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Existence of fixed points for mixed monotone operators with perturbations and applications

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Abstract. In this article we study a class of mixed monotone operators with perturbations and present some new tripled fixed point theorems by means of partial order theory, we get the existence and uniqueness of tripled fixed points without assuming the operator to be compact or continuous, which extend the existing corresponding results. As applications, we utilize the results obtained in this paper to study the existence and uniqueness of positive solutions for a fractional differential equation boundary value problem..

1. Introduction

The fixed point problems of contractive mappings in partially ordered metric spaces has been considered recently by Ran and Reurings [20], Bhaskar and Lakshmikantham [18], Nieto and lopez [13]-[34], Agarwal et al. [13], and V. Berinde, M. Borcut [19]. Later in 2006, Bhaskar and Lakshmikantham [18] introduced the concept of a coupled fixed point and studied existence and uniqueness theorems in partially ordered metric spaces. They also applied their results to problems of the existence of solution for a periodic boundary value problem. In recent years, boundary value problems of nonlinear fractional differential equations with a variety of boundary conditions have been investigated by many researchers. Fractional differential equations appear naturally in various fields of science and engineering and constitute an important field of research . It should be noted that most papers dealing with the existence of solutions of nonlinear initial value problems of fractional differential equations mainly use the techniques of nonlinear analysis such as fixed point results, the Leary-Schauder theorem, stability, etc. (see for example [1]-[12]). As a matter of fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. A significant feature of a fractional order differential operator, in contrast to its counterpart in classical calculus, is its nonlocal behaviour. It means that the future state of a dynamical system or process based on the fractional differential operator depends on its current state as well its past states. It is equivalent to saying that differential equations of arbitrary order are capable of describing

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memory and hereditary properties of certain important materials and processes. This aspect of fractional calculus has contributed towards the growing popularity of the subject.
In 2012, V. Berinde and M. Borcut in [19] introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained it's existence. Recently, CB. Zhai [32] proved some results on a class of mixed monotone operators with perturbations. Following the paper of Zhai we will study tripled fixed point theorems for a class of mixed monotone operators with perturbations on ordered Banach spaces. And then we get the existence and uniqueness of tripled fixed points without assuming the operator to be compact or continuous. This research done is important in comparison with others, as some times coupled fixed points are more perfect than other fixed points, tripled fixed points are more practical than coupled fixed points and they are applicable for most differential equations which are not solve by the application of original fixed points or coupled fixed points.
To demonstrate the applicability of our abstract results, we give, in the last section of the paper, an application to a fractional differential equation boundary value problem.

For the convenience of the reader, we present here some definitions, notations and known results.

Suppose $(E, \| . \|)$ is a Banach space which is partially ordered by a cone $P \subseteq E$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \neq y$, then we denote $x < y$ *or* $x > y$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P,\ \lambda \geq 0 \Longrightarrow \lambda x \in P$; (ii) $x \in P,\ -x \in P \Longrightarrow x = \theta$. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$. Also we define the order interval $[x_1, x_2] = \{x \in E | x_1 \leq x \leq x_2\}$ for all $x_1, x_2 \in E$. We say that and operator $A : E \to E$ is increasing whenever $x \leq y$ implies $Ax \leq Ay$.

Definition 1.1. *[21, 22] $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., u_i, v_i $(i = 1, 2) \in P$, $u_1 \leq u_2, v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. The element $x \in P$ is called a fixed point of A if $A(x, x) = x$.*

Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. On the product space $X \times X \times X$, consider the following partial order: for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w. \tag{1}$$

Definition 1.2. *[19] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say F has the mixed monotone property if for any $x, y, z \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \ \textit{implies}\ F(x_1, y, z) \leq F(x_2, y, z), \tag{2}$$
$$y_1, y_2 \in X, y_1 \leq y_2 \ \textit{implies}\ F(x, y_1, z) \geq F(x, y_2, z), \tag{3}$$
$$\textit{and}\ z_1, z_2 \in X, z_1 \leq z_2 \ \textit{implies}\ F(x, y, z_1) \leq F(x, y, z_2). \tag{4}$$

Definition 1.3. *[19] An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of a mapping $F : X \times X \times X \to X$ if $F(x, y, z) = x, F(y, x, y) = y$ and $F(z, y, x) = z$.*

