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Stability of Browder spectral properties under Riesz-type perturbations

Abdelmajid Arroud[a], Hassan Zariouh[b]

[a] *Department of Mathematics, Faculty of Science, Mohammed I University, PO Box 717, Oujda 60000 Morocco*
[b] *Centre régional des métiers de l'éducation et de la formation, B.P 458, Oujda, Morocco et Equipe de la Théorie des Opérateurs, Université Mohammed I, Faculté des Sciences d'Oujda, Dépt. de Mathématiques, Morocco*

Abstract. The properties (Bw), (Baw), (Bab) and (Bb) were introduced in [9] and [15]. In this paper we give characterizations of these spectral properties for a bounded linear operator having SVEP on the complementary of the B-Weyl spectrum. We also study their stability under commuting Riesz-type perturbations.

1. Introduction and preliminaries

For T in the Banach algebra $L(X)$ of bounded linear operators acting on a Banach space X, we will denote by $\sigma(T)$ the spectrum of T, by $\sigma_a(T)$ the approximate point spectrum of T, by $\mathcal{N}(T)$ the null space of T, by $n(T)$ the nullity of T, by $\mathcal{R}(T)$ the range of T and by $d(T)$ its defect. If $n(T) < \infty$ and $d(T) < \infty$, then T is called a *Fredholm* operator and its index is defined by $\text{ind}(T) = n(T) - d(T)$. A *Weyl* operator is a Fredholm operator of index 0 and the Weyl spectrum is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\}$.

For a bounded linear operator T and $n \in \mathbb{N}$, let $T_{[n]} : \mathcal{R}(T^n) \to \mathcal{R}(T^n)$ be the restriction of T to $\mathcal{R}(T^n)$. $T \in L(X)$ is said to be *B-Weyl* if for some integer $n \geq 0$ the range $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is Weyl; its index is defined as the index of the Weyl operator $T_{[n]}$. The respective *B-Weyl spectrum* is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}$.

The *ascent* $a(T)$ of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, and the *descent* $\delta(T)$ of T is defined by $\delta(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. According to [10], a complex number $\lambda \in \sigma(T)$ is a *pole* of the resolvent of T if $T - \lambda I$ has finite ascent and finite descent, and in this case they are equal. We recall that a complex number $\lambda \in \sigma_a(T)$ is a *left pole* of T if $a(T - \lambda I) < \infty$ and $R(T^{a(T-\lambda I)+1})$ is closed.

We summarize in the following list the usual notations and symbols needed later.

Notations and symbols:

$\mathcal{F}(X)$: the ideal of finite rank operators in $L(X)$,
$\mathcal{K}(X)$: the ideal of compact operators in $L(X)$,
$\mathcal{N}(X)$: the class of nilpotent operators on X,
$\mathcal{Q}(X)$: the class of quasi-nilpotent operators on X,

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Email addresses: arroud@hotmail.com (Abdelmajid Arroud), h.zariouh@yahoo.fr (Hassan Zariouh)

$\mathcal{R}(X)$: the class of Riesz operators acting on X,
$\operatorname{iso} A$: isolated points of a subset $A \subset \mathbb{C}$,
$\operatorname{acc} A$: accumulations points of a subset $A \subset \mathbb{C}$,
$D(0,1)$: the closed unit disc in $\mathbb{C}$,
$C(0,1)$: the unit circle of $\mathbb{C}$,
$\Pi(T)$: poles of T,
$\Pi^0(T)$: poles of T of finite rank,
$\Pi_a(T)$: left poles of T,
$\Pi_a^0(T)$: left poles of T of finite rank,
$\sigma_p(T)$: eigenvalues of T,
$\sigma_p^f(T)$: eigenvalues of T of finite multiplicity,
$E^0(T) := \operatorname{iso}\sigma(T) \cap \sigma_p^f(T)$,
$E(T) := \operatorname{iso}\sigma(T) \cap \sigma_p(T)$,
$E_a^0(T) := \operatorname{iso}\sigma_a(T) \cap \sigma_p^f(T)$,
$E_a(T) := \operatorname{iso}\sigma_a(T) \cap \sigma_p(T)$,
$\sigma_b(T) = \sigma(T) \setminus \Pi^0(T)$: Browder spectrum of T,
$\sigma_{ub}(T) = \sigma_a(T) \setminus \Pi_a^0(T)$: upper-Browder spectrum of T,
$\sigma_W(T)$: Weyl spectrum of T,
$\sigma_{BW}(T)$: B-Weyl spectrum of T.

Definition 1.1. *Let $T \in L(X)$. T is said to satisfy*
i) Weyl's theorem if $\sigma(T) \setminus \sigma_W(T) = E^0(T)$.
ii) Browder's theorem if $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$.
iii) generalized Browder's theorem if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$.
iv) property (ab) if $\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$.
v) property (aw) if $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$.

Definition 1.2. [9], [15] *Let $T \in L(X)$. We say that:*
i) T satisfies property (Bw) if $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$.
ii) T satisfies property (Bb) if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$.
iii) T satisfies property (Baw) if $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$.
iv) T satisfies property (Bab) if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$.

The relationship between properties and theorems given in the precedent definitions was studied in [15], and is summarized in the following diagram. (arrows signify implications and numbers near the arrows are references to the bibliography therein).

$$\begin{tikzcd}
\text{property } (Baw) \arrow[r, "{[15]}"] \arrow[d, "{[15]}"] & \text{property } (Bw) \arrow[r, "{[15]}"] \arrow[d, "{[15]}"] & \text{Weyl's theorem} \arrow[d, "{[3]}"] \\
\text{property } (Bab) \arrow[r, "{[15]}"'] & \text{property } (Bb) \arrow[r, "{[15]}"'] & \text{Browder's theorem}
\end{tikzcd}$$

Moreover, in [15] counterexamples were given to show that the reverse of each implication in the diagram is not true. Nonetheless, it was proved that under some additional hypothesis, these implications are equivalences as we can see in the next theorem.

