c, 0)$ is splitting exact iff $c \in A^{-1}_{left}$ is left invertible; dually $(0, a)$ is splitting exact iff $a \in A^{-1}_{right}$ is right invertible. Evidently there is now another kind of permanence in view: we shall say that $T : A \to B$ is *exactly permanent* if there is implication

2.3 $$1 \in B(Tc) + (Ta)B \subseteq B \Longrightarrow 1 \in Ac + aA \subseteq A\ .$$

Now we shall describe $H_A \subseteq A$ as [6],[7],[8] a *non commutative regularity* if, whenever $(c, a) \in A^2$ is splitting exact, there is implication

2.4 $$ca \in H_A \Longleftrightarrow \{a, c\} \subseteq H_A\ .$$

The implication (2.4) holds for each of the H of (1.1), (1.5), (1.6) and (1.7).

Alternatively we can consider the condition that

2.5 $$H_A \cdot_{com} H_A \subseteq H_A\ ,$$

where we write

2.6 $$K *_{com} L = \{k * j : jk = kj\}\ ;$$

when A is a ring we can do this separately for addition $* = +$ and for multiplication $* = \cdot$.

3. Weak exactness.

We shall describe [3],[7],[8] the ordered pair $(c, a) \in A^2$ as *weakly exact* if there is implication, for arbitrary $(u, v) \in A^2$,

3.1 $$cu = 0 = va \Longrightarrow vu = 0 .$$

For example $(c, 0)$ is weakly exact iff $c \in A$ is a *monomorphism* in the sense

3.2 $$cu = 0 \Longrightarrow u = 0 ;$$

when (3.2) holds we write

3.3 $$c \in A^o_{left} .$$

Dually $(0, a)$ is weakly exact iff $a \in A$ is an *epimorphism* in the sense

3.4 $$va = 0 \Longrightarrow v = 0 ;$$

when (3.4) holds we write

3.5 $$a \in A^o_{right} .$$

Evidently splitting exactness implies weak exactness; conversely weak exactness together with regularity implies splitting exactness; here "regularity" for $(c, a) \in A^2$ means

3.6 $$\{a, c\} \subseteq A^\cap .$$

In particular

3.7 $$A^o_{left} \cap A^\cap = A^{-1}_{left} \ ; \ A^o_{right} \cap A^\cap = A^{-1}_{right} .$$

With either $H_X = X^o_{left}$ or $H_X = X^o_{right}$ we do not in general get the functorial inclusion (1.10); however if the homomorphism $T : A \to B$ is one one we get in both cases the reverse, permanence, inclusion

3.8 $$T^{-1} H_B \subseteq H_A .$$

More generally (3.8) says that H_X is in a sense a "contravariant" functor: when $T : A \to B$ is one-one there is $H^T : H_B \to H_A$, where

3.9 $$a \in H_A \Longrightarrow H^T(Ta) = a .$$

4. Skew exactness

We call the pair $(c, a) \in A^2$ *left skew exact* if [4],[7] there is inclusion

4.1 $$a \in Aca ;$$

Evidently exactness and (left) skew exactness implies (left) invertibility:

4.2 $$(1 \in Ac + aA \ \& \ a \in Aca) \Longrightarrow 1 \in Aa ;$$

conversely left invertibility (1.5) for $a \in A$ implies the left hand side of (4.2) for $c = 1 \in A$. If $T : A \to B$ then (4.1) implies left skew exactness for $(Tc, Ta) \in B^2$, and we shall describe $T : A \to B$ as *left skew permanent* if there is implication

4.3 $$Ta \in BTcTa \Longrightarrow a \in Aca .$$

Dually we say that $(c,a) \in A^2$ is *right skew exact* if

4.4 $$a \in caA\ .$$

For "linear categories" A there is *linear exactness* defined for $(c,a) \in A^2$, where $a : X \to Y$ and $c : Y \to Z$, by the inclusion

4.5 $$c^{-1}(0) \subseteq a(X)\ ;$$

now *linear left skew exactness* says

4.6 $$c^{-1}(0) \cap a(X) = \{0\}\ ,$$

and *linear right skew exactness*

4.7 $$c^{-1}(0) + a(X) = Y\ .$$

Normed linear exactness for $(c,a) \in A^2$ says there are $k > 0$ and $h > 0$ for which

