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Weak convergence theorems of modified two-step iterations for nearly asymptotically nonexpansive mappings

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Abstract. The purpose of this paper is to study modified two-step iteration process to converge to a common fixed point for two nearly asymptotically nonexpansive mappings in the framework of uniformly Banach spaces. Also we establish some weak convergence theorems for said mappings and iteration scheme under the following assumptions on the space (i) E satisfies the Opial condition (ii) E has the Fréchet differentiable norm and (iii) the dual E^* of E has the Kadec-Klee property.

1. Introduction

Let C be a nonempty subset of a Banach space E and $T\colon C \to C$ a nonlinear mapping. We denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. The mapping T is said to be Lipschitzian [1, 17] if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x,\ y \in C$.

A Lipschitzian mapping T is said to be uniformly k-Lipschitzian if $k_n = k$ for all $n \in \mathbb{N}$ and asymptotically nonexpansive [5] if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$.

It is easy to observe that every nonexpansive mapping T (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\ y \in C$) is asymptotically nonexpansive with constant sequence $\{1\}$ and every asymptotically nonexpansive mapping is uniformly k-Lipschitzian with $k = \sup_{n\in\mathbb{N}} k_n$.

In 2005, Sahu [17] introduced the class of nearly Lipschitzian mappings as an important generalization of the class of Lipschitzian mappings.

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Let C be a nonempty subset of a Banach space E and fix a sequence $\{a_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} a_n = 0$. A mapping T is said to be nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exist constants $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad k_n(\|x-y\| + a_n) \tag{1}$$

for all $x,\ y \in C$.

The infimum of constants k_n for which the above inequality holds is denoted by $\eta(T^n)$ and is called nearly Lipschitz constant.

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T^n)\}$ is said to be
(i) nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$;
(ii) nearly uniformly k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$;
(iii) nearly uniformly k-contraction if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Example 1.1. *(See [17]) Let $E = \mathbb{R}$, $C = [0,1]$ and $T\colon C \to C$ be a mapping defined by*

$$T(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}], \\ 0, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, T is discontinuous and a non-Lipschitzian mapping. However, it is a nearly nonexpansive mapping and hence nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\} = \{\frac{1}{2^n}, 1\}$. Indeed, for a sequence $\{a_n\}$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$, we have

$$\|Tx - Ty\| \leq \|x-y\| + a_1 \text{ for all } x,\ y \in C$$

and

$$\|T^n x - T^n y\| \leq \|x-y\| + a_n \text{ for all } x,\ y \in C \text{ and } n \geq 2,$$

since

$$T^n x = \frac{1}{2} \text{ for all } x \in [0,1] \text{ and } n \geq 2.$$

In 2007, Agarwal et al. [1] have studied modified S-iteration process defined as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \ n \geq 1 \end{aligned} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and they established some weak convergence theorems under additional conditions for nearly asymptotically nonexpansive mappings in the framework of uniformly convex Banach spaces.

The asymptotic fixed point theory has a fundamental role in nonlinear functional analysis (see, [2]). The theory has been studied by many authors (see, e.g., [7], [8], [11], [13], [24]) for various classes of nonlinear mappings (e.g., Lipschitzian, uniformly k-Lipschitzian and non-Lipschitzian mappings). A branch of this theory related to asymptotically nonexpansive mappings has been developed by many authors (see, e.g., [3], [5], [6], [9], [10], [12], [13], [15], [16], [18], [19], [21], [22]) in Banach spaces with suitable geometrical structure.

The purpose of this paper to modify iteration scheme (2) for two mappings and establish some weak convergence theorems of newly proposed iteration scheme for two nearly asymptotically nonexpansive mappings in the framework of uniformly convex Banach spaces. The iteration scheme is as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n &= (1-\beta_n)S^n x_n + \beta_n T^n x_n, \ n \geq 1 \end{aligned} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

If we put $\beta_n = 0$ for all $n \in \mathbb{N}$ and $S = I$ where I is the identity mapping, then iteration scheme (3) reduces to the modified Mann iteration scheme as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n x_n, \ n \geq 1 \end{aligned} \tag{4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

2. Preliminaries

For the sake of convenience, we restate the following concepts and results.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \ \|y\| = 1, \ \varepsilon = \|x-y\|\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

We recall the following:

Let $\mathcal{S} = \{x \in E : \ \|x\| \ = \ 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The space E has:

(i) G$\hat{a}$teaux differentiable norm if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in $\mathcal{S}$.