Theorem 1.3. [15] *Let $T \in L(X)$.*
i) If $\sigma_{BW}(T) = \sigma_W(T)$, then property (Bw) holds for T if and only if Weyl's theorem holds for T; and property (Bb) holds for T if and only if Browder's theorem holds for T.
ii) If $E^0(T) = \Pi(T)$, then property (Bw) holds for T if and only if property (Bb) holds for T.

iii) If $\Pi(T) = \Pi_a^0(T)$, then property (Bab) holds for T if and only if property (Bb) holds for T.
iv) If $E^0(T) = E_a^0(T)$, then property (Baw) holds for T if and only if property (Bw) holds for T.
iv) If $E_a^0(T) = \Pi_a^0(T)$, then property (Baw) holds for T if and only if property (Bab) holds for T.

For every $T \in L(X)$ we know that $\sigma_{BW}(T) \subset \sigma_W(T)$, but generally this inclusion is proper. Indeed, let T on $\ell^2(\mathbb{N})$ defined by $T(x_1, x_2, ...) = (0, \frac{x_1}{2}, 0, 0, ...)$, then $\sigma_{BW}(T) = \emptyset \subsetneq \sigma_W(T) = \{0\}$. In the following lemma, we explicit the defect set $\sigma_W(T) \setminus \sigma_{BW}(T)$.

Lemma 1.4. *(See also [6]) Let $T \in L(X)$. Then $\sigma_W(T) = \sigma_{BW}(T) \cup \text{iso}\,\sigma_W(T)$.*

Proof. Let $\lambda_0 \in \sigma_W(T) \setminus \sigma_{BW}(T)$ be arbitrary, then $T - \lambda_0 I$ is a B-Weyl operator. From the punctured neighborhood theorem for B-Weyl operators, there exists $\varepsilon > 0$ such that if $0 < |\mu| < \varepsilon$, then $T - \lambda_0 I - \mu I$ is a Weyl operator and $\text{ind}(T - \lambda_0 I - \mu I) = \text{ind}(T - \lambda_0 I)$. Thus for every scalar z such that $0 < |z - \lambda_0| < \varepsilon$, we have $T - \lambda_0 I - (z - \lambda_0)I = T - zI$ is a Weyl operator with $\text{ind}(T - zI) = 0$. This implies that $D(\lambda_0, \varepsilon) \cap \sigma_W(T) = \{\lambda_0\}$ and as $\lambda_0 \in \sigma_W(T)$, then $\lambda_0 \in \text{iso}\,\sigma_W(T)$. Hence $\sigma_W(T) = \sigma_{BW}(T) \cup \text{iso}\,\sigma_W(T)$. □

Corollary 1.5. *Let $T \in L(X)$ such that $\text{iso}\,\sigma_W(T) = \emptyset$. The following statements hold.*
i) T satisfies property (Bw) if and only if T satisfies Weyl's Theorem.
ii) T satisfies property (Bb) if and only if T satisfies Browder's Theorem.
iii) T satisfies property (Bab) if and only if T satisfies property (ab).
iv) T satisfies property (Baw) if and only if T satisfies property (aw).

Proof. The proof of i) and ii) is a consequence of Theorem 1.3 and Lemma 1.4. The proof of iii) and iv) follows directly from Lemma 1.4, Definition 1.1 and Definition 1.2. □

The paper is organized as follows: after giving an introduction and some preliminaries in the first section, we characterize in the second section the properties (Bw), (Baw), (Bab) and (Bb) for bounded linear operators having SVEP on the complementary of the B-weyl spectrum. In the third section, we study the preservation of properties (Bw) and (Baw) under Riesz-type perturbations. Similar results are obtained for (Bb) and (Bab) in the fourth section. Several examples are given in each section to show that the results obtained fail without adequate hypothesis.

2. Browder spectral properties and SVEP

The following property has relevant role in local spectral theory: a bounded linear operator $T \in L(X)$ is said to have the *single-valued extension property* (SVEP for short) at $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the function $f \equiv 0$ is the only analytic solution of the equation $(T - \mu I)f(\mu) = 0 \quad \forall \mu \in U_\lambda$. We denote by $\mathcal{S}(T) = \{\lambda \in \mathbb{C} : T \text{ does not have SVEP at } \lambda\}$ and we say that T has SVEP if $\mathcal{S}(T) = \emptyset$. We say that T has SVEP on $A \subset \mathbb{C}$, if T has SVEP at every $\lambda \in A$.

Theorem 2.1. *Let $T \in L(X)$. If T or T^* has SVEP on $\sigma(T) \setminus \sigma_{BW}(T)$, then T satisfies property (Bb) if and only if $\Pi(T) = \Pi^0(T)$.*

Proof. $\Longrightarrow$) Assume that T satisfies property (Bb). Let $\lambda_0 \in \sigma(T) \setminus \sigma_W(T)$ be arbitrary then $\lambda_0 \in \sigma(T) \setminus \sigma_{BW}(T)$. As T satisfies property (Bb) then $\lambda_0 \in \Pi^0(T)$. Thus $\sigma(T) \setminus \sigma_W(T) \subseteq \Pi^0(T)$ and since the opposite inclusion is always true, it follows that $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$. But this is equivalent from [2, Theorem 2.1] to say that $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$. Hence $\Pi(T) = \Pi^0(T)$. Observe that in this implication, the condition of SVEP for T or T^* is not necessary.
$\Longleftarrow$) Assume that $\Pi(T) = \Pi^0(T)$. Note that T has SVEP on $\sigma(T) \setminus \sigma_{BW}(T) \Longleftrightarrow T$ has SVEP on $\sigma_{BW}(T)^C \Longleftrightarrow T^*$ has SVEP on $\sigma_{BW}(T)^C \Longleftrightarrow T^*$ has SVEP on $\sigma(T) \setminus \sigma_{BW}(T)$; where $\sigma_{BW}(T)^C$ is the complement of the B-Weyl spectrum of T. From [1, Theorem 3.2], T satisfies generalized Browder's theorem $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$. Thus T satisfies property (Bb). □

