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An addendum to: Analytically Riesz operators and Weyl and Browder type theorems

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Abstract. In this note a characterization of anallytically Riesz operators is given. This work completes the article [1].

1. Introduction

Anallytically Riesz operators, i.e., bounded and linear maps T defined on a Banach space $\mathcal{X}$ such that there exists an analytical function f defined on a neighbourhood of the spectrum of T with the property that $f(T)$ is Riesz, were studied in [3]. In addition, several spectra and some spectral properties of this class of operators were studied in [1]. In particular, when instead of an analytical funcion there exists a polynomial $P \in \mathbb{C}[X]$ such that $P(T)$ is Riesz, T is said to be a polynomially Riesz operator. The structure of polynomially Riesz operators was studied in [2] (see also [4, Theorem 2.13]). To learn more about polynomially Riesz operators see for example [4] and its reference list.

After the publication of [1], a characterization of analytically Riesz operators was obtained. In fact, similar arguments to the ones in [1] prove that necessary and sufficient for T to be analytically Riesz is that there exist $\mathcal{X}_1$ and $\mathcal{X}_2$ two closed and complemented T-invatiant subspaces of X such that if $T = T_1 \oplus T_2$, then $T_1 \in \mathcal{L}(\mathcal{X}_1)$ is an arbitray operator and $\mathcal{X}_2$ is finite dimensional or $T_2 \in \mathcal{L}(\mathcal{X}_2)$ is a polynomially Riesz operator ($T_i = T\mid_{\mathcal{X}_i}$, $i = 1, 2$). In other words, an analytically Riesz operator is essentially the direct sum of an arbitray operator and a polynomially Riesz operator. The objective of this note is to present this characterization and some other related results. This note consists in a completation of [1].

2. Results

From now on $\mathcal{X}$ will denote an infinite dimensional complex Banach space, $\mathcal{L}(\mathcal{X})$ the algebra of all bounded and linear maps defined on and with values in $\mathcal{X}$ and $I \in \mathcal{L}(\mathcal{X})$ the identity map. If $T \in \mathcal{L}(\mathcal{X})$, then $N(T)$, $R(T)$ and $\sigma(T)$ will stand for the null space, the range and the spectrum of T, respectively. In addition, $\mathcal{K}(\mathcal{X}) \subset \mathcal{L}(\mathcal{X})$ will denote the closed ideal of compact operators defined on $\mathcal{X}$, $\mathcal{C}(\mathcal{X})$ the Calkin algebra of $\mathcal{X}$ and $\pi\colon \mathcal{L}(\mathcal{X}) \to \mathcal{C}(\mathcal{X})$ the quotient map.

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Recall that $T \in L(\mathcal{X})$ is said to be a *Fredholm* operator if $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim \mathcal{X}/R(T)$ are finite dimensional. In addition, $T \in \mathcal{L}(\mathcal{X})$ is said to be a *Riesz operator*, if $T - \lambda I$ is Fredholm for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The set of all Riesz operators defined on $\mathcal{X}$ will be denoted by $\mathcal{R}(\mathcal{X})$. More generally, T will be said to be an *analytically Riesz operator*, if there exists a holomorphic function f defined on an open neighbourhood of $\sigma(T)$ such that $f(T) \in \mathcal{R}(\mathcal{X})$ ($\mathcal{H}(\sigma(T))$ will denote the algebra of germs of analytic functions defined on open neighbourhoods of $\sigma(T)$). In particular, T will be said to be *polynomially Riesz*, if there exists $P \in C[X]$ such that $P(T) \in \mathcal{R}(\mathcal{X})$ (see [4]).

To prove the main result of this note, a preliminary result is needed.

Lemma 2.1. *Let $\mathcal{X}$ be an infinite dimensional complex Banach space and consider $T \in \mathcal{R}(\mathcal{X})$. Then, if $V \in \mathcal{L}(\mathcal{X})$ is such that $TV - VT \in \mathcal{K}(\mathcal{X})$, VT and $TV \in \mathcal{R}(\mathcal{X})$.*

Proof. Recall that necessary and sufficient for $T \in \mathcal{R}(\mathcal{X})$ is that $\pi(T) \in C(\mathcal{X})$ is quasinilpotent. Since $\pi(V)$ commutes with $\pi(T)$, it is not difficult to prove that $\pi(V)\pi(T) = \pi(T)\pi(V)$ is quasinilpotent. In particular, VT and $TV \in \mathcal{R}(\mathcal{X})$. □

Next a characterization of analytically Riesz operators is given. Note that the following notation will be used. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are two closed and complemented T-invariant subspaces of the Banach space $\mathcal{X}$ ($T \in \mathcal{L}(\mathcal{X})$), then T has the decomposition $T = T_1 \oplus T_2$, where $T_i = T\mid_{\mathcal{X}_i}$, $i = 1, 2$.

