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Weighted sharp maximal function inequalities and boundedness of multilinear singular integral operator with variable Calderón-Zygmund kernel

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Abstract. In this paper, we establish the weighted sharp maximal function inequalities for the multilinear operator associated to the singular integral operator with variable Calderón-Zygmund kernel. As an application, we obtain the boundedness of the operator on weighted Lebesgue spaces.

1. Introduction

As the development of singular integral operators(see [9][20][21]), their commutators and multilinear operators have been well studied. In [6][18][19], the authors proved that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [3]) proved a similar result when singular integral operators are replaced by the fractional integral operators. In [12][17], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1][11], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained (also see [10]). In [4][5], the authors studied some multilinear singular integral operators as following (also see [7]):

$$T^b(f)(x) = \int_{R^n} \frac{R_{m+1}(b; x, y)}{|x-y|^m} K(x, y) f(y) dy,$$

and obtained some variant sharp function estimates and boundedness of the multilinear operators if $D^\alpha b \in BMO(R^n)$ for all α with $|\alpha| = m$. In [2], Calderón and Zygmund introduced some singular integral operators with variable kernel and discussed their boundedness. In [13-15][22], the authors obtained the boundedness for the commutators and multilinear operators generated by the singular integral operators with variable kernel and BMO functions. In [16], the authors prove the boundedness for the multilinear oscillatory singular integral operators generated by the operators and BMO functions. Motivated by these, in this paper, we will study the multilinear operator generated by the singular integral operator with

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variable Calderón-Zygmund kernel and the weighted Lipschitz and BMO functions, that is $D^\alpha b \in BMO(w)$ or $D^\alpha b \in Lip_\beta(w)$ for all α with $|\alpha| = m$.

2. Preliminaries

In this paper, we will study some singular integral operators as following (see [2]).

Definition 1. Let $K(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \to R$. K is said to be a Calderón-Zygmund kernel if
(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;
(b) Ω is homogeneous of degree zero;
(c) $\int_\Sigma \Omega(x)x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n.

Definition 2. Let $K(x, y) = \Omega(x, y)/|y|^n : R^n \times (R^n \setminus \{0\}) \to R$. K is said to be a variable Calderón-Zygmund kernel if
(d) $K(x, \cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;
(e) $\max_{|\gamma|\le 2n} \left\|\frac{\partial^{|\gamma|}}{\partial^\gamma y}\Omega(x, y)\right\|_{L^\infty(R^n\times\Sigma)} = L < \infty$.

Moreover, let m be the positive integer and b be the function on R^n. Set

$$R_{m+1}(b; x, y) = b(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha b(y)(x - y)^\alpha.$$

Let T be the singular integral operator with variable Calderón-Zygmund kernel as

$$T(f)(x) = \int_{R^n} K(x, x - y)f(y)dy,$$

where $K(x, x - y) = \frac{\Omega(x,x-y)}{|x-y|^n}$ and that $\Omega(x, y)/|y|^n$ is a variable Calderón-Zygmund kernel. The multilinear operator related to the operator T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{R_{m+1}(b; x, y)}{|x - y|^m} K(x, x - y)f(y)dy.$$

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the multilinear operator T^b if $m = 0$. The multilinear operator T^b are the non-trivial generalizations of the commutator. It is well known that commutators and multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [4][5][7]). The main purpose of this paper is to prove the sharp maximal inequalities for the multilinear operator T^b. As the application, we obtain the weighted L^p-boundedness for the multilinear operator T^b.

Now, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a nonnegative integrable function ω, let $\omega(Q) = \int_Q \omega(x)dx$ and $\omega_Q = |Q|^{-1} \int_Q \omega(x)dx$.

For any locally integrable function f, the sharp maximal function of f is defined by

$$M^\#(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy.$$

It is well-known that (see [9])

$$M^\#(f)(x) \approx \sup_{Q\ni x} \inf_{c\in C} \frac{1}{|Q|} \int_Q |f(y) - c|dy.$$

Let

$$M(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

For $\eta > 0$, let $M^\#_\eta(f)(x) = M^\#(|f|^\eta)^{1/\eta}(x)$ and $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \le p < \infty$ and the non-negative weight function ω, set

$$M_{\eta,p,\omega}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{\omega(Q)^{1-p\eta/n}} \int_Q |f(y)|^p \omega(y) dy \right)^{1/p}$$

and

$$M_\omega(f)(x) = \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q |f(y)| \omega(y) d(y).$$