Remark 2.2. The assumption T or T^* has SVEP on $\sigma(T) \setminus \sigma_{BW}(T)$ is essential as shown in the next example. Define the operator U on $\ell^2(\mathbb{N})$ by $U(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$.
On $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, put $T = U \oplus U^*$. Since $\sigma(U) = \sigma_{BW}(U) = D(0,1)$ and $\sigma(U^*) = \sigma_{BW}(U^*) = D(0,1)$. It follows that $\sigma(T) = D(0,1)$ and hence $\Pi(T) = \Pi^0(T) = \emptyset$. But as $n(T) = d(T) = 1$, $0 \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus property ($Bb$) does not hold for T. Notice that T and T^* do not have SVEP at 0 which lies in $\sigma(T) \setminus \sigma_{BW}(T)$, since $\mathcal{S}(T) = \mathcal{S}(T^*) = \mathcal{S}(U^*) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| < 1\}$.

Corollary 2.3. *Let $T \in L(X)$. If T or T^* has SVEP on $\sigma(T) \setminus \sigma_{BW}(T)$, then T satisfies property* (Bab) *if and only if* $\Pi(T) = \Pi_a^0(T)$.

Proof. It's a consequence of the precedent theorem and [15, Corollary 3.8]. (Note that the direct implication is always true (see [15, Corollary 3.8]). □

Corollary 2.4. *Let $T \in L(X)$. If T or T^* has SVEP on $\sigma(T) \setminus \sigma_{BW}(T)$, then T satisfies property* (Baw) *if and only if* $\Pi(T) = E_a^0(T)$.

Proof. If T satisfies (Baw) then from [15, Theorem 3.2], it satisfies property (Bab) and $\Pi_a^0(T) = E_a^0(T)$, and from Corollary 2.3 we have $\Pi(T) = E_a^0(T)$. Conversely, if $\Pi(T) = E_a^0(T)$, then $\Pi(T) = E_a^0(T) = \Pi_a^0(T)$. From Corollary 2.3 it follows that T satisfies property (Bab) and property (Baw) too. □

Remark 2.5. The assumption T or T^* has SVEP on $\sigma(T) \setminus \sigma_{BW}(T)$ is essential in corollaries 2.3 and 2.4. Indeed, the operator T given in Remark 2.2 does not satisfy property (Bb) and hence it does not satisfy the properties (Bab) and (Baw); though we have $\Pi(T) = \Pi_a^0(T) = E_a^0(T)$.

3. Properties (Baw), (Bw) and Riesz-type perturbations

We recall that an operator $R \in L(X)$ is said to be *Riesz* if $R - \mu I$ is Fredholm for every non-zero complex μ, that is, $\pi(R)$ is quasinilpotent in the Calkin algebra $C(X) = L(X)/\mathcal{K}(X)$ where π is the canonical mapping of $L(X)$ into $C(X)$.

We denote by $\mathcal{F}^0(X)$, the class of power finite rank operators as follows:

$$\mathcal{F}^0(X) = \{S \in L(X) : S^n \in \mathcal{F}(X) \text{ for some } n \in \mathbb{N}\}$$

and by $\mathcal{R}(X)$ the class of Riesz operators acting on X. Clearly,

$$\mathcal{F}(X) \cup \mathcal{N}(X) \subset \mathcal{F}^0(X) \subset \mathcal{R}(X), \text{ and } \mathcal{K}(X) \cup \mathcal{Q}(X) \subset \mathcal{R}(X).$$

We start this section by the following nilpotent perturbation result.

Proposition 3.1. *Let $T \in L(X)$ and let $N \in \mathcal{N}(X)$ which commutes with T. Then T satisfies property* (s) *if and only if $T + N$ satisfies property* (s)*; where* $(s) \in \{(Bw), (Bb), (Bab), (Baw)\}$.

Proof. Since N is nilpotent and commutes with T, we know that $\sigma(T+N) = \sigma(T)$ and $\sigma_a(T+N) = \sigma_a(T)$. From the proof of [5, Theorem 3.5], it follows that $0 < n(T+N) \Longleftrightarrow 0 < n(T)$ and $n(T+N) < \infty \Longleftrightarrow n(T) < \infty$. Thus $E_a^0(T+N) = E_a^0(T)$, $E(T+N) = E(T)$, $E_a(T+N) = E_a(T)$ and $E^0(T+N) = E^0(T)$. We also have from [4, Lemma 22] that $\Pi(T+N) = \Pi(T)$ which implies that $\Pi^0(T+N) = \Pi^0(T)$. From [16, Corollary 3.8] we know that $\Pi_a(T+N) = \Pi_a(T)$ and so $\Pi_a^0(T+N) = \Pi_a^0(T)$. On the other hand, $\sigma_{BW}(T+N) = \sigma_{BW}(T)$, see [16, Corollary 3.1]. This finishes the proof. □

Remark 3.2. We notice that the assumption of commutativity in the Proposition 3.1 is crucial.
1) Let T and N be defined on $\ell^2(\mathbb{N})$ by

$$T(x_1, x_2, \ldots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots) \text{ and } N(x_1, x_2, \ldots) = (0, \frac{-x_1}{2}, 0, 0, \ldots).$$