Theorem 2.2. *Let $\mathcal{X}$ be an infinite dimensional complex Banach space and consider $T \in \mathcal{L}(\mathcal{X})$. Then, the following statements are equivalent:*
(i) *The operator T is analytically Riesz.*
(ii) *There exist $\mathcal{X}_1$ and $\mathcal{X}_2$ two closed and complementd T-invariant subspaces of $\mathcal{X}$ with the property that, if $T = T_1 \oplus T_2$, then $T_1 \in \mathcal{L}(\mathcal{X}_1)$ is an arbitrary operator and either $\mathcal{X}_2$ is finite dimensional or $T_2 \in \mathcal{L}(\mathcal{X}_2)$ is polynomially Riesz.*

Proof. Suppose that statement (i) holds. Let $f \in \mathcal{H}(\sigma(T))$ be such that $f(T) \in \mathcal{R}(\mathcal{X})$. According to [3, Theorem 1], there are two closed disjoint sets S_1 and S_2 such that $\sigma(T) = S_1 \cup S_2$, f is locally zero at each point of S_1 but it is not at any point of S_2. In addition, if $\mathcal{X}_1$ and $\mathcal{X}_2$ are two closed and complemented T-invariant subspaces of $\mathcal{X}$ associated to the decompositon defined by S_1 and S_2, then $f(T_1) = 0$ and either $\mathcal{X}_2$ is a finite dimensional space or T_2 can be decomposed as a direct sum of operators (see the proof of [3, Theorem 1] for details).

Suppose then that $\dim \mathcal{X}_2$ is infinite. Let $f_2 \in \mathcal{H}(\sigma(T_2))$ be defined as the restriction of f to an open set containing S_2 but disjoint to S_1 (recall that $\sigma(T_1) = S_1$ and $\sigma(T_2) = S_2$). Since $f(T) = 0 \oplus f_2(T_2)$, $T_2 \in \mathcal{L}(\mathcal{X}_2)$ is analytically Riesz. Since f is analytically zero at no point of S_2, the set $f_2^{-1}(0) \cap S_2$ is finite. In particular, there exist $n \in \mathbb{N}$, $k_i \in \mathbb{N}$ and $\lambda_i \in f_2^{-1}(0) \cap S_2$ ($i = 1, \dots, n$) such that $f_2(z) = (z - \lambda_1)^{k_1} \dots (z - \lambda_n)^{k_n} g(z)$, where $g \in \mathcal{H}(\sigma(T_2))$ is such that $g(z) \neq 0$ for all $z \in \sigma(T_2)$.

Now, since $(T_2 - \lambda_1)^{k_1} \dots (T_2 - \lambda_n)^{k_n} g(T_2) = f_2(T_2) \in \mathcal{R}(\mathcal{X}_2)$, and $g(T_2) \in \mathcal{L}(\mathcal{X}_2)$ is an invertible operator, which commutes with $(T_2 - \lambda_1)^{k_1} \dots (T_2 - \lambda_n)^{k_n}$, according to Lemma 2.1, $(T_2 - \lambda_1)^{k_1} \dots (T_2 - \lambda_n)^{k_n} \in \mathcal{R}(\mathcal{X}_2)$. Consequently, T_2 is polinomially Riesz.

To prove the converse, consider two closed and disjoint sets $S_1, S_2 \subset \mathbb{C}$ such that $S_1 \cup S_2 = \sigma(T)$, $\sigma(T_i) = S_i$, $i = 1, 2$. Let U_i be two disjoint open sets such that $S_i \subset U_i$, $i = 1, 2$, and define $f \in \mathcal{H}(\sigma(T))$ as follows: $f\mid_{U_1} = 0$ and when $\mathcal{X}_2$ is finite dimensional, $f\mid_{U_2} = z$, while when $\dim \mathcal{X}_2$ is infinite, $f\mid_{U_2} = P$, where $P \in \mathbb{C}[X]$ is such that $P(T) \in \mathcal{R}(\mathcal{X})$. Therefore, $f(T) = 0 \oplus T_2$ ($\dim \mathcal{X}_2 < \infty$) or $f(T) = 0 \oplus P(T)$ ($\dim \mathcal{X}_2$ infinite). In both cases T is analytically Riesz. □

Recall that polynomially Riesz operators were characterized in [2, 4] (see in paricular [4, Theorems 2.2, 2.3, 2.13]).

Corollary 2.3. *Let $\mathcal{X}$ be an infinite dimensional complex Banach space and consider $T \in \mathcal{L}(\mathcal{X})$. Suppose that there is $f \in \mathcal{H}(\sigma(T))$ such that $f(T) \in \mathcal{R}(\mathcal{X})$ and for each $x \in \sigma(T)$, f is not locally zero at x. Then, T is polynomially Riesz.*

Proof. Using the same notation as in Theorem 2.2, according to the proof of this Theorem and [3, Theorem 1], $S_1 = \emptyset$ and $\mathcal{X} = \mathcal{X}_2$. Since dim $\mathcal{X}_2$ is infinite, T is polynomially Riesz. $\square$

Note that according to Corollary 2.3, the operators considered in [1] are polynomially Riesz. Compare with [4].

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