(ii) Fr$\acute{e}$chet differentiable norm [23] if for each x in $\mathcal{S}$, the above limit exists and is attained uniformly for y in $\mathcal{S}$ and in this case, it is also well-known that

$$\begin{aligned} \langle h, \ J(x)\rangle + \frac{1}{2}\|x\|^2 &\leq \frac{1}{2}\|x+h\|^2 \\ &\leq \langle h, \ J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|x\|) \qquad (*) \end{aligned}$$

for all x, $h \in E$, where J is the Fr$\acute{e}$chet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in E$, $\langle . . \rangle$ is the pairing between E and E^*, and b is an increasing function defined on $[0, \infty)$ such that $\lim_{t \to 0} \frac{b(t)}{t} = 0$.

(*iii*) Opial condition [14] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p (1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial condition.

A mapping $T \colon K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.

A Banach space E has the Kadec-Klee property [20] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Next we state the following useful lemmas to prove our main results.

Lemma 2.1. *([22]) Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + r_n, \ \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 2.2. *([18]) Let E be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = a$ hold for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.3. *([20]) Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and p, $q \in w_w(x_n)$ (where $w_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1 - t)p - q\|$ exists for all $t \in [0, 1]$. Then $p = q$.*

Lemma 2.4. *([20]) Let K be a nonempty convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing continuous convex function $\phi \colon [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for each Lipschitzian mapping $T \colon K \to K$ with the Lipschitz constant L,*

$$\|tTx + (1 - t)Ty - T(tx + (1 - t)y)\| \leq L\phi^{-1}\Big(\|x - y\| - \frac{1}{L}\|Tx - Ty\|\Big)$$

for all x, $y \in K$ and all $t \in [0, 1]$.

3. Main Results

In this section, we prove some weak convergence theorems of modified two-step iteration scheme for two nearly asymptotically nonexpansive mappings in the framework of uniformly convex Banach spaces. First, we shall need the following lemmas.

Lemma 3.1. *Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $S, T \colon C \to C$ be two nearly asymptotically nonexpansive mappings with sequences $\{a'_n, \eta(S^n)\}$ and $\{a''_n, \eta(T^n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} \Big((\eta(S^n)\eta(T^n))^2 - 1\Big) < \infty$. Let $\{x_n\}$ be the modified two-step iteration scheme defined by (3), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\delta, 1 - \delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0, 1)$. If $F = F(S) \cap F(T) \neq \emptyset$ and $q \in F$, then $\lim_{n\to\infty} \|x_n - q\|$ exists.*

Proof. Let $q \in F$. For the sake of convenience, set

$$B_n x = (1 - \beta_n)S^n x + \beta_n T^n x$$

and

$$H_n x = (1 - \alpha_n)T^n x + \alpha_n S^n B_n x.$$

Then $y_n = B_n x_n$ and $x_{n+1} = H_n x_n$. Moreover, it is clear that q is a fixed point of H_n for all n. Let $\eta = \sup_{n\in\mathbb{N}} \eta(S^n) \vee \sup_{n\in\mathbb{N}} \eta(T^n)$ and $a_n = \max\{a'_n, a''_n\}$ for all n.

Consider

$$\begin{aligned}
\|B_n x - B_n y\| &= \|((1-\beta_n)S^n x + \beta_n T^n x) - ((1-\beta_n)S^n y + \beta_n T^n y)\| \\
&= \|(1-\beta_n)(S^n x - S^n y) + \beta_n (T^n x - T^n y)\| \\
&\le (1-\beta_n)\eta(S^n)(\|x-y\| + a'_n) + \beta_n \eta(T^n)(\|x-y\| + a''_n) \\
&\le (1-\beta_n)\eta(S^n)(\|x-y\| + a_n) + \beta_n \eta(T^n)(\|x-y\| + a_n) \\
&\le (1-\beta_n)\eta(S^n)\eta(T^n)\|x-y\| + \beta_n \eta(S^n)\eta(T^n)\|x-y\| \\
&\quad +(1-\beta_n)a_n \eta(S^n)\eta(T^n) + \beta_n a_n \eta(S^n)\eta(T^n) \\
&= \eta(S^n)\eta(T^n)\|x-y\| + a_n \eta(S^n)\eta(T^n).
\end{aligned} \tag{5}$$