2. Main results

Now we consider the mixed monotone operator $A : P \times P \times P \to P$. The following conditions will be assumed:
(A_1) there exists $h \in P$ with $h \neq \theta$ such that $A(h, h, h) \in P_h$,
(A_2) for any $u, v, w \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that

$$A(tu, t^{-1}v, tw) \geq \frac{\varphi(t)}{t} A(u, v, w). \tag{5}$$

Lemma 2.1. *Assume (A_1), (A_2) hold. Then $A : P_h \times P_h \times P_h \to P_h$; and there exist $u_0, v_0, w_0 \in P_h$ and $r \in (0, 1)$ such that*

$$rv_0 \leq u_0 \leq w_0 < v_0, u_0 \leq A(u_0, v_0, w_0) \leq A(v_0, u_0, w_0) \leq v_0, A(w_0, u_0, w_0) \geq w_0.$$

Proof. Firstly, from condition (A_2) we get

$$A(t^{-1}x, ty, t^{-1}z) \leq \frac{t}{\varphi(t)} A(x, y, z), \ \ \forall\, t \in (0, 1), \ x, y, z \in P. \tag{6}$$

For any $u, v, w \in P_h$, there exist $\mu_1, \mu_2, \mu_3 \in (0, 1)$, such that

$$\mu_1 h \leq u \leq \frac{1}{\mu_1} h, \ \mu_2 h \leq v \leq \frac{1}{\mu_2} h, \ \mu_3 h \leq w \leq \frac{1}{\mu_3} h.$$

Let $\mu = \min\{\mu_1, \mu_2, \mu_3\}$. Then $\mu \in (0, 1)$. From (6) and the mixed monotone properties of operator A, we have

$$A(u, v, w) \leq A(\frac{1}{\mu_1} h, \mu_2 h, \frac{1}{\mu_3} h) \leq A(\frac{1}{\mu} h, \mu h, \frac{1}{\mu} h) \leq \frac{\mu}{\varphi(\mu)} A(h, h, h) \leq \frac{1}{\varphi(\mu)} A(h, h, h),$$
$$A(u, v, w) \geq A(\mu_1 h, \frac{1}{\mu_2} h, \mu_3 h) \geq A(\mu h, \frac{1}{\mu} h, \mu h) \geq \frac{\varphi(\mu)}{\mu} A(h, h, h) \geq \varphi(\mu) A(h, h, h).$$

It follows from $A(h, h, h) \in P_h$ that $A(u, v, w) \in P_h$. Hence we have $A : P_h \times P_h \times P_h \longrightarrow P_h$. Since $A(h, h, h) \in P_h$, we can choose a sufficiently small number $t_0 \in (0, 1)$ such that

$$t_0 h \leq A(h, h, h) \leq \frac{1}{t_0} h. \tag{7}$$

Noting that $t_0 < \varphi(t_0) \leq 1$, we can choose $s_0 \in (0, 1)$ and take a positive integer k such that

$$t_0 \leq s_0 \leq \varphi(t_0) \leq 1, \qquad\qquad (\frac{\varphi(t_0)}{t_0})^k \geq \frac{1}{t_0}. \tag{8}$$

Put $u_0 = {t_0}^k h, v_0 = \frac{1}{t_0^k} h, w_0 = s_0^k h$. Evidently, $u_0, v_0, w_0 \in P_h$ and $u_0 = {t_0}^{2k} v_0 < v_0$.
Take any $r \in (0, {t_0}^{2k}]$, then $r \in (0, 1)$ and

$$u_0 \geq r v_0, \ u_0 \leq w_0, v_0 = \frac{1}{t_0^k} h \geq \frac{1}{s_0^k} h > {s_0}^k h = w_0,$$

and hence $w_0 < v_0$. By the mixed monotone properties of A, we have $A(u_0, v_0, w_0) \leq A(v_0, u_0, w_0)$. Further, combining condition (A_2) with (7),(8), we have

