An addendum to: Analytically Riesz operators and Weyl and Browder type theorems

Enrico Boasso

Via Cristoforo Cancellieri 2, 34137 Trieste-TS, Italy

Abstract. In this note a characterization of analytically Riesz operators is given. This work completes the article [1].

1. Introduction

Anallytically Riesz operators, i.e., bounded and linear maps $T$ defined on a Banach space $X$ such that there exists an analytical function $f$ defined on a neighbourhood of the spectrum of $T$ with the property that $f(T)$ is Riesz, were studied in [3]. In addition, several spectra and some spectral properties of this class of operators were studied in [1]. In particular, when instead of an analytical function there exists a polynomial $P \in \mathbb{C}[X]$ such that $P(T)$ is Riesz, $T$ is said to be a polynomially Riesz operator. The structure of polynomially Riesz operators was studied in [2] (see also [4, Theorem 2.13]). To learn more about polynomially Riesz operators see for example [4] and its reference list.

After the publication of [1], a characterization of analytically Riesz operators was obtained. In fact, similar arguments to the ones in [1] prove that necessary and sufficient for $T$ to be analytically Riesz is that there exist $X_1$ and $X_2$ two closed and complemented $T$-invariant subspaces of $X$ such that if $T = T_1 \oplus T_2$, then $T_1 \in \mathcal{L}(X_1)$ is an arbitrary operator and $X_2$ is finite dimensional or $T_2 \in \mathcal{L}(X_2)$ is a polynomially Riesz operator ($T_i = T \big|_{X_i}$, $i = 1, 2$). In other words, an analytically Riesz operator is essentially the direct sum of an arbitrary operator and a polynomially Riesz operator. The objective of this note is to present this characterization and some other related results. This note consists in a completion of [1].

2. Results

Fron now on $X$ will denote an infinite dimensional complex Banach space, $\mathcal{L}(X)$ the algebra of all bounded and linear maps defined on and with values in $X$ and $I \in \mathcal{L}(X)$ the identity map. If $T \in \mathcal{L}(X)$, then $N(T)$, $R(T)$ and $\sigma(T)$ will stand for the null space, the range and the spectrum of $T$, respectively. In addition, $\mathcal{K}(X) \subset \mathcal{L}(X)$ will denote the closed ideal of compact operators defined on $X$, $C(X)$ the Calkin algebra of $X$ and $\pi: \mathcal{L}(X) \to C(X)$ the quotient map.
Recall that $T \in L(X)$ is said to be a Fredholm operator if $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$ are finite dimensional. In addition, $T \in \mathcal{L}(X)$ is said to be a Riesz operator, if $T - \lambda I$ is Fredholm for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The set of all Riesz operators defined on $X$ will be denoted by $\mathcal{R}(X)$. More generally, $T$ will be said to be an analytically Riesz operator, if there exists a holomorphic function $f$ defined on an open neighbourhood of $\sigma(T)$ such that $f(T) \in \mathcal{R}(X)$ ($\mathcal{H}(\sigma(T))$) will denote the algebra of germs of analytic functions defined on open neighbourhoods of $\sigma(T)$. In particular, $T$ will be said to be polynomially Riesz, if there exists $P \in \mathbb{C}[X]$ such that $P(T) \in \mathcal{R}(X)$ (see [4]).

To prove the main result of this note, a preliminary result is needed.

**Lemma 2.1.** Let $X$ be an infinite dimensional complex Banach space and consider $T \in \mathcal{R}(X)$. Then, if $V \in \mathcal{L}(X)$ is such that $TV - VT \in \mathcal{K}(X)$, $VT$ and $TV \in \mathcal{R}(X)$.

**Proof.** Recall that necessary and sufficient for $T \in \mathcal{R}(X)$ is that $\pi(T) \in \mathbb{C}(X)$ is quasinilpotent. Since $\pi(V)$ commutes with $\pi(T)$, it is not difficult to prove that $\pi(V)\pi(T) = \pi(T)\pi(V)$ is quasinilpotent. In particular, $VT$ and $TV \in \mathcal{R}(X)$. \(\square\)

Next a characterization of analytically Riesz operators is given. Note that the following notation will be used. If $X_1$ and $X_2$ are two closed and complemented $T$-invariant subspaces of the Banach space $X$ ($T \in \mathcal{L}(X)$), then $T$ has the decomposition $T = T_1 \oplus T_2$, where $T_i = T |_{X_i}, i = 1, 2$.

**Theorem 2.2.** Let $X$ be an infinite dimensional complex Banach space and consider $T \in \mathcal{L}(X)$. Then, the following statements are equivalent:

(i) The operator $T$ is analytically Riesz.