Choosing $x = x_n$ and $y = q$, we get

$$\|y_n - q\| \le \eta(S^n)\eta(T^n)\|x_n - q\| + a_n \eta(S^n)\eta(T^n). \tag{6}$$

Now, consider

$$\begin{aligned}
\|H_n x - H_n y\| &= \|((1-\alpha_n)T^n x + \alpha_n S^n B_n x) - ((1-\alpha_n)T^n y + \alpha_n S^n B_n y)\| \\
&= \|(1-\alpha_n)(T^n x - T^n y) + \alpha_n (S^n B_n x - S^n B_n y)\| \\
&\le (1-\alpha_n)\eta(T^n)(\|x-y\| + a''_n) + \alpha_n \eta(S^n)(\|B_n x - B_n y\| + a'_n) \\
&\le (1-\alpha_n)\eta(T^n)(\|x-y\| + a_n) + \alpha_n \eta(S^n)(\|B_n x - B_n y\| + a_n) \\
&\le (1-\alpha_n)\eta(T^n)\|x-y\| + \alpha_n \eta(S^n)\|B_n x - B_n y\| \\
&\quad +(1-\alpha_n)a_n \eta(T^n) + \alpha_n a_n \eta(S^n).
\end{aligned} \tag{7}$$

Now using (5) in (7), we get

$$\begin{aligned}
\|H_n x - H_n y\| &\le (1-\alpha_n)\eta(T^n)\|x-y\| + \alpha_n \eta(S^n)[\eta(S^n)\eta(T^n)\|x-y\| \\
&\quad +a_n \eta(S^n)\eta(T^n)] + (1-\alpha_n)a_n \eta(T^n) + \alpha_n a_n \eta(S^n) \\
&\le (1-\alpha_n)(\eta(T^n)\eta(S^n))^2\|x-y\| + \alpha_n(\eta(T^n)\eta(S^n))^2\|x-y\| \\
&\quad +(1-\alpha_n+\alpha_n)a_n \eta(T^n)\eta(S^n) + \alpha_n a_n (\eta(T^n)\eta(S^n))^2 \\
&\le (\eta(T^n)\eta(S^n))^2\|x-y\| + a_n \eta(T^n)\eta(S^n) \\
&\quad +a_n(\eta(T^n)\eta(S^n))^2 \\
&\le (\eta(T^n)\eta(S^n))^2\|x-y\| + a_n\eta^2 + a_n\eta^4 \\
&= \Big[1 + \Big((\eta(T^n)\eta(S^n))^2 - 1\Big)\Big]\|x-y\| + a_n\eta^2(1+\eta^2) \\
&= (1+\delta_n)\|x-y\| + \theta_n,
\end{aligned} \tag{8}$$

where $\delta_n = \Big((\eta(T^n)\eta(S^n))^2 - 1\Big)$ and $\theta_n = a_n\eta^2(1+\eta^2)$. Since by hypothesis $\sum_{n=1}^{\infty}\Big((\eta(S^n)\eta(T^n))^2 - 1\Big) < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$. It follows that $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}\theta_n < \infty$.