The A_p weight is defined by (see [9])

$$A_p = \left\{ 0 < \omega \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}, \ \ 1 < p < \infty,$$

and

$$A_1 = \{0 < \omega \in L^p_{loc}(R^n) : M(\omega)(x) \le Cw(x), a.e.\}.$$

Given a non-negative weight function ω. For $1 \le p < \infty$, the weighted Lebesgue space $L^p(R^n, \omega)$ is the space of functions f such that

$$\|f\|_{L^p(\omega)} = \left(\int_{R^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty.$$

Given the non-negative weight function ω. The weighted BMO space $BMO(\omega)$ is the space of functions b such that

$$\|b\|_{BMO(\omega)} = \sup_Q \frac{1}{\omega(Q)} \int_Q |b(y) - b_Q| dy < \infty.$$

For $0 < \beta < 1$, the weighted Lipschitz space $Lip_\beta(\omega)$ is the space of functions b such that

$$\|b\|_{Lip_\beta(\omega)} = \sup_Q \frac{1}{\omega(Q)^{\beta/n}} \left(\frac{1}{\omega(Q)} \int_Q |b(y) - b_Q|^p \omega(x)^{1-p} dy \right)^{1/p} < \infty.$$

Remark.(1). It has been known that(see [8]), for $b \in Lip_\beta(\omega)$, $\omega \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^kQ}| \le Ck\|b\|_{Lip_\beta(\omega)} \omega(x) \omega(2^k Q)^{\beta/n}.$$

(2). Let $b \in Lip_\beta(\omega)$ and $\omega \in A_1$. By [8], we know that spaces $Lip_\beta(\omega)$ coincide and the norms $\|b\|_{Lip_\beta(\omega)}$ are equivalent with respect to different values $1 \le p < \infty$.

We give some Preliminary lemmas.

Lemma 1.([9, p.485]) Let $0 < p < q < \infty$ and for any function $f \ge 0$. We define that, for $1/r = 1/p - 1/q$,

$$\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, \qquad N_{p,q}(f) = \sup_Q \|f\chi_Q\|_{L^p} / \|\chi_Q\|_{L^r},$$

where the sup is taken for all measurable sets Q with $0 < |Q| < \infty$. Then

$$\|f\|_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p} \|f\|_{WL^q}.$$

Lemma 2.(see [2]) Let T be the singular integral operator as **Definition 2**. Then T is bounded on $L^p(R^n, \omega)$ for $\omega \in A_p$ with $1 < p < \infty$, and weak (L^1, L^1) bounded.

Lemma 3.(see [1]) Let $b \in BMO(\omega)$. Then

$$|b_Q - b_{2^jQ}| \le Cj\|b\|_{BMO(\omega)} \omega_{Q_j},$$

where $\omega_{Q_j} = \max_{1 \le i \le j} |2^i Q|^{-1} \int_{2^i Q} \omega(x) dx$.

Lemma 4.(see [1]) Let $\omega \in A_p$, $1 < p < \infty$. Then there exists $\varepsilon > 0$ such that $\omega^{-r/p} \in A_r$ for any $p' \le r \le p' + \varepsilon$.

Lemma 5.(see [1]) Let $b \in BMO(\omega)$, $\omega = (\mu\nu^{-1})^{1/p}$, $\mu, \nu \in A_p$ and $p > 1$. Then there exists $\varepsilon > 0$ such that for $p' \le r \le p' + \varepsilon$,

$$\int_Q |b(x) - b_Q|^r \mu(x)^{-r/p} dx \le C\|b\|^r_{BMO(\omega)} \int_Q \nu(x)^{-r/p} dx.$$

Lemma 6.(see [1]) Let $\omega \in A_p$, $1 < p < \infty$. Then there exists $0 < \delta < 1$ such that $\omega^{1-r'/p} \in A_{p/r'}(d\mu)$ for any $p' < r < p'(1+\delta)$, where $d\mu = \omega^{r'/p}dx$.

Lemma 7.(see [1]) Let $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$, $1 < p < \infty$. Then there exists $1 < q < p$ such that

$$\omega_Q(\nu_Q)^{1/q}\left(\frac{1}{|Q|}\int_Q \omega(x)^{-q'}\nu(x)^{-q'/q}dx\right)^{1/q'} \le C.$$

Lemma 8.(see [3][9]) Let $0 \le \eta < n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $\omega \in A_1$. Then

$$\|M_{\eta,s,\omega}(f)\|_{L^q(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Lemma 9.(see [9]). Let $0 < p, \eta < \infty$ and $\omega \in \cup_{1\le r<\infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,

$$\int_{R^n} M_\eta(f)(x)^p\omega(x)dx \le C\int_{R^n} M^\#_\eta(f)(x)^p\omega(x)dx.$$

Lemma 10.(see [5]) Let b be a function on R^n and $D^\alpha A \in L^s(R^n)$ for all α with $|\alpha| = m$ and any $s > n$. Then

$$|R_m(b; x, y)| \le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^s dz\right)^{1/s},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

3. Theorems and Proofs

We shall prove the following theorems.