Clearly N is nilpotent and does not commute with T. The properties (Baw) and (Bw) are satisfied by T, since $\sigma(T) = \{0\} = \sigma_{BW}(T)$ and $E_a^0(T) = \emptyset$. But $T + N$ does not satisfy neither property (Bw) nor property (Baw) as we have $\sigma(T + N) = \sigma_{BW}(T + N) = \{0\}$ and $\{0\} = E^0(T + N)$.
2) Let T and N be defined by

$$T = \begin{pmatrix} U^* & U^* \\ 0 & U \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & U^* \\ 0 & 0 \end{pmatrix},$$

where U is defined in Remark 2.2. Obviously, N is nilpotent and does not commute with T. Since $\sigma_a(U^*) \subset \sigma_a(T) \subset \sigma_a(U^*) \cup \sigma_a(U)$, see [8, Proposition 1.1], it follows that $\sigma_a(T) = D(0,1)$ and since U has SVEP, then $\sigma_{BW}(T) = \sigma(T) = \sigma(U^*) \cup \sigma(U) = D(0,1)$. Moreover, $\Pi^0(T) = \Pi_a^0(T) = \emptyset$. Consequently, T satisfies properties (Bab) and (Bb). But $T - N$ does not satisfy neither property (Bb) nor property (Bab). To see this, as $T - N = U^* \oplus U$ then $\sigma_a(T - N) = \sigma(T - N) = D(0,1)$ and $\Pi_a^0(T - N) = \Pi^0(T - N) = \emptyset$. But from Remark 2.2 we have $\sigma(T - N) \setminus \sigma_{BW}(T - N) \neq \emptyset$.

Corollary 3.3. *Let $T \in \mathcal{Q}(X)$ be an injective quasi-nilpotent and let $F \in \mathcal{F}(X)$ which commutes with T. Then T satisfies property (s) if and only if $T + F$ satisfies property (s); where $(s) \in \{(Bw), (Bb), (Bab), (Baw)\}$.*

Proof. If T is injective, as TF is a finite rank quasi-nilpotent operator, then TF is a nilpotent operator. Since T is injective, then F is nilpotent. Thus the result follows from Proposition 3.1. □

The stability of properties (Baw) and (Bw) showed in Proposition 3.1 cannot be extended to commuting quasi-nilpotent operators, as we can see in the next remark.

Remark 3.4. In general, the properties (Bw) and (Baw) are unstable under quasi-nilpotent perturbations. For this we consider the operators T and R defined on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by

$$T = 0 \oplus Q \text{ and } R = Q \oplus 0,$$

where Q is defined on $\ell^2(\mathbb{N})$ by $Q(x_1, x_2, \ldots) = (\frac{x_2}{2}, \frac{x_3}{3}, \ldots)$. Clearly R is compact and quasi-nilpotent and verifies $TR = RT = 0$. On the other hand, T satisfies properties (Bw) and (Baw), because $\sigma(T) = \{0\} = \sigma_{BW}(T)$ and $E_a^0(T) = \emptyset$. But $T + R = Q \oplus Q$ does not satisfy neither property (Bw) nor property (Baw), since $\sigma(T + R) = \{0\} = \sigma_{BW}(T + R)$ and $E^0(T + R) = \{0\}$. Note that here $\Pi(T + R) = \emptyset$.

However, in Theorem 3.6 below we give necessary and sufficient conditions to ensure the stability of these properties under commuting perturbations by Riesz operators which are not necessary nilpotent. The case of nilpotent operators is studied in Proposition 3.1. But before that we need the following lemma in the proof of the next main results.

Lemma 3.5. *Let $T \in L(X)$. If $S \in \mathcal{F}^0(X)$ and $R \in \mathcal{R}(X)$ are commuting operators with T, then the following statements hold.*
i) T satisfies Browder's theorem if and only if $T + R$ satisfies Browder's theorem.
ii) If T satisfies property (Bb), then $\Pi(T + S) = \Pi^0(T + S)$. In particular, this equality holds if T satisfies property (Bab) or (Bw).

Proof. i) As T satisfies Browder's theorem, then $\sigma_b(T) = \sigma_W(T)$. Since $TR = RT$ then from [14] we have $\sigma_b(T + R) = \sigma_b(T)$ and from [13, Lemma 2.2] we have $\sigma_W(T) = \sigma_W(T + R)$. So $\sigma_b(T + R) = \sigma_W(T + R)$. Thus $T + R$ satisfies Browder's theorem, and consequently $T + R$ satisfies generalized Browder's theorem too. Conversely, assume that $T + R$ satisfies Browder's theorem. Since $(T + R)R = R(T + R)$ and $T = (T + R) - R$, we conclude similarly.

ii) The inclusion $\Pi^0(T + S) \subset \Pi(T + S)$ is always true. Conversely let $\lambda \in \Pi(T + S)$, as T satisfies property (Bb), then from [15, Theorem 2.4] we have $\sigma_{BW}(T) = \sigma_W(T)$. Since $S \in \mathcal{F}^0(X)$ and $TS = ST$, then from [16, Theorem 2.8] we have $\sigma_{BW}(T) = \sigma_{BW}(T + S)$. Hence $\lambda \notin \sigma_{BW}(T + S) = \sigma_{BW}(T) = \sigma_W(T + S)$. So $n(T + S - \lambda I) < \infty$. In particular, if T satisfies property (Bab) or (Bw) then it satisfies property (Bb). □

Theorem 3.6. *Let $R \in \mathcal{R}(X)$ and let $T \in L(X)$ which commutes with R.*
i) If T satisfies property (Bw), then $T + R$ satisfies property (Bw) if and only if $\Pi(T + R) = E^0(T + R)$.
ii) If T satisfies property (Baw), then $T + R$ satisfies property (Baw) if and only if $\Pi(T + R) = E^0_a(T + R)$.