$$\begin{aligned}
A(u_0, v_0, w_0) &= A(t_0^k h, \frac{1}{t_0^k} h, {s_0}^k h) \\
&\geq A(t_0^k h, \frac{1}{t_0^k} h, {t_0}^k h) = A(t_0.t_0^{k-1} h, \frac{1}{t_0}.\frac{1}{t_0^{k-1}} h, t_0.t_0^{k-1} h) \\
&\geq \frac{\varphi(t_0)}{t_0} A(t_0^{k-1} h, \frac{1}{t_0^{k-1}} h, t_0^{k-1} h) = \frac{\varphi(t_0)}{t_0} A(t_0.t_0^{k-2} h, \frac{1}{t_0}.\frac{1}{t_0^{k-2}} h, t_0.t_0^{k-2} h) \\
&\geq \frac{\varphi(t_0)}{t_0} \frac{\varphi(t_0)}{t_0} A(t_0^{k-2} h, \frac{1}{t_0^{k-2}} h, t_0^{k-2} h) \geq \ldots \\
&\geq (\frac{\varphi(t_0)}{t_0})^k A(h, h, h) \geq (\frac{\varphi(t_0)}{t_0})^k t_0 h \geq \frac{1}{t_0} t_0 h \geq h \geq {t_0}^k h = u_0.
\end{aligned}$$

By the mixed monotone properties of A and from (6) we get

$$\begin{aligned}
A(v_0,u_0,w_0) &= A(\frac{1}{t_0^k}h, t_0^k h, s_0^k h) \le A(\frac{1}{t_0^k}h, t_0^k h, \frac{1}{s_0^k}h) \le A(\frac{1}{t_0^k}h, t_0^k h, \frac{1}{t_0^k}h)\\
&= A(\frac{1}{t_0}.\frac{1}{t_0^{k-1}}h, t_0.t_0^{k-1}h, \frac{1}{t_0}.\frac{1}{t_0^{k-1}}h)\\
&\le \frac{t_0}{\varphi(t_0)}A(\frac{1}{t_0^{k-1}}h, t_0^{k-1}h, \frac{1}{t_0^{k-1}}h)\\
&= \frac{t_0}{\varphi(t_0)}A(\frac{1}{t_0}.\frac{1}{t_0^{k-2}}h, t_0.t_0^{k-2}h, \frac{1}{t_0}.\frac{1}{t_0^{k-2}}h)\\
&\le \frac{t_0}{\varphi(t_0)}.\frac{t_0}{\varphi(t_0)}A(\frac{1}{t_0^{k-2}}h, t_0^{k-2}h, \frac{1}{t_0^{k-2}}h)\\
&\le \ldots \le (\frac{t_0}{(\varphi(t_0))})^k A(h,h,h) \le (\frac{t_0}{(\varphi(t_0))})^k \frac{1}{t_0}h.
\end{aligned}$$

An application of (8) implies that

$$A(v_0,u_0,w_0) \le (\frac{t_0}{\varphi(t_0)})^k \frac{1}{t_0}h \le \frac{1}{t_0}t_0 h = h \le \frac{1}{{t_0}^k}h = v_0.$$

Thus we have

$$u_0 \le A(u_0,v_0,w_0) \le A(v_0,u_0,w_0) \le v_0.$$

We prove $A(w_0,u_0,w_0) \ge w_0$,

$$\begin{aligned}
A(w_0,u_0,w_0) &= A({s_0}^k h, {t_0}^k h, {s_0}^k h \ge A(t_0^k h, \frac{1}{t_0^k}h, {s_0}^k h)\\
&\ge A(t_0^k h, \frac{1}{t_0^k}h, {t_0}^k h) = A(t_0.t_0^{k-1}h, \frac{1}{t_0}.\frac{1}{t_0^{k-1}}h, t_0.t_0^{k-1}h)\\
&\ge \frac{\varphi(t_0)}{t_0}A(t_0^{k-1}h, \frac{1}{t_0^{k-1}}h, t_0^{k-1}h) = \frac{\varphi(t_0)}{t_0}A(t_0.t_0^{k-2}h, \frac{1}{t_0}.\frac{1}{t_0^{k-2}}h, t_0.t_0^{k-2}h)\\
&\ge \frac{\varphi(t_0)}{t_0}\frac{\varphi(t_0)}{t_0}A(t_0^{k-2}h, \frac{1}{t_0^{k-2}}h, t_0^{k-2}h) \ge \ldots\\
&\ge (\frac{\varphi(t_0)}{t_0})^k A(h,h,h) \ge (\frac{\varphi(t_0)}{t_0})^k t_0 h \ge \frac{1}{t_0}t_0 h \ge h \ge {s_0}^k h = w_0.
\end{aligned}$$