Theorem 1. Let T be the singular integral operator as **Definition 2**, $1 < p < \infty$, $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$, $0 < \eta < 1$ and $D^\alpha b \in BMO(\omega)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$, $\varepsilon > 0$, $0 < \delta < 1$, $1 < q < p$ and $p' < r < \min(p' + \varepsilon, p'(1+\delta))$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$\begin{aligned} M^\#_\eta(T^b(f))(\tilde{x}) &\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)} \\ &\times\left([M_\nu(|\omega T(f)|^q)(\tilde{x})]^{1/q} + [M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'} + [M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}\right). \end{aligned}$$

Theorem 2. Let T be the singular integral operator as **Definition 2**, $\omega \in A_1$, $0 < \eta < 1$, $1 < r < \infty$, $0 < \beta < 1$ and $D^\alpha b \in Lip_\beta(\omega)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^\#_\eta(T^b(f))(\tilde{x}) \le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(\omega)}\omega(\tilde{x})M_{\beta,r,\omega}(f)(\tilde{x}).$$

Theorem 3. Let T be the singular integral operator as **Definition 2**, $1 < p < \infty$, $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$ and $D^\alpha b \in BMO(\omega)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^p(R^n, \mu)$ to $L^p(R^n, \nu)$.

Theorem 4. Let T be the singular integral operator as **Definition 2**, $\omega \in A_1$, $0 < \beta < 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $D^\alpha b \in Lip_\beta(\omega)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^p(R^n, \omega)$ to $L^q(R^n, \omega^{1-q})$.

Corollary. Let $[b, T](f) = bT(f) - T(bf)$ be the commutator generated by the singular integral operator T as **Definition 2** and b. Then Theorems 1-4 hold for $[b, T]$.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^\eta dx\right)^{1/\eta} \le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)}$$
$$\times\left([M_\nu(|\omega T(f)|^q)(\tilde{x})]^{1/q} + [M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'} + [M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}(x) = b(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b)_{\tilde{Q}} x^\alpha$, then $R_m(b; x, y) = R_m(\tilde{b}; x, y)$ and $D^\alpha \tilde{b} = D^\alpha b - (D^\alpha b)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned}
T^b(f)(x) &= \int_{R^n} \frac{R_m(\tilde{b}; x, y)}{|x-y|^m} K(x, x-y) f_1(y) dy \\
&- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-y)^\alpha D^\alpha \tilde{b}(y)}{|x-y|^m} K(x, x-y) f_1(y) dy \\
&+ \int_{R^n} \frac{R_{m+1}(\tilde{b}; x, y)}{|x-y|^m} K(x, x-y) f_2(y) dy \\
&= T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right) - T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right) + T^{\tilde{b}}(f_2)(x),
\end{aligned}$$

then

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0)\right|^\eta dx\right)^{1/\eta} \\
\le\ & C\left(\frac{1}{|Q|}\int_Q \left|T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q \left|T\left(\sum_{|\alpha|=m} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\
+\ & C\left(\frac{1}{|Q|}\int_Q \left|T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)\right|^\eta dx\right)^{1/\eta} \\
=\ & I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, noting that $\omega \in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^{p_0} dx\right)^{1/p_0} \le \frac{C}{|Q|}\int_Q \omega(x) dx$$

for all cube Q and some $1 < p_0 < \infty$ (see [9]). We take $s = rp_0/(r + p_0 - 1)$ in Lemma 10 and have $1 < s < r$

and $p_0 = s(r-1)/(r-s)$, then by the Lemma 10 and Hölder's inequality, we gain

$$\begin{aligned}
|R_m(b;x,y)| &\le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\alpha b(z)|^s dz\right)^{1/s}\\
&\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/s}\left(\int_{\tilde{Q}(x,y)}|D^\alpha b(z)|^s\omega(z)^{s(1-r)/r}\omega(z)^{s(r-1)/r}dz\right)^{1/s}\\
&\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/s}\left(\int_{\tilde{Q}(x,y)}|D^\alpha b(z)|^r\omega(z)^{1-r}dz\right)^{1/r}\left(\int_{\tilde{Q}(x,y)}\omega(z)^{s(r-1)/(r-s)}dz\right)^{(r-s)/rs}\\
&\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/s}\|D^\alpha b\|_{BMO(\omega)}\omega(\tilde{Q})^{1/r}|\tilde{Q}|^{(r-s)/rs}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\omega(z)^{p_0}dz\right)^{(r-s)/rs}\\
&\le C|x-y|^m \sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}|\tilde{Q}|^{-1/q}\omega(\tilde{Q})^{1/r}|\tilde{Q}|^{1/s-1/r}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\omega(z)dz\right)^{(r-1)/r}\\
&\le C|x-y|^m \sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}|\tilde{Q}|^{-1/q}\omega(\tilde{Q})^{1/r}|\tilde{Q}|^{1/s-1/r}\omega(\tilde{Q})^{1-1/r}|\tilde{Q}|^{1/r-1}\\
&\le C|x-y|^m \sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\frac{\omega(\tilde{Q})}{|\tilde{Q}|},
\end{aligned}$$

