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A note on some results related to infinite matrices on weighted ℓ_1 spaces

Ivana Djolović[a], Eberhard Malkowsky[b]

[a]*Technical Faculty in Bor, University of Belgrade, VJ 12, 19210 Bor, Serbia*
[b]*Department of Mathematics, Faculty of Science, Fatih University, Büyükçekmece 34500, Istanbul, Turkey c/o Državni Univerzitet u Novom Pazaru, Vuka Karadžića bb, 36300 Novi Pazar, Serbia*

Abstract. We were motivated by the results in [1] and present another approach to obtain some of the results in [1], applying the theory of matrix transformations which provides completely different techniques of the proofs. This also demonstrates one more application of infinite matrices.

1. Introduction and Notation

We demonstrate how applications of the theory of matrix transformations, mainly results of the *BK* space theory, yield alternative proofs of the fundamental results in [1]. Among other things, we give a simple direct proof of [1, Corollary 1], which is the basis for the results in [1, Sections 3 and 4]. We also obtain some of the results in [4].

As usual, let ω, ℓ_∞ and ϕ denote the sets of all complex, bounded and finite sequences $x = (x_k)_{k=0}^\infty$, respectively. Furthermore, we write cs, bs and ℓ_1 for the sets convergent, bounded and absolutely convergent series.

A set X of sequences is said to be *normal* or *solid*, if $x \in X$ and $\tilde{x} \in \omega$ with $|\tilde{x}_k| \leq |x_k|$ for all k imply $\tilde{x} \in X$.

Let X and Y be subsets of ω and $z \in \omega$. Then we use the notation ([11, Definition 4.3.4])

$$z^{-1} * Y = \left\{x \in \omega : x \cdot z = (x_k z_k)_{k=0}^\infty \in Y\right\},$$

and write

$$M(X, Y) = \bigcap_{x \in X} x^{-1} * Y$$

for multiplier space of X and Y; the special cases where $Y = \ell_1$ and $Y = cs$ are called the α– and β–duals of X, denoted by $X^\alpha = M(X, \ell_1)$ and $X^\beta = M(X, cs)$. It is clear that $X^\alpha \subset X^\beta$ for all $X \subset \omega$ and $X^\beta \subset X^\alpha$ whenever X is normal.

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Email addresses: zucko@mts.rs (Ivana Djolović), Eberhard.Malkowsky@math.uni-giessen.de; ema@Bankerinter.net (Eberhard Malkowsky)

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers, X and Y be subsets of ω and $x \in \omega$. We write $A_n = (a_{nk})_{k=0}^{\infty}$ and $A^{(k)} = (a_{nk})_{n=0}^{\infty}$ for the sequences in the n–th row and k–th column of A, $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ and $Ax = (A_n x)_{n=0}^{\infty}$ (provided all the series $A_n x$ converge). The set $X_A = \{x \in \omega : Ax \in X\}$ is called the matrix domain of A in X and (X, Y) denotes the class of all matrices that map X into Y, that is, the class of all matrices A such that $X \subset Y_A$; hence, $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for all n and $Ax \in Y$ for all $x \in X$.

A matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is said to be a triangle if $t_{nk} = 0$ for all $k > n$ and $t_{nn} \neq 0$ $(n = 0, 1 \ldots)$. Throughout, let T denote a triangle, S its inverse and $R = S^t$, the transpose of S. We remark that the inverse of a triangle exists, is unique and a triangle ([11, 1.4.8, p. 9] and [9, Remark 22 (a), p. 22]).

If X and Y are Banach spaces, then we write, as usual, $\mathcal{B}(X, Y)$ for the set of all bounded linear operators $L : X \to Y$ with the operator norm $\|\cdot\|$ defined by $\|L\| = \sup\{\|L(x)\| : \|x\| = 1\}$.

A Banach space $X \subset \omega$ is a *BK* space if each coordinate $P_n : X \to \mathbb{C}$ with $P_n(x) = x_n$ for all $x = (x_k)_{k=0}^{\infty} \in X$ is continuous. A *BK* space $X \supset \phi$ is said to have *AK* if $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \to x$ $(m \to \infty)$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, where $e^{(n)}$ $(n = 0, 1, \ldots)$ is the sequence with $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$.