□

Theorem 2.2. *Suppose that P is a normal cone of E, and $(A_1), (A_2)$ hold. Then operator A has a unique fixed point x in P_h. Moreover, for any initial $x_0, y_0, z_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}, z_{n-1}), y_n = A(y_{n-1}, x_{n-1}, z_{n-1}), z_n = A(z_{n-1}, x_{n-1}, z_{n-1})$$
$$n = 1, 2, \ldots,$$

we have $\| x_n - x^* \| \to 0$, $\| y_n - x^* \| \to 0$ and $\| z_n - x^* \| \to 0$ as $n \to \infty$.

Proof. From Lemma (2.1), there exist $u_0, v_0, w_0 \in P_h$ and $r \in (0,1)$ such that

$$rv_0 \le u_0 \le w_0 < v_0, u_0 \le A(u_0,v_0,w_0) \le A(v_0,u_0,w_0) \le v_0, A(w_0,u_0,w_0) \ge w_0.$$

Construct successively the sequences

$$u_n = A(u_{n-1}, v_{n-1}, w_{n-1}), v_n = A(v_{n-1}, u_{n-1}, w_{n-1}), w_n = A(w_{n-1}, u_{n-1}, w_{n-1}),$$

$n = 1, 2, \ldots.$
Evidently $u_1 \leq v_1$ and $w_1 \geq w_0$. By the mixed monotone properties of A, we obtain $u_n \leq v_n$ and $w_n \geq \ldots \geq w_1 \geq w_0,\ n = 1, 2, \ldots$. It also follows from Lemma 2.1 and the mixed monotone properties of A that

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq w_0 \leq w_1 \leq \ldots \leq w_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0, \tag{9}$$

Noting that $u_0, w_0 \geq rv_0$. We can get $u_n \geq u_0 \geq rv_0 \geq rv_n,\ n = 1, 2, \ldots$. Let

$$t_n = \sup\{t > 0 | u_n \geq tv_n\} \ \ n = 1, 2, \ldots.$$

Thus we have $u_n \geq t_n v_n,\ w_n \geq t_n v_n, n = \ldots$, and then

$$u_{n+1} \geq u_n \geq t_n v_n \geq t_n v_{n+1}, n = 1, 2, \ldots.$$

Therefore, $t_{n+1} \geq tn,\ i.e., t_n$ is increasing with $t_n \subset (0, 1]$. Suppose $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. Otherwise, $0 < t^* < 1$. Then from condition (A_2) and $t_n \leq t^*$, we have

$$\begin{aligned} u_{n+1} &= A(u_n, v_n, w_n) \geq A(t_n v_n, \frac{1}{t_n} u_n, t_n v_n) = A(\frac{t_n}{t^*} t^* v_n, \frac{t^*}{t_n} \frac{1}{t^*} u_n, \frac{t_n}{t^*} t^* w_n) \\ &\geq \frac{t_n}{t^*} A(t^* v_n, \frac{1}{t^*} u_n, t^* w_n) \geq \frac{t_n}{t^*} \frac{\varphi(t^*)}{t^*} A(v_n, u_n, w_n) \geq \frac{t_n}{t^*} \varphi(t^*) A(v_n, u_n, w_n) \\ &= \frac{t_n}{t^*} \varphi(t^*) v_{n+1}. \end{aligned}$$

By the definition of $t_n,\ t_{n+1} \geq \frac{t_n}{t^*} . \varphi(t^*)$. Let $n \to \infty$, we get $t^* \geq \varphi(t^*) > t^*$, which is a contradiction. Thus, $\lim_{n\to\infty} t_n = 1$. For any natural number p we have

$$\begin{aligned} &\theta \leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - t_n v_n = (1 - t_n) v_n \leq (1 - t_n) v_0, \\ &\theta \leq v_n - v_{n+p} \leq v_n - u_n \leq (1 - t_n) v_0, \\ &\theta \leq w_n - w_{n+p} \leq v_n - u_n \leq v_n - t_n v_n = (1 - t_n) v_n \leq (1 - t_n) v_0. \end{aligned}$$