thus, by Lemma 7, we obtain

$$\begin{aligned}
I_1 &\le \frac{C}{|Q|}\int_Q\left|T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right)\right|dx\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\frac{\omega(\tilde{Q})}{|\tilde{Q}|}\frac{1}{|Q|}\int_Q|T(f)(y)|\omega(y)\nu(y)^{1/q}\omega(y)^{-1}\nu(y)^{-1/q}dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\omega_{\tilde{Q}}\left(\frac{1}{|Q|}\int_Q|\omega(y)T(f)(y)|^q\nu(y)dy\right)^{1/q}\\
&\quad\times\left(\frac{1}{|Q|}\int_Q\omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\omega_Q(\nu_Q)^{1/q}\left(\frac{1}{\nu(Q)}\int_Q|\omega(y)T(f)(y)|^q\nu(y)dy\right)^{1/q}\\
&\quad\times\left(\frac{1}{|Q|}\int_Q\omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}[M_\nu(|\omega T(f)|^q)(\tilde{x})]^{1/q}\\
&\quad\times\omega_Q(\nu_Q)^{1/q}\left(\frac{1}{|Q|}\int_Q\omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}[M_\nu(|\omega T(f)|^q)(\tilde{x})]^{1/q}.
\end{aligned}$$

For I_2, we know $\nu^{-r/p} \in A_r$ by Lemma 4, thus

$$\left(\frac{1}{|Q|}\int_Q\nu(x)^{-r/p}dx\right)^{1/r} \le C\left(\frac{1}{|Q|}\int_Q\nu(x)^{r'/p}dx\right)^{-1/r'},$$

then, by the weak (L^1, L^1) boundedness of T(see Lemma 2) and Kolmogoro's inequality(see Lemma 1), we obtain, by Lemma 5,

$$
\begin{aligned}
I_2 &\leq C\sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_Q |T(D^\alpha\tilde{b}f_1)(x)|^\eta dx\right)^{1/\eta}\\
&\leq C\sum_{|\alpha|=m}\frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}}\frac{\|T(D^\alpha\tilde{b}f_1)\chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}}\\
&\leq C\sum_{|\alpha|=m}\frac{1}{|Q|}\|T(D^\alpha\tilde{b}f_1)\|_{WL^1}\\
&\leq C\sum_{|\alpha|=m}\frac{1}{|Q|}\int_{R^n}|D^\alpha\tilde{b}(x)f_1(x)|dx\\
&= C\sum_{|\alpha|=m}\frac{1}{|Q|}\int_{\tilde{Q}}|D^\alpha b(x)-(D^\alpha b)_{\tilde{Q}}|\mu(x)^{-1/p}|f(x)|\omega(x)\nu(x)^{1/p}dx\\
&\leq C\sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|(D^\alpha b(x)-(D^\alpha b)_{\tilde{Q}})|^r\mu(x)^{-r/p}dx\right)^{1/r}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^{r'}\omega(x)^{r'}\nu(x)^{r'/p}dx\right)^{1/r'}\\
&\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\nu(x)^{-r/p}dx\right)^{1/r}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)\omega(x)|^{r'}\nu(x)^{r'/p}dx\right)^{1/r'}\\
&\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\nu(x)^{r'/p}dx\right)^{-1/r'}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)\omega(x)|^{r'}\nu(x)^{r'/p}dx\right)^{1/r'}\\
&\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left(\frac{1}{\nu(\tilde{Q})^{r'/p}}\int_{\tilde{Q}}|f(x)\omega(x)|^{r'}\nu(x)^{r'/p}dx\right)^{1/r'}\\
&\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}[M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'}.
\end{aligned}
$$