Choosing $x = x_n$ and $y = q$ in (8), we get

$$\|x_{n+1} - q\| = \|H_n x_n - q\| \le (1+\delta_n)\|x_n - q\| + \theta_n. \tag{9}$$

Applying Lemma 2.1 in (9), we have $\lim_{n\to\infty}\|x_n - q\|$ exists. □

Lemma 3.2. *Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $S, T\colon C \to C$ be two nearly asymptotically nonexpansive mappings with sequences $\{a'_n, \eta(S^n)\}$, $\{a''_n, \eta(T^n)\}$ and $F = F(S)\cap F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\Big((\eta(S^n)\eta(T^n))^2 - 1\Big) < \infty$. Let $\{x_n\}$ be the modified two-step* *iteration scheme defined by (3), where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[\delta, 1-\delta]$ *for all* $n \in \mathbb{N}$ *and for some* $\delta \in (0,1)$. *If* $\|x - Sx\| \le \|Tx - Sx\|$ *for all* $x \in C$, *then* $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$. Let $a_n = \max\{a'_n, a''_n\}$ for all n and set $\lim_{n\to\infty} \|x_n - q\| = c$. Then $c > 0$ otherwise there is nothing to prove.

Now (6) implies that

$$\limsup_{n\to\infty} \|y_n - q\| \le c. \tag{10}$$

Also

$$\begin{aligned} \|T^n x_n - q\| &\le \eta(T^n)\Big(\|x_n - q\| + a''_n\Big) \\ &\le \eta(T^n)\Big(\|x_n - q\| + a_n\Big), \end{aligned}$$

for all $n = 1, 2, \ldots$, and

$$\begin{aligned} \|S^n x_n - q\| &\le \eta(S^n)\Big(\|x_n - q\| + a'_n\Big) \\ &\le \eta(S^n)\Big(\|x_n - q\| + a_n\Big), \end{aligned}$$

for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty} \|T^n x_n - q\| \le c. \tag{11}$$

and

$$\limsup_{n\to\infty} \|S^n x_n - q\| \le c. \tag{12}$$

Next,

$$\begin{aligned} \|S^n y_n - q\| &\le \eta(S^n)\Big(\|y_n - q\| + a'_n\Big) \\ &\le \eta(S^n)\Big(\|y_n - q\| + a_n\Big) \end{aligned}$$

gives by virtue of (10) that

$$\limsup_{n\to\infty} \|S^n y_n - q\| \le c. \tag{13}$$

Since

$$c = \|x_{n+1} - q\| = \|(1-\alpha_n)(T^n x_n - q) + \alpha_n(S^n y_n - q)\|.$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|T^n x_n - S^n y_n\| = 0. \tag{14}$$

From (3) and (14), we have

$$\begin{aligned} \|x_{n+1} - T^n x_n\| &= \alpha_n \|S^n y_n - T^n x_n\| \\ &\le \|S^n y_n - T^n x_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{15}$$

Hence

$$\begin{aligned} \|x_{n+1} - S^n y_n\| &\le \|x_{n+1} - T^n x_n\| + \|T^n x_n - S^n y_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{16}$$

Now

$$\begin{aligned} \|x_{n+1} - q\| &\leq \|x_{n+1} - S^n y_n\| + \|S^n y_n - q| \\ &\leq \|x_{n+1} - S^n y_n\| + \eta(S^n)\Big(\|y_n - q| + a'_n\Big) \\ &\leq \|x_{n+1} - S^n y_n\| + \eta(S^n)\Big(\|y_n - q| + a_n\Big) \end{aligned} \tag{17}$$

which gives from (17) that

$$c \leq \liminf_{n\to\infty} \|y_n - q\|. \tag{18}$$

From (10) and (18), we obtain

$$c = \|y_n - q\| = \|(1 - \beta_n)(T^n x_n - q) + \beta_n(S^n x_n - q)\|.$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|T^n x_n - S^n x_n\| = 0. \tag{19}$$

Now

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - S^n x_n\| + \|S^n x_n - x_n\| \\ &\leq \|T^n x_n - S^n x_n\| + \|S^n x_n - T^n x_n\| \\ &= 2\|T^n x_n - S^n x_n\| \to 0 \text{ as } n \to \infty, \end{aligned} \tag{20}$$

and

$$\|S^n x_n - x_n\| \leq \|S^n x_n - T^n x_n\| + \|T^n x_n - x_n\|$$

by (19) and (20), we obtain

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0. \tag{21}$$

By (15) and (20), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{22}$$

Finally, we make use of the fact that every nearly asymptotically nonexpansive mapping is nearly k-Lipschitzian, we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| + \|T^{n+1} x_n - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + \eta(T^{n+1})\Big(\|x_{n+1} - x_n\| + a''_{n+1}\Big) \\ &\quad + k\,\|T^n x_n - x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + \eta(T^{n+1})\Big(\|x_{n+1} - x_n\| + a_{n+1}\Big) \\ &\quad + k\,\|T^n x_n - x_n\|. \end{aligned} \tag{23}$$