Since the cone P is normal, we have

$$\begin{aligned} &\| u_{n+p} - u_n \| \leq N(1 - t_n) \| v_0 \| \to 0, \ \ \| v_n - v_{n+p} \| \leq N(1 - t_n) \| v_0 \| \to 0, \\ &\| w_n - w_{n+p} \| \leq N(1 - t_n) \| v_0 \| \to 0. \ \ (n \to \infty), \end{aligned}$$

where N is the normality constant of P . So we can claim that u_n and v_n are Cauchy sequences. Because E is complete, there exist u^*, v^*, w^* such that $u_n \to u^*, v_n \to v^*, w_n \to w^*$ as $n \to \infty$. By (9), we know that $u_n \leq u^* \leq w^* \leq v^* \leq v_n$ with $u^*, v^*, w^* \in P_h$ and

$$\begin{aligned} &\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - t_n) v_0, \ \ \theta \leq w^* - v^* \leq v_n - u_n \leq (1 - t_n) v_0 \\ &\theta \leq u^* - w^* \leq v_n - u_n \leq (1 - t_n) v_0. \end{aligned}$$

Further

$$\begin{aligned} &\| v^* - u^* \| \leq N(1 - t_n) \| v_0 \| \to 0 \ \ (n \to \infty), \\ &\| w^* - v^* \| \leq N(1 - t_n) \| v_0 \| \to 0 \ \ (n \to \infty), \\ &\| u^* - w^* \| \leq N(1 - t_n) \| v_0 \| \to 0 \ \ (n \to \infty), \end{aligned}$$

and thus $u^* = v^* = w^*$. Let $x^* := u^* = v^* = w^*$ and then we obtain

$$u_{n+1} = A(u_n, v_n, w_n) \leq A(x^*, x^*, x^*) \leq A(v_n, u_n, w_n) = v_{n+1}.$$

Let $n \to \infty$, then we get $x^* = A(x^*, x^*, x^*)$. That is, x^* is a fixed point of A in P_h. In the following, we prove that x^* is the unique fixed point of A in P_h. In fact, suppose $\bar{x}$ is a fixed point of A in P_h.Since $x^*, \bar{x} \in P_h$, there exists positive numbers $\bar{\mu}_1, \bar{\mu}_2, \bar{\lambda}_1, \bar{\lambda}_2 > 0$ such that

$$\bar{\mu}_1 h \leq x^* \leq \bar{\lambda}_1, \quad \bar{\mu}_2 h \leq \bar{x} \leq \bar{\lambda}_2 h.$$

Then we obtain

$$\bar{x} \leq \bar{\lambda}_2 h = \frac{\bar{\lambda}_2}{\bar{\mu}_1}.\bar{\mu}_1 h \leq \frac{\bar{\lambda}_2}{\bar{\mu}_1}.x^*, \;\; \bar{x} \geq \bar{\lambda}_2 h = \frac{\bar{\mu}_2}{\bar{\lambda}_1}.\bar{\lambda}_1 h \geq \frac{\bar{\mu}_2}{\bar{\lambda}_1} x^*.$$

Let $e_1 = \sup\{t > 0 | tx^* \leq \bar{x} \leq t^{-1}x^*\}$. Evidently, $0 < e_1 \leq 1, e_1 x^* \leq \bar{x} \leq \frac{1}{e_1} x^*$. Next we prove $e_1 = 1$. If $0 < e_1 < 1$, then

$$\begin{aligned}
\bar{x} &= A(\bar{x}, \bar{x}, \bar{x}) \geq A(e_1 x^*, \frac{1}{e_1} x^*, e_1 x^*) \\
&\geq \frac{\varphi(e_1)}{e_1} A(x^*, x^*, x^*) \geq \varphi(e_1) A(x^*, x^*, x^*) \\
&= \varphi(e_1) x^*.
\end{aligned}$$

Since $\varphi(e_1) > e_1$, this contradicts the definition of e_1. Hence $e_1 = 1$, and we get $\bar{x} = x^*$. Therefore, A has a unique fixed point x^* in P_h. Note that $[u_0, v_0] \subset P_h$, then we know that x^* is the unique fixed point of A in $[u_0, v_0]$.