For I_3, note that $|x-y|\approx|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus Q$, we write

$$
\begin{aligned}
&|T^{\tilde{b}}(f_2)(x)-T^{\tilde{b}}(f_2)(x_0)|\\
\leq\ & \int_{R^n}\left|R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)\right|\frac{|\Omega(x,x-y)|}{|x-y|^{m+n}}|f_2(y)|dy\\
+\ & \int_{R^n}\left|\frac{\Omega(x,x-y)}{|x-y|^{n+m}}-\frac{\Omega(x_0,x_0-y)}{|x_0-y|^{n+m}}\right||R_m(\tilde{b};x_0,y)||f_2(y)|dy\\
+\ & \sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left|\frac{\Omega(x,x-y)}{|x-y|^{n+m}}-\frac{\Omega(x_0,x_0-y)}{|x_0-y|^{n+m}}\right||(x-y)^\alpha||D^\alpha\tilde{b}(y)||f_2(y)|dy\\
+\ & \sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left|\frac{(x-y)^\alpha}{|x-y|^{m+n}}-\frac{(x_0-y)^\alpha}{|x_0-y|^{m+n}}\right||\Omega(x_0,x_0-y)||D^\alpha\tilde{b}(y)||f_2(y)|dy\\
=\ & I_3^{(1)}(x)+I_3^{(2)}(x)+I_3^{(3)}(x)+I_3^{(4)}(x).
\end{aligned}
$$

For $I_3^{(1)}$, by the formula (see [5]):

$$R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)=\sum_{|\gamma|<m}\frac{1}{\gamma!}R_{m-|\gamma|}(D^\gamma\tilde{b};x,x_0)(x-y)^\gamma$$

and Lemma 10, we have, similar to the proof of I_1 and for $k \geq 0$,

$$|R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \leq C \sum_{|\gamma|<m} \sum_{|\alpha|=m} |x - x_0|^{m-|\gamma|}|x - y|^{|\gamma|}\|D^\alpha b\|_{BMO(\omega)} \frac{\omega(2^k\tilde{Q})}{|2^k\tilde{Q}|},$$

thus

$$\begin{aligned}
I_3^{(1)}(x) &\leq \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left|R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)\right| \frac{|\Omega(x, x-y)|}{|x-y|^{m+n}}|f(y)|dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \sum_{k=0}^{\infty} \frac{\omega(2^{k+1}\tilde{Q})}{|2^{k+1}\tilde{Q}|} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \sum_{k=1}^{\infty} \omega_{2^k\tilde{Q}} \frac{d}{(2^k d)^{n+1}} \int_{2^k\tilde{Q}} |f(y)|\omega(y)\nu(y)^{1/q}\omega(y)^{-1}\nu(y)^{-1/q}dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \sum_{k=1}^{\infty} 2^{-k}\omega_{2^k\tilde{Q}} \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |\omega(y)f(y)|^q\nu(y)dy\right)^{1/q} \\
&\quad \times \left(\frac{1}{|2^kQ|}\int_{2^k\tilde{Q}} \omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \sum_{k=1}^{\infty} 2^{-k}\omega_{2^k\tilde{Q}}(\nu_{2^k\tilde{Q}})^{1/q} \left(\frac{1}{\nu(2^k\tilde{Q})}\int_{2^k\tilde{Q}} |\omega(y)f(y)|^q\nu(y)dy\right)^{1/q} \\
&\quad \times \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} \omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)}[M_\nu(|\omega f|^q)(\tilde{x})]^{1/q} \sum_{k=1}^{\infty} k2^{-k} \\
&\quad \times \omega_{2^kQ}(\nu_{2^kQ})^{1/q} \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} \omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)}[M_\nu(|\omega f|^q)(\tilde{x})]^{1/q} \sum_{k=1}^{\infty} k2^{-k} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)}[M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}.
\end{aligned}$$

For $I_3^{(2)}$, by [2], we know that

$$\frac{|\Omega(x, x-y)|}{|x-y|^{m+n}} \leq C \sum_{u=1}^{\infty} \sum_{v=1}^{g_u} a_{uv}(x) \frac{Y_{uv}(x-y)}{|x-y|^n},$$

where $g_u \leq Cu^{n-2}$, $\|a_{uv}\|_{L^\infty} \leq Cu^{-2n}$, $|Y_{uv}(x-y)| \leq Cu^{n/2-1}$ and

$$\left|\frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right| \leq Cu^{n/2}|x-x_0|/|x_0-y|^{n+1}$$

for $|x-y|>2|x_0-x|>0$. Thus, we get

$$\begin{aligned}
I_3^{(2)}(x) &\le C\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|R_m(\tilde{b};x_0,y)|\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}|a_{uv}(x)|\left|\frac{Y_{uv}(x-y)}{|x-y|^{n+m}}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^{n+m}}\right||f(y)|dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\sum_{u=1}^{\infty}u^{-2n}\cdot u^{n/2}\sum_{k=0}^{\infty}\frac{\omega(2^{k+1}\tilde{Q})}{|2^{k+1}\tilde{Q}|}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}[M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}\sum_{k=1}^{\infty}k2^{-k}\\
&\quad\times\omega_{2^k\tilde{Q}}(\nu_{2^k\tilde{Q}})^{1/q}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}[M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}.
\end{aligned}$$