Proof. i) If $T+R$ satisfies (Bw), then from [9, Theorem 2.5] we have $E^0(T+R) = \Pi(T+R)$. Conversely, suppose that $E^0(T + R) = \Pi(T + R)$. Since T satisfies (Bw) then from [9, Theorem 2.4], it satisfies Browder's theorem. From Lemma 3.5, $T + R$ satisfies generalized Browder's theorem, that is $\sigma(T + R) \setminus \sigma_{BW}(T + R) = \Pi(T + R)$. So $T + R$ satisfies property (Bw).

ii) If $T + R$ satisfies (Baw), then from [15] we have $E^0_a(T + R) = \Pi(T + R)$. Conversely, suppose that $E^0_a(T+R) = \Pi(T+R)$. Since T satisfies (Baw) then from [15, Corollary 3.5], it satisfies property (Bw). Hence $T+R$ satisfies generalized Browder's theorem $\sigma(T+R)\setminus\sigma_{BW}(T+R) = \Pi(T+R)$. So $\sigma(T+R)\setminus\sigma_{BW}(T+R) = E^0_a(T+R)$. □

Now if we restrict to the class $\mathcal{F}^0(X)$, we obtain the following perturbation result concerning property (Bw). The same result holds for property (Baw) with similar proof.

Theorem 3.7. *Let $S \in \mathcal{F}^0(X)$. If $T \in L(X)$ satisfies property (Bw) and commutes with S, then the following statements are equivalent.*
i) $T + S$ satisfies property (Bw);
ii) $\Pi(T + S) = E^0(T + S)$.
iii) $E^0(T + S) \cap \sigma(T) \subset \Pi^0(T)$.

Proof. i) $\Longleftrightarrow$ ii) Since $\mathcal{F}^0(X) \subset \mathcal{R}(X)$, this equivalence follows from Theorem 3.6.

ii) $\Longrightarrow$ iii) Suppose that $\Pi(T + S) = E^0(T + S)$ and let $\lambda_0 \in E^0(T + S) \cap \sigma(T)$ be arbitrary. Then $\lambda_0 \in \Pi^0(T + S) \cap \sigma(T)$ and so $\lambda_0 \notin \sigma_b(T + S) = \sigma_b(T)$. Thus $\lambda_0 \in \Pi^0(T)$. This proves that $E^0(T + S) \cap \sigma(T) \subset \Pi^0(T)$.

iii) $\Longrightarrow$ ii) Suppose that $E^0(T+S)\cap\sigma(T) \subset \Pi^0(T)$. Firstly, we show that $E^0(T+S) \subset \Pi(T+S)$. Let $\mu_0 \in E^0(T+S)$ be arbitrary. We distinguish two cases: the first is $\mu_0 \in \sigma(T)$. Then $\mu_0 \in E^0(T + S) \cap \sigma(T) \subset \Pi^0(T)$. It follows that $\mu_0 \notin \sigma_b(T) = \sigma_b(T + S)$ and since $\mu_0 \in \sigma(T + S)$, then $\mu_0 \in \Pi(T + S)$. The second case is $\mu_0 \notin \sigma(T)$. This implies that $\mu_0 \notin \sigma_b(T) = \sigma_b(T + S)$. Thus $\mu_0 \in \Pi^0(T + S) \subset \Pi(T + S)$. Consequently, $E^0(T + S) \subset \Pi(T + S)$. From Lemma 3.5, we conclude that $\Pi(T + S) = E^0(T + S)$. □

Theorem 3.8. *Let $S \in \mathcal{F}^0(X)$. If $T \in L(X)$ satisfies property (Baw) and commutes with S, then the following statements are equivalent.*
i) $T + S$ satisfies property (Baw);
ii) $\Pi(T + S) = E^0_a(T + S)$.
iii) $E^0_a(T + S) \cap \sigma(T) \subset \Pi^0(T)$.

Proof. Goes similarly with the proof of Theorem 3.7. □

The following example proves that in general, property (Baw) is not preserved under commuting finite rank power perturbations.

Example 3.9. On $\ell^2(\mathbb{N})$, let U defined in Remark 2.2. For fixed $0 < \varepsilon < 1$, let F_ε be the finite rank operator defined on $\ell^2(\mathbb{N})$ by $F_\epsilon(x_1, x_2, x_3, \ldots) = (-\varepsilon x_1, 0, 0, 0, \ldots)$. We consider the operators T and F defined by $T = U \oplus I$ and $F = 0 \oplus F_\varepsilon$, respectively. Then F is a finite rank operator and $TF = FT$. We have,

$$\sigma(T) = \sigma(U) \cup \sigma(I) = D(0, 1),\ \sigma_a(T) = \sigma_a(U) \cup \sigma_a(I) = C(0, 1),\ \sigma_{BW}(T) = D(0, 1),$$

$$\sigma(T + F) = \sigma(U) \cup \sigma(I + F_\varepsilon) = D(0, 1),\ \sigma_{BW}(T + F) = D(0, 1) \text{ and}$$

$$\sigma_a(T + F) = \sigma_a(U) \cup \sigma_a(I + F_\varepsilon) = C(0, 1) \cup \{1 - \varepsilon\}.$$

Moreover, $E^0_a(T) = \emptyset$ and $E^0_a(T + F) = \{1 - \varepsilon\}$. Thus T satisfies property (Baw), but $T + F$ does not satisfy property (Baw). Note that here $\Pi(T + F) = \emptyset$.