Using (20) and (22) in (23), we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{24}$$

Similarly

$$\begin{aligned}\|x_n - Sx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^{n+1}x_{n+1}\| \\ &\quad + \|S^{n+1}x_{n+1} - S^{n+1}x_n\| + \|S^{n+1}x_n - Sx_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^{n+1}x_{n+1}\| \\ &\quad + \eta(S^{n+1})\Big(\|x_{n+1} - x_n\| + a'_{n+1}\Big) \\ &\quad + k\,\|S^n x_n - x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^{n+1}x_{n+1}\| \\ &\quad + \eta(S^{n+1})\Big(\|x_{n+1} - x_n\| + a_{n+1}\Big) \\ &\quad + k\,\|S^n x_n - x_n\|. \end{aligned} \tag{25}$$

Using (21) and (22) in (25), we obtain

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{26}$$

This completes the proof. □

Lemma 3.3. *Assume that the conditions of Lemma 3.2 are satisfied. Then, for any $p_1,\ p_2 \in F$, $\lim_{n\to\infty}\langle x_n,\ J(p_1-p_2)\rangle$ exists; in particular, $\langle p-q,\ J(p_1-p_2)\rangle = 0$ for all $p,\ q \in w_w(x_n)$.*

Proof. Suppose that $x = p_1 - p_2$ with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in inequality (∗). Then, we get

$$\begin{aligned} & t\,\langle x_n - p_1,\ J(p_1 - p_2)\rangle + \frac{1}{2}\|p_1 - p_2\|^2 \\ &\leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\ &\leq t\,\langle x_n - p_1,\ J(p_1 - p_2)\rangle + \frac{1}{2}\|p_1 - p_2\|^2 \\ &\quad + b(t\|x_n - p_1\|). \end{aligned}$$

Since $\sup_{n\geq 1}\|x_n - p_1\| \leq K_1$ for some $K_1 > 0$, we have

$$\begin{aligned} & t\,\limsup_{n\to\infty}\langle x_n - p_1,\ J(p_1 - p_2)\rangle + \frac{1}{2}\|p_1 - p_2\|^2 \\ &\leq \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|^2 \\ &\leq t\,\liminf_{n\to\infty}\langle x_n - p_1,\ J(p_1 - p_2)\rangle + \frac{1}{2}\|p_1 - p_2\|^2 \\ &\quad + b(tK_1). \end{aligned}$$

That is,

$$\begin{aligned} & \limsup_{n\to\infty}\langle x_n - p_1,\ J(p_1 - p_2)\rangle \\ &\leq \liminf_{n\to\infty}\langle x_n - p_1,\ J(p_1 - p_2)\rangle + \frac{b(tK_1)}{tK_1}K_1. \end{aligned}$$

If $t \to 0$, then $\lim_{n\to\infty}\langle x_n - p_1,\ J(p_1 - p_2)\rangle$ exists for all $p_1,\ p_2 \in F$; in particular, we have $\langle p - q,\ J(p_1 - p_2)\rangle = 0$ for all $p,\ q \in w_w(x_n)$. This completes the proof. □

Theorem 3.4. *Let E be a uniformly convex Banach space satisfying Opial's condition and C be a nonempty closed convex subset of E. Let $S,\ T\colon C \to C$ be two nearly asymptotically nonexpansive mappings with sequences $\{a'_n, \eta(S^n)\}$, $\{a''_n, \eta(T^n)\}$ and $F = F(S) \cap F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\Big((\eta(S^n)\eta(T^n))^2 - 1\Big) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Let $\{x_n\}$ be the modified two-step iteration scheme defined by (3). If the mappings $I - S$ and $I - T$, where I denotes the identity mapping, are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T.*