The important next result is well known.

Lemma 1.1. *Let X and Y be BK spaces.*
(a) Then we have $(X, Y) \subset \mathcal{B}(X, Y)$*, that is, every* $A \in (X, Y)$ *defines an operator* $L_A \in \mathcal{B}(X, Y)$ *where* $L_A(x) = Ax$ *for all* $x \in X$ ([5, Theorem 1.23] or [11, Theorem 4.2.8]).
(b) If X has AK then we have $\mathcal{B}(X, Y) \subset (X, Y)$*, that is, every* $L \in \mathcal{B}(X, Y)$ *is given by a matrix* $A \in (X, Y)$ *such that* $Ax = L(x)$ *for all* $x \in X$ ([3, Theorem 1.9.]).

Since ℓ_1 is a *BK* space with *AK* with respect to its natural norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{k=0}^{\infty} |x_k|$ for all $x = (x_k)_{k=0}^{\infty} \in \ell_1$ ([11, p. 55 and Example 4.2.13]), we obtain $\mathcal{B}(\ell_1, \ell_1) = (\ell_1, \ell_1)$, in particular, that every bounded linear operator $L : \ell_1 \to \ell_1$ is given by an infinite matrix $A \in (\ell_1, \ell_1)$.

2. Weighted ℓ_1 Spaces

Let $\mathcal{U} = \{u \in \omega : u_k \neq 0 \text{ for all } k\}$ and $\mathbf{r} \in \mathcal{U}$ throughout; we write $1/u = (1/u_k))_{k=0}^{\infty}$ for $u \in \mathcal{U}$. We consider the sets

$$\ell_1(\mathbf{r}) = \mathbf{r}^{-1} * \ell_1 = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k r_k| < \infty \right\}.$$

associated with the sequence $\mathbf{r}$. Since the set ℓ_1 is normal, this coincides with the definition of the weighted ℓ_1 spaces $\ell_1(\mathbf{r})$ for positive sequences $\mathbf{r}$ in [1]. These spaces were also studied in [4].

Since $(\ell_1, \|\cdot\|_1)$ is a *BK* space, we obtain as an immediate consequence of [11, Theorems 4.3.6 and 4.3.12]

Remark 2.1. *The set* $\ell_1(\mathbf{r})$ *is a BK space with AK with respect to its natural norm* $\|\cdot\|_{\mathbf{r}}$ *given by*

$$\|x\|_{\mathbf{r}} = \|x \cdot \mathbf{r}\|_1 = \sum_{k=0}^{\infty} |x_k r_k| \text{ for all } x \in \ell_1(\mathbf{r}).$$

Furthermore, we observe that if $\mathbf{r}, \mathbf{s} \in \mathcal{U}$ and X and Y are arbitrary subsets of ω then it is clear from the definition of the of the sets $\mathbf{r}^{-1} * X$ and $\mathbf{s}^{-1} * Y$ that

$$a \in M(\mathbf{r}^{-1} * X, \mathbf{s}^{-1} * Y) \text{ if and only if } a \in (\mathbf{s}/\mathbf{r})^{-1} * M(X, Y), \text{ where } \mathbf{s}/\mathbf{r} = \left(\frac{s_k}{r_k}\right)_{k=0}^{\infty} \tag{1}$$

and

$$A \in (\mathbf{r}^{-1} * X, \mathbf{s}^{-1} * Y) \text{ if and only if } B = (b_{nk})_{n,k=0}^{\infty} \in (X, Y) \text{ where } b_{nk} = \frac{s_n a_{nk}}{r_k} \tag{2}$$

for all n and k. It is well known that $\ell_1^\beta = \ell_\infty$ ([11, Example 7.3.1]). Applying (1) with $\mathbf{s} = e = (1, 1, \dots)$, we obtain

$$(\ell_1(\mathbf{r}))^\beta = (1/\mathbf{r})^{-1} * \ell_\infty = \left\{ a \in \omega : \sup_k \left| \frac{a_k}{r_k} \right| < \infty \right\}.$$

We may also consider the set $\ell_1(\mathbf{r})$ as the matrix domain in ℓ_1 of the triangle $T = D(\mathbf{r})$, the diagonal matrix with the sequence $\mathbf{r}$ on its diagonal, that is, $D(\mathbf{r})$ is the matrix with the rows $(D(\mathbf{r}))_n = r_n e^{(n)}$. So we have $\ell_1(\mathbf{r}) = (\ell_1)_{D(\mathbf{r})}$.