4.8 $$\|vu\| \leq k\|v\|\,\|cu\| + h\|va\|\,\|u\|\ ;$$

for the induced "strong monomorphisms" $c \in A^\bullet_{left}$ and "strong epimorphisms" $a \in A^\bullet_{right}$ there are $k > 0$ and $h > 0$ for which

4.9 $$\|u\| \leq k\|cu\|\ ;\ \|v\| \leq h\|va\|\ .$$

Skew exactness is here given by

4.10 $$\|a\| \leq k\|ca\|\ ;\ \|c\| \leq h\|ca\|\ .$$

When $T : A \to B$ is bounded below then (3.8) and (3.9) hold with $H_X = X^\bullet_{left}$ and with $H_X = X^\bullet_{right}$.

5. Composite permanence

If $T : A \to B$ and $S : B \to D$ with

5.1 $$T(H_A) \subseteq H_B\ ,\ S(H_B) \subseteq H_D\ ,$$

so that also

$$ST(H_A) \subseteq H_D\ ,$$

then if

5.2 $$T^{-1}H_B \subseteq H_A\ \&\ S^{-1}H_D \subseteq H_B\ ,$$

it also follows

5.3 $$(ST)^{-1}H_D \subseteq H_A\ ;$$

in turn (5.3) implies the first half of (5.2). In words "H permanence" for each of S and T implies "H permanence" for ST, which in turn implies "H permanence" for T . It is a nice problem to decide whether splitting exactness of the pair (S,T) of homomorphisms is enough, together with H permanence for ST, to ensure H permanence for S. The fact that permanence properties of a product ST are transmitted to the factor T guarantees that *left invertible* homomorphisms have all the permanence properties we can think of.

These conditions are valid for

5.4 $$H_X \in \{X^{-1}, X^{-1}_{left}, X^{-1}_{right}, X^\cap\}\ .$$

When the homomorphisms are one one we can add

5.5 $$H_X \in \{X^o_{left}, X^o_{right}\}\ ;$$

When the homomorphisms are bounded below we can also add

5.6 $$H_X \in \{X^\bullet_{left}, X^\bullet_{right}\}\ .$$

6. Spectral permanence

When the ring A is a (complex) linear algebra then we have the concept of *spectrum*:

6.1 $$\sigma(a) \equiv \sigma_A(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}\}\ ;$$

more generally H_A gives rise to

6.2 $$\omega = \varpi_H : a \mapsto \varpi_H(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin H_A\}\ .$$

Conversely a "spectrum" ω on A gives rise to a regularity

6.3 $$H = R^\omega = \{a \in A : 0 \notin \omega(a)\}\ .$$

Now if $T : A \to B$ is a linear algebra homomorphism then the fundamental inclusion (1.10) takes the form

6.4 $$\mathrm{AND}_{a\in A} : \varpi_B(Ta) \subseteq \varpi_A(a)\ ;$$

the spectral permanence condition (3.8) is the opposite inclusion

6.5 $$\mathrm{AND}_{a\in A} : \varpi_A(a) \subseteq \varpi_B(Ta)\ ;$$

Thus in particular, when $H_X = X^{-1}$ so that $\varpi_H = \sigma$ then "spectral permanence" is what it says on the tin:

$$\sigma_B(Ta) = \sigma_A(a)\ .$$

When the linear algebra homomorphism $T : A \to B$ is one one then (3.8) holds:

6.6 $$\pi^{left}_A(a) \subseteq \pi^{left}_B(Ta)\ ;\ \pi^{right}_A(a) \subseteq \pi^{right}_B(Ta)\ .$$

Here

6.7 $$H_A = A^o_{left} \Longrightarrow \varpi_H = \pi^{left}\ ;\ H_A = A^o_{right} \Longrightarrow \varpi_H = \pi^{right}\ .$$

Specializing further to Banach algebras, if $T : A \to B$ is bounded below then

6.8 $$\tau^{left}_A(a) \subseteq \tau^{left}_B(Ta) \subseteq \sigma^{left}_B(Ta) \subseteq \sigma^{left}_A(a)$$

and

6.9 $$\tau^{right}_A(a) \subseteq \tau^{right}_B(Ta) \subseteq \sigma^{right}_B(Ta) \subseteq \sigma^{right}_A(a)\ .$$