For $I_3^{(3)}$ and $I_3^{(4)}$, by using the same arguments as $I_3^{(2)}$ and I_2, we have

$$\begin{aligned}
&I_3^{(3)}(x)+I_3^{(4)}(x)\le C\sum_{|\alpha|=m}\sum_{u=1}^{\infty}u^{-2n}\cdot u^{n/2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)||D^\alpha\tilde{b}(y)|dy\\
&+\ C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)||D^\alpha\tilde{b}(y)|dy\\
&\le\ C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\frac{d}{(2^kd)^{n+1}}\int_{2^k\tilde{Q}}|D^\alpha b(y)-(D^\alpha b)_{2^k\tilde{Q}}||f(y)|dy\\
&+\ C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\frac{d}{(2^kd)^{n+1}}|(D^\alpha b)_{2^k\tilde{Q}}-(D^\alpha b)_{\tilde{Q}}|\int_{2^k\tilde{Q}}|f(y)|dy\\
&\le\ C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\sum_{k=1}^{\infty}2^{-k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\nu(y)^{-r/p}dy\right)^{1/r}\\
&\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)\omega(y)|^{r'}\nu(y)^{r'/p}dy\right)^{1/r'}\\
&+\ C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}[M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}\sum_{k=1}^{\infty}k2^{-k}\\
&\quad\times\omega_{2^k\tilde{Q}}(\nu_{2^k\tilde{Q}})^{1/q}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\omega(y)^{-q'}\nu(y)^{-q'/q}dy\right)^{1/q'}\\
&\le\ C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left([M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'}+[M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}\right).
\end{aligned}$$

Thus

$$I_3\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left([M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'}+[M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}\right).$$

These complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q\left|T^b(f)(x)-C_0\right|^\eta dx\right)^{1/\eta}\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(\omega)}w(\tilde{x})M_{\beta,r,\omega}(f)(\tilde{x}).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$
\begin{aligned}
& \left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0)\right|^\eta dx\right)^{1/\eta} \\
\leq\ & C\left(\frac{1}{|Q|}\int_Q \left|T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right)\right|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q \left|T\left(\sum_{|\alpha|=m}\frac{(x-\cdot)^\alpha D^\alpha\tilde{b}}{|x-\cdot|^m}f_1\right)\right|^\eta dx\right)^{1/\eta} \\
+\ & C\left(\frac{1}{|Q|}\int_Q \left|T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)\right|^\eta dx\right)^{1/\eta} \\
=\ & J_1 + J_2 + J_3.
\end{aligned}
$$

For J_1, by using the same argument as in the proof of Theorem 1, we get

$$
\begin{aligned}
& |R_m(b;x,y)| \leq C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/s}\left(\int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^s \omega(z)^{s(1-r)/r}\omega(z)^{s(r-1)/r}dz\right)^{1/s} \\
\leq\ & C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/s}\left(\int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^r \omega(z)^{1-r}dz\right)^{1/r}\left(\int_{\tilde{Q}(x,y)} \omega(z)^{s(r-1)/(r-s)}dz\right)^{(r-s)/rs} \\
\leq\ & C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/s}||D^\alpha b||_{Lip_\beta(\omega)}\omega(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{(r-s)/rs}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} \omega(z)^{p_0}dz\right)^{(r-s)/rs} \\
\leq\ & C|x-y|^m \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)}|\tilde{Q}|^{-1/s}\omega(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/s-1/r}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} \omega(z)dz\right)^{(r-1)/r} \\
\leq\ & C|x-y|^m \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)}|\tilde{Q}|^{-1/s}\omega(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/s-1/r}\omega(\tilde{Q})^{1-1/r}|\tilde{Q}|^{1/r-1} \\
\leq\ & C|x-y|^m \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)}\frac{\omega(\tilde{Q})^{\beta/n+1}}{|\tilde{Q}|} \\
\leq\ & C|x-y|^m \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)}\omega(\tilde{Q})^{\beta/n}\omega(\tilde{x}),
\end{aligned}
$$

thus, by the L^s-boundedness of T for $1 < s < r$ and $w \in A_1 \subseteq A_{r/s}$, we obtain