Proof. Let $q \in F$, from Lemma 3.1 the sequence $\{\|x_n - q\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q^* \in C$. From Lemma 3.2, we have

$$\lim_{k\to\infty}\|x_{n_k} - Sx_{n_k}\| = 0 \text{ and } \lim_{k\to\infty}\|x_{n_k} - Tx_{n_k}\| = 0.$$

Since the mappings $I - S$ and $I - T$ are demiclosed at zero, therefore $Sq^* = q^*$ and $Tq^* = q^*$, which means $q^* \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to q^*. Suppose on contrary that there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $p^* \in C$ and $q^* \neq p^*$. Then by the same method as given above, we can also prove that $p^* \in F$. From Lemma 3.1, the limits $\lim_{n\to\infty}\|x_n - q^*\|$ and $\lim_{n\to\infty}\|x_n - p^*\|$ exist. By virtue of the Opial condition of E, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - q^*\| &= \lim_{n_k\to\infty}\|x_{n_k} - q^*\| \\
&< \lim_{n_k\to\infty}\|x_{n_k} - p^*\| \\
&= \lim_{n\to\infty}\|x_n - p^*\| \\
&= \lim_{n_j\to\infty}\|x_{n_j} - p^*\| \\
&< \lim_{n_j\to\infty}\|x_{n_j} - q^*\| \\
&= \lim_{n\to\infty}\|x_n - q^*\|
\end{aligned}$$

which is a contradiction so $q^* = p^*$. Thus $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T. This completes the proof. □

It is well known that there exist classes of uniformly convex Banach spaces with out the Opial condition (e.g., L_p spaces, $p \neq 2$). Therefore, Theorem 3.4 is not true for such Banach spaces. We now show that Theorem 3.4 is valid if the assumption that E satisfies the Opial condition is replaced by either (i) E has Fréchet differentiable norm or (ii) E^* has the Kadec-Klee property (KK-property).

Theorem 3.5. *Let E be a real uniformly convex Banach space which has a Fréchet differentiable norm and C be a nonempty closed convex subset of E. Let $S,\ T\colon C \to C$ be two nearly asymptotically nonexpansive mappings with sequences $\{a'_n, \eta(S^n)\}$ and $\{a''_n, \eta(T^n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\Big((\eta(S^n)\eta(T^n))^2 - 1\Big) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Let $\{x_n\}$ be the modified two-step iteration scheme defined by (3). If $F = F(S) \cap F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T.*

Proof. By Lemma 3.3, $\langle p - q,\ J(p_1 - p_2)\rangle = 0$ for all $p,\ q \in w_w(x_n)$. Therefore $\|q^* - p^*\|^2 = \langle q^* - p^*,\ J(q^* - p^*)\rangle = 0$ implies $q^* = p^*$. Consequently, $\{x_n\}$ converges weakly to a point in F. This completes the proof. □

Lemma 3.6. *Under the conditions of Lemma 3.2 and for any $p,\ q \in F$, $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - z\|$ exists for all $z \in F$ and therefore $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|tx_n + (1-t)p - q\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p-q\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q\|$ exists by Lemma 3.1. It, therefore, remains to prove the Lemma 3.6 for $t \in (0,1)$. For all $x \in C$, we define the mapping $H_n\colon C \to C$ by:

$$B_n x = (1-\beta_n)S^n x + \beta_n T^n x$$

and

$$H_n x = (1-\alpha_n)T^n x + \alpha_n S^n B_n x.$$

Then it follows that $x_{n+1} = H_n x_n$, $H_n p = p$ for all $p \in F$ and we have shown earlier in Lemma 3.1 that

$$\|H_n x - H_n y\| \leq L_n\|x-y\| + \theta_n \tag{27}$$

for all $x, y \in C$, where $L_n = 1+\delta_n$, $\delta_n = \Big((\eta(T^n)\eta(S^n))^2 - 1\Big)$ and $\theta_n = a_n\eta^2(1+\eta^2)$ with $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\theta_n < \infty$ and $L_n \to 1$ as $n \to \infty$. Setting

$$R_{n,m} = H_{n+m-1}H_{n+m-2}\ldots H_n,\ m \geq 1 \tag{28}$$

and

$$b_{n,m} = \|R_{n,m}(tx_n + (1-t)p) - (tR_{n,m}x_n + (1-t)R_{n,m}q)\|. \tag{29}$$