We say that an operator $T \in L(X)$ is finitely polaroid if $\operatorname{iso}\sigma(T) = \Pi^0(T)$ and is said to be finitely a-polaroid if $\operatorname{iso}\sigma_a(T) = \Pi^0_a(T)$.

Lemma 3.10. *Let $T \in L(X)$ and let $S \in \mathcal{F}^0(X)$ which commutes with T.*
i) T is finitely polaroid if and only if $T+S$ is finitely polaroid.
ii) T is finitely a-polaroid if and only if $T+S$ is finitely a-polaroid.

Proof. i) Let T be finitely polaroid and $S \in \mathcal{F}^0(X)$. Then $\operatorname{acc}\sigma(T) = \sigma_b(T)$. Since S commutes with T we have $\sigma_b(T+S) = \sigma_b(T)$ and from [17, Theorem 2.2] we know that $\operatorname{acc}\sigma(T+S) = \operatorname{acc}\sigma(T)$. So $\sigma_b(T+S) = \operatorname{acc}\sigma(T+S)$ and $T+S$ is finitely polaroid. The proof of the reverse implication is similar, since $T = (T+S) - S$ and $T+S$ commutes with $-S$.

ii) Proof similar to the first assertion since $\sigma_{ub}(T+S) = \sigma_{ub}(T)$, see [14] and $\operatorname{acc}\sigma_a(T+S) = \operatorname{acc}\sigma_a(T)$, see [17, Theorem 2.2]. □

Corollary 3.11. *Let $T \in L(X)$ and let $S \in \mathcal{F}^0(X)$ which commutes with T.*
i) If T is finitely polaroid, then T satisfies property (Bw) if and only if $T+S$ satisfies property (Bw).
ii) If T is finitely a-polaroid, then T satisfies property (Baw) if and only if $T+S$ satisfies property (Baw).

Proof. i) Suppose that T satisfies property (Bw). Let $\lambda_0 \in E^0(T+S) \cap \sigma(T)$ be arbitrary, then $\lambda_0 \notin \operatorname{acc}\sigma(T+S) = \operatorname{acc}\sigma(T)$. So $\lambda_0 \in \operatorname{iso}\sigma(T) = \Pi^0(T)$. Hence $E^0(T+S) \cap \sigma(T) \subset \Pi^0(T)$, but this is equivalent by Theorem 3.7 to say that $T+S$ satisfies property (Bw). The proof of the reverse is similar, since $T+S$ is finitely polaroid.
ii) Let $\lambda_0 \in E^0_a(T+S) \cap \sigma(T)$ be arbitrary, then $\lambda_0 \notin \operatorname{acc}\sigma_a(T+S) = \operatorname{acc}\sigma_a(T)$. So $\lambda_0 \in \operatorname{iso}\sigma_a(T) = \Pi^0_a(T)$. As T satisfies (Baw), then from [15, Corollary 3.8] we have $\Pi^0_a(T) = \Pi^0(T)$. Hence $E^0_a(T+S) \cap \sigma(T) \subset \Pi^0(T)$, but this is equivalent by Theorem 3.8 to say that $T+S$ satisfies property (Baw). Analogously, we prove the reverse, since $T+S$ is finitely a-polaroid. □

4. Properties (Bab), (Bb) and Riesz-type perturbations

We begin this section by the following proposition in which, we improve Proposition 3.1 and show that the property (Bb) is stable under commuting perturbations by operators of power finite rank.

Proposition 4.1. *If $T \in L(X)$ satisfies property (Bb) and if $S \in \mathcal{F}^0(X)$ commutes with T, then $T+S$ satisfies property (Bb). In particular, if $S \in \mathcal{F}(X)$ and commutes with T then $T+S$ satisfies property (Bb).*

Proof. Since $S \in \mathcal{F}^0(X)$ and $ST = TS$, then from Lemma 3.5 we have $\Pi(T+S) = \Pi^0(T+S)$. As T satisfies property (Bb), then from [15, Theorem 2.4], it satisfies generalized Browder's theorem and from Lemma 3.5, $T+S$ satisfies generalized Browder's theorem. Thus $\sigma(T+S) \setminus \sigma_{BW}(T+S) = \Pi(T+S) = \Pi^0(T+S)$. So $T+S$ satisfies property (Bb). □

As we have observed in the precedent section, we also cannot extend Proposition 3.1 concerning properties (Bab) and (Bb) to commuting quasi-nilpotent perturbations, as shown in the next example.

Example 4.2. *Let T be the operator defined on $\ell^2(\mathbb{N})$ by $T(x_1, x_2, \ldots) = (\frac{x_2}{2}, \frac{x_3}{3}, \ldots)$. Put $R = -T$, clearly R is quasi-nilpotent, compact and commutes with T. Moreover, we have $\sigma(T) = \{0\} = \sigma_{BW}(T)$ and $\Pi^0(T) = \Pi^0_a(T) = \emptyset$. It follows that T satisfies properties (Bab) and (Bb). But $T+R = 0$ does not satisfy neither property (Bab) nor property (Bb). Indeed, $\sigma(T+R) = \{0\}$, $\sigma_{BW}(T+R) = \emptyset$, $\Pi^0(T+R) = \Pi^0_a(T+R) = \emptyset$. Note also that $\Pi(T+R) = \{0\}$.*

This example shows that the result obtained in Proposition 4.1 cannot be extended to commuting Riesz operators. Nonetheless, we have the next result.

Theorem 4.3. *Let $T \in L(X)$ and let $R \in \mathcal{R}(X)$ which commutes with T. We have:*
i) If T satisfies property (Bb), then $T+R$ satisfies property (Bb) if and only if $\Pi(T+R) = \Pi^0(T+R)$.
ii) If T satisfies property (Bab), then $T+R$ satisfies property (Bab) if and only if $\Pi(T+R) = \Pi^0_a(T+R)$.