The following useful known results concern matrix transformations between matrix domains of triangles.

Lemma 2.2. ([5, Theorem 3.8]) *Let T be triangle.*
(a) Then, for arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.
(b) If X and Y are BK spaces and $A \in (X, Y_T)$, then $\|L_A\| = \|L_B\|$.

Lemma 2.3. ([6, Theorem 3.6]) *Let X and Y be BK spaces and X have AK. If $A \in (X_T, Y)$ then we have*

$$\|L_A\| = \|L_{\hat{A}}\| \tag{3}$$

where $\hat{A} \in (X, Y)$ is the matrix with the rows $\hat{A}_n = RA_n$ for $(n = 0, 1, \dots)$.

The characterization of the class (ℓ_1, ℓ_1) is a classical result (cf. for instance, [7], [10, **77**] or [11, 8.4.1D]). Here we state the result in the form needed in the sequel; it contains [1, Lemma 1].

Theorem 2.4. ([5, Theorem 2.27]) *We have $L \in \mathcal{B}(\ell_1, \ell_1)$ if and only if*

$$\|A\|_{(1,1)} = \sup_k \sum_{n=0}^\infty |a_{nk}| < \infty, \tag{4}$$

where $A \in (\ell_1, \ell_1)$ is the matrix that represents L (Lemma 1.1); moreover

$$\|L\| = \|A\|_{(1,1)}. \tag{5}$$

Theorem 2.5. ([2, Theorem 2.6]) *Let T and $\tilde{T}$ be triangles. Then we have $A \in ((\ell_1)_T, (\ell_1)_{\tilde{T}})$ if and only if*

$$\sup_k \sum_{n=0}^\infty \left| \sum_{j=k}^\infty s_{jk} \sum_{i=0}^n \tilde{t}_{ni} a_{ij} \right| < \infty, \tag{6}$$

and

$$\sup_{m,k} \left| \sum_{j=m}^\infty s_{jk} a_{nj} \right| < \infty \textit{ for all } n = 0, 1, \dots \tag{7}$$

3. Main Results - New Approach to the General Results

In this section we are going to prove the main results of [1, Section 2] in a new way.
We start with a useful result which is easily obtained from Lemmas 2.2 and 2.3.

Proposition 3.1. *Let X and Y be BK space and X have AK, and T and $\tilde{T}$ be triangles. If $A \in (X_T, Y_{\tilde{T}})$ then*

$$\|L_A\| = \|L\|_{\hat{B}}, \textit{ where } \hat{B} = R(\tilde{T}A) \in (X, Y). \tag{8}$$

Proof. Let $A \in (X_T, Y_{\tilde{T}})$. Then it follows by Lemmas 2.2 and 2.3 that $B = \tilde{T}A \in (X_T, Y)$ and $\|L_A\| = \|L_B\| = \|L_{\hat{B}}\|$, where $\hat{B} = RB = R(\tilde{T}A) \in (X, Y)$. □

Remark 3.2. *It was shown in [2, Remark 2.5] that if $A \in (X_T, Y_{\tilde{T}})$ then $\hat{A} \in (X, Y_T)$ by [6, Theorem 3.4], and then $\hat{C} = \tilde{T}\hat{A} = \tilde{T}(RA) \in (X, Y)$ and $\hat{B} = \hat{C}$.*

Theorem 3.3. ([1, Theorem 1]) *Given two sequences $\mathbf{r}, \mathbf{s} \in \mathcal{U}$, and an infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$, then $L_A \in \mathcal{B}(\ell_1(\mathbf{r}), \ell_1(\mathbf{s}))$ if and only if*

$$\|A\|_{(\mathbf{r},\mathbf{s})} = \sup_k \sum_{n=0}^{\infty} \left| \frac{s_n a_{nk}}{r_k} \right| < \infty. \tag{9}$$

In this case, the operator norm of L_A is given by

$$\|L_A\|_{(\mathbf{r},\mathbf{s})} = \|A\|_{(\mathbf{r},\mathbf{s})}. \tag{10}$$

Proof. We are going to prove the theorem in two different simple ways.