Here

6.10 $$H_A = A^\bullet_{left} \Longrightarrow \varpi_H = \tau^{left}\ ;\ H_A = A^\bullet_{right} \Longrightarrow \varpi_H = \tau^{right}\ .$$

Since also in Banach algebras

6.11 $$\partial\sigma^{left}(a) \subseteq \tau^{right}(a)\ ;\ \partial\sigma^{right}(a) \subseteq \tau^{left}(a)\ ,$$

it follows that if $T : A \to B$ is bounded below then

6.12 $$\partial\sigma_A(a) \subseteq \sigma_B(Ta) \subseteq \sigma_A(a) .$$

If we only assume that $T : A \to B$ is one one then we still get

6.13 $$\text{iso}\ \sigma_A(a) \subseteq \sigma_B(Ta) \subseteq \sigma_A(a) .$$

7. Invariant subspaces

Suppose $Y \subseteq X$ is a linear subspace, alternarively a closed linear subspace of a Banach space, with quotient $Z = X/Y$, and then write

7.1 $$B = L(X)\ ,\ D = L(Y)\ ,\ E = L(Z) \equiv L(X/Y)\ ,$$

(alternatively $B(X)$, $B(Y)$, $B(Z)$) and finally

7.2 $$A = B_Y \equiv \{a \in B : a(Y) \subseteq Y\}\ :$$

then there are homomorphisms

7.3 $$J : A \to B\ ,\ L : A \to D\ ,\ K : A \to E\ .$$

The *natural embedding* $J : a \mapsto a$ is one-one; $L : a \mapsto a_Y$ is the *restriction*, and then the *quotient* $K : a \mapsto a_{/Y}$ is onto. If $a \in A$ there is [1] implication

7.4 $$\big(L(a)\ \textit{one one}\ \&\ K(a)\ \textit{one one}\big) \Longrightarrow J(a)\ \textit{one one} \Longrightarrow L(a)\ \textit{one one}\ ;$$

7.5 $$\big(L(a)\ \textit{onto}\ \&\ K(a)\ \textit{onto}\big) \Longrightarrow J(a)\ \textit{onto} \Longrightarrow K(a)\ \textit{onto}\ ;$$

7.6 $$\big(J(a)\ \textit{one one}\ \&\ L(a)\ \textit{onto}\big) \Longrightarrow K(a)\ \textit{one one}\ ;$$

7.7 $$\big(J(a)\ \textit{onto}\ \&\ K(a)\ \textit{one one}\big) \Longrightarrow L(a)\ \textit{onto}\ .$$

It follows that [7] the conditions

7.8 $$J(a) \in B^{-1}\ ;\ L(a) \in D^{-1}\ ;\ K(a) \in E^{-1}$$

"form a democracy".

8. Hyperinvariant subspaces

With

8.1 $$A' = \{a \in B : \text{comm}(a)(Y) \subseteq Y\} = \{a \in B : \text{comm}(a) \subseteq A\}\ ,$$

8.2 $$A'' = \{a \in B : \text{comm}^2(a) \subseteq A\}\ ,$$

8.3 $$A''' = \{a \in B : a - \lambda \in B^{-1} \Longrightarrow (a - \lambda)^{-1} \in A\}\ ,$$

there is [1] inclusion

8.4 $$A' \subseteq A'' \subseteq A''' \subseteq A \ ;$$

each of these three inclusions is liable to be proper. Since

8.5 $$\mathrm{comm}^2(a) = \mathrm{comm}^2(a^{-1}) \ ,$$

the inclusion $A' \subseteq B$ has spectral permanence:

8.6 $$A' \cap B^{-1} \subseteq (A')^{-1} \ .$$

Of course the subset $A' \subseteq A \subseteq B$ is not in general a subring: indeed, since $B \in \{L(X), B(X)\}$ is *irreducible* there is implication

8.7 $$1 \in A' \Longrightarrow B \subseteq A \Longrightarrow Y \in \{O.X\} \ .$$

There is by definition spectral permanence for the inclusion $A''' \subseteq B$:

8.8 $$A''' \cap B^{-1} \subseteq (A''')^{-1} \ .$$

It is plausible that $A'' \subseteq B$ satisfy the commuting product condition (2.5), at least if there is inclusion

8.9 $$A''' \subseteq A'' \ .$$

Jerry Koliha has noticed [10] that (8.9) holds for finite dimensional X.

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