Proof. i) If $T+R$ satisfies (Bb), then from [15, Theorem 2.4] we have $\Pi^0(T+R) = \Pi(T+R)$. Conversely, suppose that $\Pi^0(T+R) = \Pi(T+R)$. Since T satisfies property (Bb) then it satisfies Browder's theorem. By Lemma 3.5, $T+R$ satisfies generalized Browder theorem. Thus $\sigma(T+R)\setminus\sigma_{BW}(T+R) = \Pi(T+R) = \Pi^0(T+R)$. So $T+R$ satisfies property (Bb).

ii) If $T+R$ satisfies property (Bab), then $\Pi(T+R) = \Pi_a^0(T+R)$, see [15, Theorem 3.6]. Conversely, assume that $\Pi(T+R) = \Pi_a^0(T+R)$. Since T satisfies property (Bab) then it satisfies property (Bb) and therefore Browder's theorem holds for $T+R$ and generalized Browder's theorem, that is: $\sigma(T+R)\setminus\sigma_{BW}(T+R) = \Pi(T+R)$. So $T+R$ satisfies property (Bab). □

As an application of Theorem 4.3 to the class of quasi-nilpotent operators, we give two corollaries.

Corollary 4.4. *Let $T \in L(X)$ and let $Q \in \mathcal{Q}(X)$ such that $TQ = QT$.*
i) If T satisfies property (Bb), then the statements a), b) and c) are equivalent:
a) $T+Q$ satisfies property (Bb);
b) $\Pi(T+Q) = \Pi^0(T)$;
c) $\sigma_{BW}(T+Q) = \sigma_{BW}(T)$.
ii) If $\operatorname{iso}\sigma_W(T) = \emptyset$ or $\operatorname{iso}\sigma_b(T) = \emptyset$, then T satisfies property (Bb) if and only if $T+Q$ satisfies property (Bb).

Proof. i) a) $\Longleftrightarrow$ b) Since T commutes with Q, we know that $\sigma(T+Q) = \sigma(T)$ and $\sigma_a(T+Q) = \sigma_a(T)$. As it was already mentioned, we have $\sigma_b(T+Q) = \sigma_b(T)$. So $\Pi^0(T+Q) = \sigma(T+Q)\setminus\sigma_b(T+Q) = \sigma(T)\setminus\sigma_b(T) = \Pi^0(T)$. Hence the equivalence between statements a) and b) is a consequence of Theorem 4.3.

a) $\Longleftrightarrow$ c) If $T+Q$ satisfies property (Bb) then $\sigma_{BW}(T+Q) = \sigma(T+Q)\setminus\Pi^0(T+Q) = \sigma(T)\setminus\Pi^0(T) = \sigma_{BW}(T)$, since T satisfies property (Bb). Conversely, assume that $\sigma_{BW}(T+Q) = \sigma_{BW}(T)$. Then $\sigma(T+Q)\setminus\sigma_{BW}(T+Q) = \sigma(T)\setminus\sigma_{BW}(T) = \Pi^0(T) = \Pi^0(T+Q)$. So $T+Q$ satisfies property (Bb).

ii) Case 1. $\operatorname{iso}\sigma_W(T) = \emptyset$: assume that T satisfies property (Bb). The condition $\operatorname{iso}\sigma_W(T) = \emptyset$ implies from [6, Proposition 2.4] that $\sigma_{BW}(T+Q) = \sigma_{BW}(T)$. So from the assertion i), it follows that $T+Q$ satisfies property (Bb). Conversely, assume that $T+Q$ satisfies property (Bb). Since $\operatorname{iso}\sigma_W(T+Q) = \emptyset$ and $T = (T+Q)-Q$, we conclude similarly.

Case 2. $\operatorname{iso}\sigma_b(T) = \emptyset$: assume that T satisfies property (Bb). The condition $\operatorname{iso}\sigma_b(T) = \emptyset$ implies from [6, Corollary 2.10] that $\Pi(T+Q) = \Pi(T)$, and since T satisfies property (Bb) then $\Pi(T) = \Pi^0(T)$ as seen in the proof of Theorem 2.1. So $\Pi(T+Q) = \Pi^0(T)$ and hence $T+Q$ satisfies property (Bb). We obtain the proof of the converse analogously, since $\operatorname{iso}\sigma_b(T+Q) = \emptyset$. □

In the following corollary, we give conditions ensuring the stability of property (Bab) under commuting perturbations by quasi-nilpotent operators.

Corollary 4.5. *Let $T \in L(X)$ and let $Q \in \mathcal{Q}(X)$ such that $TQ = QT$.*
i) If T satisfies property (Bab), then the statements a), b) and c) are equivalent:
a) $T+Q$ satisfies property (Bab);
b) $\Pi(T+Q) = \Pi_a^0(T)$;
c) $\sigma_{BW}(T+Q) = \sigma_{BW}(T)$.
ii) If $\operatorname{iso}\sigma_W(T) = \emptyset$ or $\operatorname{iso}\sigma_b(T) = \emptyset$, then T satisfies property (Bab) if and only if $T+Q$ satisfies property (Bab).

Proof. Is similar to the proof of the precedent corollary, since $\Pi_a^0(T+Q) = \Pi_a^0(T)$ and since the equality $\Pi(T) = \Pi_a^0(T)$ holds for every operator T satisfying property (Bab). □

If we restrict to the class $\mathcal{F}^0(X)$, we obtain the following perturbation result concerning property (Bab).