Now we construct successively the sequences

$$\begin{aligned}
&x_n = A(x_{n-1}, y_{n-1}, z_{n-1}), y_n = A(y_{n-1}, x_{n-1}, z_{n-1}), \\
&z_n = A(z_{n-1}, x_{n-1}, z_{n-1}), n = 1, 2, \ldots,
\end{aligned}$$

for any initial points $x_0, y_0, z_0 \in P_h$. Since $x_0, y_0, z_0 \in P_h$ we can choose small numbers $e_2, e_3, e_4 \in (0, 1)$ such that

$$e_2 h \leq x_0 \leq \frac{1}{e_2} h, \quad e_3 h \leq y_0 \leq \frac{1}{e_3} h, \quad e_4 h \leq z_0 \leq \frac{1}{e_4} h.$$

Let $e^* = \min\{e_2, e_3, e_4\}$. Then $e^* \in (0, 1)$ and

$$e^* h \leq x_0, \quad y_0 \leq \frac{1}{e^*} h, \quad e^* h \leq z_0.$$

We can choose a sufficiently large positive integer m such that

$$[\frac{\varphi(e^*)}{e^*}]^m \geq \frac{1}{e^*},$$

and we choose $e_1^* \in (0, 1)$ such that $e^* \leq e_1^* \leq \varphi(e^*) \leq 1$.

Put $\bar{u}_0 = e^{*m} h, \bar{v}_0 = \frac{1}{e^{*m}} h, \bar{w}_0 = e_1^{*m} h$. It easy to see that $\bar{u}_0, \bar{v}_0, \bar{w}_0 \in P_h$ and $\bar{u}_0 < x_0, y_0 < \bar{v}_0, w_0 < \bar{z}_0$. Let

$$\begin{aligned}
&\bar{u}_n = A(\bar{u}_{n-1}, \bar{v}_{n-1}, \bar{w}_{n-1}), \bar{v}_n = A(\bar{v}_{n-1}, \bar{u}_{n-1}, \bar{w}_{n-1}), \\
&\bar{w}_n = A(\bar{w}_{n-1}, \bar{u}_{n-1}, \bar{w}_{n-1}), n = 1, 2, \ldots.
\end{aligned}$$

Similarly, it follows that there exists $y^* \in P_h$ such that $A(y^*, y^*, y^*) = y^*, \lim_{n\to\infty} \bar{u}_n = \lim_{n\to\infty} \bar{v}_n = \lim_{n\to\infty} \bar{w}_n = y^*$. By the uniqueness of fixed point of operator A in P_h. We get $x^* = y^* = z^*$ and by induction $\bar{u}_n \leq x_n, y_n \leq \bar{v}_n, \bar{w}_n \leq z_n,\; n = 1, 2, \ldots$. Since cone P is normal we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x^*$. □

3. Application

We study the existence and uniqueness of a solution for the fractional differential equation

$$\frac{D^\alpha}{Dt}u(r,s,t)+f(r,s,t,u(r,s,t))=0,$$
$$(0<\epsilon<T,\ T\geq 1,\ t\in[\epsilon,T],\ 0<\alpha<1,\ s\in[a,b],\ t\in[c,d]) \tag{10}$$

subject to condition

$$u(s,r,\zeta)=u(s,r,T),\quad (r,s,\zeta)\in[a,b]\times[c,d]\times(\epsilon,t), \tag{11}$$

where D^α is the Riemann-Liouville fractional derivative of order α. We will suppose that $a,b,c,d\in(0,\infty)$, $a<b,\ c<d$.
Let

$$E=C([a,b]\times[c,d]\times[\epsilon,T]).$$

Consider the Banach space of continuous functions on $[a,b]\times[c,d]\times[\epsilon,T]$ with sup norm and set

$$P=\{y\in C([a,b]\times[c,d]\times[\epsilon,T]):\min_{(s,r,t)\in[a,b]\times[c,d]\times[\epsilon,T]}y(s,r,t)\geq 0\}.$$

Then P is a normal cone.