From (30) and (31), we have

$$\begin{aligned}
\|R_{n,m}x - R_{n,m}y\| &= \|H_{n+m-1}H_{n+m-2}\ldots H_n x - H_{n+m-1}H_{n+m-2}\ldots H_n y\| \\
&\leq L_{n+m-1}\|H_{n+m-2}\ldots H_n x - H_{n+m-2}\ldots H_n y\| + \theta_{n+m-1} \\
&\leq L_{n+m-1}L_{n+m-2}\|H_{n+m-3}\ldots H_n x - H_{n+m-3}\ldots H_n y\| \\
&\quad +\theta_{n+m-1} + \theta_{n+m-2} \\
&\ \vdots \\
&\leq \Big(\prod_{j=n}^{n+m-1} L_j\Big)\|x-y\| + \sum_{j=n}^{n+m-1}\theta_j \\
&= V_n\|x-y\| + \sum_{j=n}^{n+m-1}\theta_j
\end{aligned} \tag{30}$$

for all $x, y \in C$, where $V_n = \prod_{j=n}^{n+m-1} L_j$ and $R_{n,m}x_n = x_{n+m}$, $R_{n,m}p = p$ for all $p \in F$. Thus

$$\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\
&\leq b_{n,m} + \|R_{n,m}(tx_n + (1-t)p) - q\| \\
&\leq b_{n,m} + V_n a_n(t) + \sum_{j=n}^{n+m-1}\theta_j.
\end{aligned} \tag{31}$$

By using [[4], Theorem 2.3], we have

$$\begin{aligned}
b_{n,m} &\leq \varphi^{-1}\Big(\|x_n - u\| - \|R_{n,m}x_n - R_{n,m}u\|\Big) \\
&\leq \varphi^{-1}\Big(\|x_n - u\| - \|x_{n+m} - u + u - R_{n,m}u\|\Big) \\
&\leq \varphi^{-1}\Big(\|x_n - u\| - \Big(\|x_{n+m} - u\| - \|R_{n,m}u - u\|\Big)\Big)
\end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0, i.e., $b_{n,m} \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty} V_n = 1$ and $\sum_{n=1}^{\infty}\theta_n < \infty$, that is, $\theta_n \to 0$ as $n \to \infty$, therefore from (31), we have

$$\limsup_{n\to\infty} a_n(t) \leq \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} a_n(t) + 0 = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is, $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. This completes the proof. □

Now we prove a weak convergence theorem for the spaces whose dual have Kadec-Klee property (KK-property).

Theorem 3.7. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and C be a nonempty closed convex subset of E. Let $S, T\colon C \to C$ be two nearly asymptotically nonexpansive mappings with sequences $\{a'_n, \eta(S^n)\}$, $\{a''_n, \eta(T^n)\}$ and $F = F(S) \cap F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} \Big((\eta(S^n)\eta(T^n))^2 - 1\Big) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Let $\{x_n\}$ be the modified two-step iteration scheme defined by (3). If the mappings $I - S$ and $I - T$, where I denotes the identity mapping, are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T.*

Proof. By Lemma 3.1, we know that $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in C$. By Lemma 3.2, we have

$$\lim_{j\to\infty} \|x_{n_j} - Sx_{n_j}\| = 0 \text{ and } \lim_{j\to\infty} \|x_{n_j} - Tx_{n_j}\| = 0.$$

Since by hypothesis the mappings $I - S$ and $I - T$ are demiclosed at zero, therefore $Sp = p$ and $Tp = p$, which means $p \in F$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in C$. By the same method as above, we have $q \in F$ and $p, q \in w_w(x_n)$. By Lemma 3.6, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$ and so $p = q$ by Lemma 2.3. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof. □

4. Conclusion

In this paper, we study newly introduced modified two step iteration scheme which contains modified Mann iteration scheme and establish some weak convergence theorems using Opial's condition, Fréchet differentiable norm and the dual of the space has Kadec-Klee property (KK-property) for more general class of nonexpansive and asymptotically nonexpansive mappings. Our results extend and improve several results from the existing literature.

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