$$
\begin{aligned}
J_1 &\leq \frac{C}{|Q|}\int_Q \left| T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m} f_1\right)\right| dx \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(\omega)} \omega(\tilde{Q})^{\beta/n}\omega(\tilde{x}) \left(\frac{1}{|Q|}\int_{R^n} |T(f_1)(x)|^s dx\right)^{1/s} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(\omega)} \omega(\tilde{Q})^{\beta/n}\omega(\tilde{x})|Q|^{-1/s} \left(\int_{R^n} |f_1(x)|^s dx\right)^{1/s} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(\omega)} \omega(\tilde{Q})^{\beta/n}\omega(\tilde{x})|Q|^{-1/s} \left(\int_{\tilde{Q}} |f(x)|^r \omega(x) dx\right)^{1/r} \left(\int_{\tilde{Q}} \omega(x)^{-s/(r-s)} dx\right)^{(r-s)/rs} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(\omega)} \omega(\tilde{x})|\tilde{Q}|^{-1/s}\omega(\tilde{Q})^{1/r} \left(\frac{1}{\omega(\tilde{Q})^{1-r\beta/n}}\int_{\tilde{Q}} |f(x)|^r \omega(x) dx\right)^{1/r} \\
&\quad \times \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \omega(x)^{-s/(r-s)} dx\right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \omega(x) dx\right)^{1/r} |\tilde{Q}|^{1/s}\omega(\tilde{Q})^{-1/r} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).
\end{aligned}
$$

For J_2, by the weak (L^1, L^1) boundedness of T and Kolmogoro's inequality, we obtain

$$
\begin{aligned}
J_2 &\leq C \sum_{|\alpha|=m} \left(\frac{1}{|Q|}\int_Q |T(D^\alpha \tilde{b} f_1)(x)|^\eta dx\right)^{1/\eta} \\
&\leq C \sum_{|\alpha|=m} \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}} \frac{\|T(D^\alpha \tilde{b} f_1)\chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}} \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|T(D^\alpha \tilde{b} f_1)\|_{WL^1} \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{R^n} |D^\alpha \tilde{b}(x) f_1(x)| dx \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}| \omega(x)^{-1/r} |f(x)| \omega(x)^{1/r} dx \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \left(\int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}|^{r'} \omega(x)^{1-r'} dx\right)^{1/r'} \left(\int_{\tilde{Q}} |f(x)|^r \omega(x) dx\right)^{1/r} \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|D^\alpha b\|_{Lip_\beta(\omega)} \omega(\tilde{Q})^{\beta/n+1/r'} \omega(\tilde{Q})^{1/r-\beta/n} \left(\frac{1}{\omega(\tilde{Q})^{1-r\beta/n}}\int_{\tilde{Q}} |f(x)|^r \omega(x) dx\right)^{1/r} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(\omega)} \frac{\omega(\tilde{Q})}{|\tilde{Q}|} M_{\beta,r,\omega}(f)(\tilde{x}) \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).
\end{aligned}
$$

For J_3, we have

$$
|R_m(\tilde{b};x,y) - R_m(\tilde{b};x_0,y)| \leq C \sum_{|\gamma|<m} \sum_{|\alpha|=m} |x-x_0|^{m-|\gamma|} |x-y|^{|\gamma|} \|D^\alpha b\|_{Lip_\beta(\omega)} \omega(\tilde{x}) \omega(2^k \tilde{Q})^{\beta/n},
$$