Lemma 3.1. *Let $(s,r,t)\in[a,b]\times[c,d]\times[\epsilon,T]$, $(s,r,\zeta)\in[a,b]\times[c,d]\times(\epsilon,t)$ and $0<\alpha<1$. Then the problem*

$$\frac{D^\alpha}{Dt}u(s,r,t)+f(s,r,t,u(s,r,t))=0$$

with the boundary value condition $u(s,r,\zeta)=u(s,r,T)$ has a solution u_0 if and only if u_0 is a solution of the fractional integral equation

$$u(s,r,t)=\int_\epsilon^T G(t,\xi)f(s,r,\xi,u(s,r,\xi))d\xi,$$

where,

$$G(t,\xi)=\begin{cases}\frac{t^{\alpha-1}(\zeta-\xi)^{\alpha-1}-t^{\alpha-1}(T-\xi)^{\alpha-1}}{(\zeta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)}-\frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & \epsilon\leq\xi\leq\zeta\leq t\leq T,\\ \frac{-t^{\alpha-1}-(T-\xi)^{\alpha-1}}{(\zeta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)}-\frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & \epsilon\leq\zeta\leq\xi\leq t\leq T,\\ \frac{-t^{\alpha-1}(T-\xi)^{\alpha-1}}{(\zeta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)}, & \epsilon\leq\zeta\leq t\leq\xi\leq T.\end{cases}$$

Proof. From $\frac{D^\alpha}{Dt}u(s,r,t)+f(s,r,t,u(s,r,t))=0$ and the boundary condition, it is easy to see that $u(s,r,t)-c_1t^{\alpha-1}=-I_\epsilon^\alpha f(s,r,t,u(s,r,t))$. By the definition of a fractional integral, we get

$$u(s,r,t)=c_1t^{\alpha-1}-\int_\epsilon^\zeta\frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}f(s,r,\xi,u(s,r,\xi))d\xi,$$
$$u(s,r,\zeta)=c_1T^{\alpha-1}-\int_\epsilon^\zeta\frac{(\zeta-\xi)^{\alpha-1}}{\Gamma(\alpha)}f(s,r,\xi,u(s,r,\xi))d\xi,$$

and

$$u(s,r,T)=c_1T^{\alpha-1}-\int_\epsilon^T\frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}f(s,r,\xi,u(s,r,\xi))d\xi.$$

Since $u(s,r,\zeta)=u(s,r,T)$, we obtain

$$c_1=\frac{1}{\zeta^{\alpha-1}-T^{\alpha-1}}\int_\epsilon^\zeta\frac{(\zeta-\xi)^{\alpha-1}}{\Gamma(\alpha)}f(s,r,\xi,u(s,r,\xi))d\xi$$
$$-\frac{1}{\zeta^{\alpha-1}-T^{\alpha-1}}\int_\epsilon^T\frac{(T-\xi)^{\alpha-1}}{\Gamma(\alpha)}f(s,r,\xi,u(s,r,\xi))d\xi.$$

Hence

$$\begin{aligned} u(s,r,t) &= \frac{t^{\alpha-1}}{\zeta^{\alpha-1}-T^{\alpha-1}}\int_{\epsilon}^{\zeta}\frac{(\zeta-\xi)^{\alpha-1}}{\Gamma(\alpha)}f(s,r,\xi,u(s,r,\xi))d\xi \\ &- \frac{t^{\alpha-1}}{\zeta^{\alpha-1}-T^{\alpha-1}}\int_{\epsilon}^{T}\frac{(T-\xi)^{\alpha-1}}{\Gamma(\alpha)}f(s,r,\xi,u(s,r,\xi))d\xi \\ &- \int_{\epsilon}^{t}\frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}f(s,r,\xi,u(s,r,\xi))d\xi = \int_{\epsilon}^{T}G(t,\xi)f(s,r,\xi,u(s,r,\xi))d\xi. \end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $0<\epsilon<T$ be given and*
$f(s,r,t,u(s,r,t),v(s,r,t),\eta(s,r,t)) \in C([a,b],[c,d],[\epsilon,T],[0,\infty],[0,\infty],[0,\infty])$ and $c\in(0,1), s,r,t\in P$, there exists $\varphi(t)\in(t,1]$ such that
$f(s,r,t,cu(s,r,t),c^{-1}v(s,r,t),c\eta(s,r,t)) \geq \frac{\varphi(t)}{t}f(s,r,t,u(s,r,t),v(s,r,t),\eta(s,r,t))$ and
$f(s,r,t,u(s,r,t),v(s,r,t),\eta(s,r,t)) = 0$ whenever $G(s,t)<0$.
Also assume that there exist $M_1, M_2>0$ and $\theta\neq h\in P$ such that