(ii) There exist $X_1$ and $X_2$ two closed and complemented $T$-invariant subspaces of $X$ with the property that, if $T = T_1 \oplus T_2$, then $T_1 \in \mathcal{L}(X_1)$ is an arbitrary operator and either $X_2$ is finite dimensional or $T_2 \in \mathcal{L}(X_2)$ is polynomially Riesz.

**Proof.** Suppose that statement (i) holds. Let $f \in \mathcal{H}(\sigma(T))$ be such that $f(T) \in \mathcal{R}(X)$. According to [3, Theorem 1], there are two closed disjoint sets $S_1$ and $S_2$ such that $\sigma(T) = S_1 \cup S_2$, $f$ is locally zero at each point of $S_1$ but it is not at any point of $S_2$. In addition, if $X_1$ and $X_2$ are two closed and complemented $T$-invariant subspaces of $X$ associated to the decompositon defined by $S_1$ and $S_2$, then $f(T_{1}) = 0$ and either $X_2$ is a finite dimensional space or $T_2 \in \mathcal{L}(X_2)$ can be decomposed as a direct sum of operators (see the proof of [3, Theorem 1] for details).

Suppose then that $\dim X_2$ is infinite. Let $f_2 \in \mathcal{H}(\sigma(T_2))$ be defined as the restriction of $f$ to an open set containing $S_2$ but disjoint to $S_1$ (recall that $\sigma(T_1) = S_1$ and $\sigma(T_2) = S_2$). Since $f(T) = 0 \oplus f_2(T_2), T_2 \in \mathcal{L}(X_2)$ is analytically Riesz. Since $f$ is analytically zero at no point of $S_2$, the set $f_2^{-1}(0) \cap S_2$ is finite. In particular, there exist $n \in \mathbb{N}, k_i \in \mathbb{N}$ and $\lambda_i \in f_2^{-1}(0) \cap S_2 (i = 1, \ldots, n)$ such that $f_2(z) = (z - \lambda_1)^{k_1} \cdots (z - \lambda_n)^{k_n}g(z)$, where $g \in \mathcal{H}(\sigma(T_2))$ is such that $g(z) \neq 0$ for all $z \in \sigma(T_2)$.

Now, since $(T_2 - \lambda_1)^{k_1} \cdots (T_2 - \lambda_n)^{k_n}g(T_2) = f_2(T_2) \in \mathcal{R}(X_2)$, and $g(T_2) \in \mathcal{L}(X_2)$ is an invertible operator, which commutes with $(T_2 - \lambda_1)^{k_1} \cdots (T_2 - \lambda_n)^{k_n}$, according to Lemma 2.1, $(T_2 - \lambda_1)^{k_1} \cdots (T_2 - \lambda_n)^{k_n} \in \mathcal{R}(X_2)$. Consequently, $T_2$ is polynomially Riesz.

To prove the converse, consider two closed and disjoint sets $S_1, S_2 \subset \mathbb{C}$ such that $S_1 \cup S_2 = \sigma(T), \sigma(T_2) = S_i, i = 1, 2$. Let $U_i$ be two disjoint open sets such that $S_i \subset U_i, i = 1, 2$, and define $f \in \mathcal{H}(\sigma(T))$ as follows: $f |_{U_1} = 0$ and when $\dim X_2$ is finite dimensional, $f |_{U_2} = z$, while when $\dim X_2$ is infinite, $f |_{U_2} = P$, where $P \in \mathbb{C}[X]$ is such that $P(T) \in \mathcal{R}(X)$. Therefore, $f(T) = 0 \oplus T_2 (\dim X_2 < \infty)$ or $f(T) = 0 \oplus P(T) (\dim X_2 \infty)$. In both cases $T$ is analytically Riesz. \(\square\)

Recall that polynomially Riesz operators were characterized in [2, 4] (see in particular [4, Theorems 2.2, 2.3, 2.13]).

**Corollary 2.3.** Let $X$ be an infinite dimensional complex Banach space and consider $T \in \mathcal{L}(X)$. Suppose that there is $f \in \mathcal{H}(\sigma(T))$ such that $f(T) \in \mathcal{R}(X)$ and for each $x \in \sigma(T)$, $f$ is not locally zero at $x$. Then, $T$ is polynomially Riesz.
Proof. Using the same notation as in Theorem 2.2, according to the proof of this Theorem and [3, Theorem 1], $S_1 = \emptyset$ and $\mathcal{X} = \mathcal{X}_2$. Since $\dim \mathcal{X}_2$ is infinite, $T$ is polynomially Riesz.

Note that according to Corollary 2.3, the operators considered in [1] are polynomially Riesz. Compare with [4].

References