thus

$$
\begin{aligned}
& |T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)| \\
\leq\ & \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left| R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y) \right| \frac{|\Omega(x, x-y)|}{|x-y|^{m+n}} |f(y)| dy \\
+\ & \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left| \frac{\Omega(x, x-y)}{|x-y|^{n+m}} - \frac{\Omega(x_0, x_0-y)}{|x_0-y|^{n+m}} \right| |R_m(\tilde{b}; x_0, y)| |f(y)| dy \\
+\ & C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left| \frac{\Omega(x, x-y)}{|x-y|^{n+m}} - \frac{\Omega(x_0, x_0-y)}{|x_0-y|^{n+m}} \right| |(x-y)^\alpha| |D^\alpha \tilde{b}(y)| |f(y)| dy \\
+\ & C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left| \frac{(x-y)^\alpha}{|x-y|^{m+n}} - \frac{(x_0-y)^\alpha}{|x_0-y|^{m+n}} \right| |\Omega(x_0, x_0-y)| |D^\alpha \tilde{b}(y)| |f(y)| dy \\
\leq\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)} \omega(\tilde{x}) \sum_{k=0}^{\infty} \omega(2^{k+1}\tilde{Q})^{\beta/n} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}} |f(y)| dy \\
+\ & C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \int_{2^k\tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^k\tilde{Q}}| \omega(y)^{-1/r} |f(y)| \omega(y)^{1/r} dy \\
+\ & C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \int_{2^k\tilde{Q}} |(D^\alpha b)_{2^k\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| |f(y)| \omega(y)^{1/r} \omega(y)^{-1/r} dy \\
\leq\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)} \omega(\tilde{x}) \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \omega(2^k\tilde{Q})^{\beta/n} \left(\int_{2^k\tilde{Q}} |f(y)|^r \omega(y) dx \right)^{1/r} \\
& \times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y) dy \right)^{1/r} |2^k\tilde{Q}| \omega(2^k\tilde{Q})^{-1/r} \\
+\ & C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \left(\int_{2^k\tilde{Q}} |(D^\alpha b(y) - (D^\alpha b)_{2^k\tilde{Q}})|^{r'} \omega(y)^{1-r'} dy \right)^{1/r'} \left(\int_{2^k\tilde{Q}} |f(y)|^r \omega(y) dy \right)^{1/r} \\
+\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)} \omega(\tilde{x}) \sum_{k=1}^{\infty} k \omega(2^k\tilde{Q})^{\beta/n} \frac{d}{(2^k d)^{n+1}} \left(\int_{2^k\tilde{Q}} |f(y)|^r \omega(y) dx \right)^{1/r} \\
& \times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y) dy \right)^{1/r} |2^k\tilde{Q}| \omega(2^k\tilde{Q})^{-1/r} \\
\leq\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)} \omega(\tilde{x}) \sum_{k=1}^{\infty} k 2^{-k} \left(\frac{1}{\omega(2^k\tilde{Q})^{1-r\beta/n}} \int_{2^k\tilde{Q}} |f(y)|^r \omega(y) dx \right)^{1/r} \\
+\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)} \sum_{k=1}^{\infty} 2^{-k} \frac{\omega(2^k\tilde{Q})}{|2^k\tilde{Q}|} \left(\frac{1}{\omega(2^k\tilde{Q})^{1-r\beta/n}} \int_{2^k\tilde{Q}} |f(y)|^r \omega(y) dx \right)^{1/r} \\
\leq\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Notice $\nu^{r'/p} \in A_{r'+1-r'/p} \subset A_p$ and $\nu(x)dx \in A_{p/r'}(\nu(x)^{r'/p}dx)$ by Lemma 6, thus, by

Theorem 1, Lemmas 2 and 9,

$$\begin{aligned}
&\int_{R^n} |T^b(f)(x)|^p \nu(x)dx \le \int_{R^n} |M_\eta(T^b(f))(x)|^p \nu(x)dx \le C \int_{R^n} |M^\#_\eta(T^b(f))(x)|^p \nu(x)dx \\
\le\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{BMO(\omega)} \int_{R^n} \left([M_\nu(|\omega T(f)|^q)(x)]^{p/q} + [M_{\nu^{r'/p}}(|\omega f|^{r'})(x)]^{p/r'} + [M_\nu(|\omega f|^q)(x)]^{p/q} \right) \nu(x)dx \\
\le\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{BMO(\omega)} \left(\int_{R^n} |\omega(x) f(x)|^p \nu(x)dx + \int_{R^n} |\omega(x) T(f)(x)|^p \nu(x)dx \right) \\
=\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{BMO(\omega)} \left(\int_{R^n} |f(x)|^p \mu(x)dx + \int_{R^n} |T(f)(x)|^p \mu(x)dx \right) \\
\le\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{BMO(\omega)} \int_{R^n} |f(x)|^p \mu(x)dx.
\end{aligned}$$

This completes the proof of Theorem 3.

Proof of Theorem 4. Choose $1 < r < p$ in Theorem 2 and notice $\omega^{1-q} \in A_1$, then we have, by Lemmas 8 and 9,

$$\begin{aligned}
&||T^b(f)||_{L^q(\omega^{1-q})} \le ||M_\eta(T^b(f))||_{L^q(\omega^{1-q})} \le C||M^\#_\eta(T^b(f))||_{L^q(\omega^{1-q})} \\
\le\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)} ||\omega M_{\beta,r,\omega}(f)||_{L^q(\omega^{1-q})} \\
=\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)} ||M_{\beta,r,\omega}(f)||_{L^q(\omega)} \\
\le\ & C \sum_{|\alpha|=m} ||D^\alpha b||_{Lip_\beta(\omega)} ||f||_{L^p(\omega)}.
\end{aligned}$$

This completes the proof of Theorem 4.

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