$$M_1 h \leq \int_{\epsilon}^{T} G(t,\xi)f(s,r,\xi,h(s,r,\xi),h(s,r,\xi),h(s,r,\xi))d\xi \leq M_2 h,$$

for all $(s,r,t)\in[a,b]\times[c,d]\times[\epsilon,T]$, where $G(t,\xi)$ is the green function defined in lemma (3.1). Then the problem (10) with the boundary condition (11) has a unique solution in P_h. Moreover, for any initial $u_0, v_0, \eta_0\in P_h$, constructing successively the sequences

$$\begin{aligned} u_{n+1} &= \int_{\epsilon}^{T} G(t,\xi)f(s,r,\xi,u_n(s,r,\xi),v_n(s,r,\xi),\eta_n(s,r,\xi))d\xi, \\ v_{n+1} &= \int_{\epsilon}^{T} G(t,\xi)f(s,r,\xi,v_n(s,r,\xi),u_n(s,r,\xi),\eta_n(s,r,\xi))d\xi, \\ \eta_{n+1} &= \int_{\epsilon}^{T} G(t,\xi)f(s,r,\xi,\eta_n(s,r,\xi),u_n(s,r,\xi),\eta_n(s,r,\xi))d\xi, \end{aligned}$$

we have $\| u_n-u^* \|\to 0, \| v_n-u^* \|\to 0, \| \eta_n-u^* \|\to 0$.

Proof. By using Lemma (2.1), the problem is equivalent to the integral equation

$$u(s,r,t) = \int_{\epsilon}^{T} G(t,\xi)f(s,r,\xi,u(s,r,\xi),v(s,r,\xi),\eta(s,r,\xi))d\xi,$$

where

$$G(t,\xi) = \begin{cases} \frac{t^{\alpha-1}(\zeta-\xi)^{\alpha-1}-t^{\alpha-1}(T-\xi)^{\alpha-1}}{(\zeta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & \epsilon\leq\xi\leq\zeta\leq t\leq T, \\ \frac{-t^{\alpha-1}-(T-\xi)^{\alpha-1}}{(\zeta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)} - \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & \epsilon\leq\zeta\leq\xi\leq t\leq T, \\ \frac{-t^{\alpha-1}(T-\xi)^{\alpha-1}}{(\zeta^{\alpha-1}-T^{\alpha-1})\Gamma(\alpha)}, & \epsilon\leq\zeta\leq t\leq\xi\leq T. \end{cases}$$

Define the operator $A: P\times P\times P\to P$ by the following,
$A(u(s,r,t),v(s,r,t),\eta(s,r,t)) = \int_{\epsilon}^{T} G(t,\xi)f(s,r,\xi,u(s,r,\xi),v(s,r,\xi),\eta(s,r,\xi))d\xi$.
Then u is solution for the problem if and only if $u=A(u,u,u)$.

For $c \in (0,1), s, r, t \in P$, there exists $\varphi(t) \in (t, 1]$ such that

$$\begin{aligned}
&A(cu(s,r,t), c^{-1}v(s,r,t), c\eta(s,r,t)) \\
&= \int_{\epsilon}^{T} G(t,\xi) f(, s, r, \xi, cu(s,r,\xi), c^{-1}v(s,r,\xi), c\eta(s,r,\xi)) d\xi \\
&\geq \frac{\varphi(t)}{t} \int_{\epsilon}^{T} G(t,\xi) f(s,r,\xi, u(s,r,\xi), , v(s,r,\xi), \eta(s,r,\xi)) d\xi \\
&= \frac{\varphi(t)}{t} A(u(s,r,t), v(s,r,t), \eta(s,r,t)).
\end{aligned}$$

Since

$$M_1 h \leq A(h,h,h) = \int_{\epsilon}^{T} G(t,\xi) f(s,r,\xi, h(s,r,\xi), h(s,r,\xi), h(s,r,\xi)) d\xi \leq M_2 h,$$

for all $(s,r,t) \in [a,b] \times [c,d] \times [\epsilon, T]$, we get $A(h,h,h) \in P_h$. Therefore A satisfies all conditions of Theorem (2.2), and so, the operator A has a unique positive solution (u^*, u^*, u^*) such that $A(u^*, u^*, u^*) = u^*$. This completes the proof. □

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