Theorem 4.6. *Let $S \in \mathcal{F}^0(X)$. If $T \in L(X)$ satisfies property (Bab) and commutes with S, then the following statements are equivalent.*
i) $T+S$ satisfies property (Bab);
ii) $\Pi(T+S) = \Pi_a^0(T+S)$;
iii) $\Pi_a^0(T+S)\cap\sigma(T) \subset \Pi^0(T)$.

Proof. i) $\Longleftrightarrow$ ii) Since $\mathcal{F}^0(X) \subset \mathcal{R}(X)$, this equivalence is a consequence of Theorem 4.3.
ii) $\Longrightarrow$ iii) Suppose that $\Pi(T+S) = \Pi_a^0(T+S)$ and let $\lambda_0 \in \Pi_a^0(T+S)\cap\sigma(T)$ be arbitrary. Then $\lambda_0 \in \Pi^0(T+S)\cap\sigma(T)$ and so $\lambda_0 \notin \sigma_b(T+S) = \sigma_b(T)$. Thus $\lambda_0 \in \Pi^0(T)$. This proves that $\Pi_a^0(T+S)\cap\sigma(T) \subset \Pi^0(T)$.
iii) $\Longrightarrow$ ii) Suppose that $\Pi_a^0(T+S)\cap\sigma(T) \subset \Pi^0(T)$. Firstly, we show that $\Pi_a^0(T+S) \subset \Pi(T+S)$. Let $\mu_0 \in \Pi_a^0(T+S)$ be arbitrary. We distinguish two cases: the first is $\mu_0 \in \sigma(T)$. Then $\mu_0 \in \Pi_a^0(T+S)\cap\sigma(T) \subset \Pi^0(T)$. It follows that $\mu_0 \notin \sigma_b(T) = \sigma_b(T+S)$ and since $\mu_0 \in \sigma(T+S)$, then $\mu_0 \in \Pi(T+S)$. The second case is $\mu_0 \notin \sigma(T)$. This implies that $\mu_0 \notin \sigma_b(T) = \sigma_b(T+S)$. As $\mu_0 \in \sigma(T+S)$ then $\mu_0 \in \Pi^0(T+S) \subset \Pi(T+S)$. Therefore $\Pi_a^0(T+S) \subset \Pi(T+S)$. From Lemma 3.5, we conclude that $\Pi(T+S) = \Pi_a^0(T+S)$. □

Generally, the property (*Bab*) is not preserved under commuting power finite rank perturbations. For this, we consider the operators T and F defined in Example 3.9. T satisfies property (*Baw*) and then property (*Bab*). But $T+F$ does not satisfy property (*Bab*) since $\sigma(T+F)\setminus\sigma_{BW}(T+F) = \emptyset$ and $\Pi_a^0(T+F) = \{1-\varepsilon\}$. Here $\Pi_a^0(T+F)\cap\sigma(T) = \{1-\varepsilon\}$ and $\Pi^0(T) = \emptyset$.

From Theorem 4.6, we obtain immediately the following corollary:

Corollary 4.7. *Let $T \in L(X)$ be finitely a-polaroid. If $S \in \mathcal{F}^0(X)$ commutes with T, then T satisfies property* (*Bab*) *if and only if $T+S$ satisfies property* (*Bab*).

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References

[1] P. Aiena and O. Garcia, *Generalized Browder's Theorem and SVEP*, Mediterr. J. Math. **4** (2007), 215–228.
[2] M. Amouch, H. Zguitti, *On the equivalence of Browder's and generalized Browder's theorem*, Glasgow Math. J. **48** (2006), 179–185.
[3] B. A. Barnes, *Riesz points and Weyl's theorem*, Integr. Equ. and Oper. Theory **34** (1999), 187–196.
[4] M. Berkani, H. Zariouh, *Extended Weyl type theorems and perturbations*, Math. Proc. R. Ir. Acad. **110**A (2010), 73–82.
[5] M. Berkani, H. Zariouh, *Perturbation results for Weyl type theorems*, Acta Math. Univ. Comenianae **80** (2011), 119–132.
[6] M. Berkani, H. Zariouh, *B-Fredholm spectra and Riesz perturbations*, Mat. Vesnik **67** (2015), 155–165.
[7] D. S. Djordjević, *Operators obeying a-Weyls theorem*, Publ. Math. Debrecen **55** (1999), 283–298.
[8] H. Elbjaoui and E. H. Zerouali, *Local spectral theory for* 2×2 *operator matrices*, Inter. J. Math. and Math. Sc. **42** (2003), 2667–2672.
[9] A. Gupta and N. Kashyap, *Property (Bw) and Weyl type theorems*, Bull. Math. Anal. Appl. **3** (2011), 1–7.
[10] H. G. Heuser, *Functional Analysis*, John Wiley & Sons Inc, Chichester, 1982.
[11] K. B. Laursen and M. M. Neumann, *An introduction to Local Spectral Theory*, the Clarendon Press, Oxford university press, New York, 2000.
[12] W. Y. Lee, S. H. Lee, *On Weyl's theorem II*, Math. Japonica **43** (1996), 549–553.
[13] K. K. Oberai, *Spectral mapping theorem for essential spectra*, Rev. Roum. Math. Pures et Appl. **25** (1980), 365–373.
[14] V. Rakočević, *Semi-Browder operators and perturbations*, Studia Math. **122** (1997), 131–137.
[15] H. Zariouh and H. Zguitti, *Variations on Browder's Theorem*, Acta Math. Univ. Comenianae **81** (2012), 255–264.
[16] Q. P. Zeng, Q. F. Jiang and H. J. Zhong, *Spectra originating from semi-B-Fredholm theory and commuting perturbations*, Studia Math. **219** (2013), 1–18.
[17] Q. P. Zeng, H. J. Zhong and K. Yan, *An extension of a result of Djordjević and its applications*, to appear in Linear and Multilinear Algebra (2015).