First, if we consider the spaces $\ell_1(\mathbf{r})$ and $\ell_1(\mathbf{s})$ as the matrix domains of the triangles $D(\mathbf{r})$ and $D(\mathbf{s})$ in ℓ_1, respectively. Putting $T = D(\mathbf{r})$ and $\tilde{T} = D(\mathbf{s})$ in Theorem 2.5, we obtain $A \in (\ell_1(\mathbf{r}), \ell_1(\mathbf{s}))$ if and only if (9) holds, and $\sup_k |a_{nk}/r_k| < \infty$ for $n = 0, 1, \ldots$, which clearly is redundant.

The identity for the operator norm in (10) follows from (8), since obviously the entries of the matrix $\hat{B}$ are given by $\hat{b}_{nk} = s_n a_{nk}/r_k$ for all n and k.

The second way of proof is more elementary. We use $\ell_1(\mathbf{r}) = \mathbf{r}^{-1} * \ell_1$ and $\ell_1(\mathbf{s}) = \mathbf{s}^{-1} * \ell_1(\mathbf{s})$ and obtain directly from (2) and (4) that $L_A \in \mathcal{B}(\ell_1(\mathbf{r}), \ell_1(\mathbf{s}))$ if and only if (9) is satisfied. Furthermore, since $x \in \ell_1(\mathbf{r})$ if and only if $y = \mathbf{r}\cdot x \in \ell_1$, we obtain by the definition of the norm $\|\cdot\|_{\mathbf{s}}$ and (2) that $\|L_A(x)\|_s = \|\mathbf{s}\cdot L(x)\|_1 = \|L_B(y)\|_1$ where B is the matrix with the entries $b_{nk} = s_n a_{nk}/r_k$ for all n and k. Now the identity for the operator norm in (10) follows from (9), the definition of the operator norm, and the fact that $\|x\|_{\mathbf{r}} = \|y\|_1$. □

Remark 3.4. *Since $\ell_1(\mathbf{r})$ is a BK space with AK by Remark 2.1, it follows from Lemma 1.1 that $(\ell_1(\mathbf{r}), \ell_1(\mathbf{s})) = \mathcal{B}(\ell_1(\mathbf{r}), \ell_1(\mathbf{s}))$.*

Theorem 3.5. ([1, Lemma 3]) *Given any infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ and a sequence $\mathbf{r}$, then $L_A \in \mathcal{B}(\ell_1(\mathbf{r}), \ell_1(\mathbf{s}))$ for some sequence $\mathbf{s}$ if and only if*

$$s_n(A) = \|A_n/\mathbf{r}\|_\infty = \sup_k \left| \frac{a_{nk}}{r_k} \right| < \infty \text{ for all } n = 0, 1, \ldots. \tag{11}$$

Proof. The necessity of the condition in (11) is trivial, since $L_A \in \mathcal{B}(\ell_1(\mathbf{r}), \ell_1(\mathbf{s}))$ means, in particular, $A_n \in (\ell_1(\mathbf{r}))^\beta = (1/\mathbf{r})^{-1} * \ell_\infty$ for all n.

To show the sufficiency of the condition, we assume that the condition in (11) is satisfied. Then we have $A_n \in (1/\mathbf{r})^{-1} * \ell_\infty = (\ell_1(\mathbf{r}))^\beta$ for $n = 0, 1, \ldots$. We define the sequence $s = (s_n)_{n=0}^{\infty}$ by

$$s_n = \begin{cases} \dfrac{1}{s_n(A)(n+1)^2} & \text{if } A_n \in \mathcal{U} \\ \dfrac{1}{(n+1)^2} & \text{otherwise.} \end{cases}$$

Now let $x \in \ell_1(\mathbf{r})$ be given. Then we obtain

$$\begin{aligned} \|Ax\|_{\mathbf{s}} &= \sum_{n=0}^{\infty} |A_n x s_n| \le \sum_{n=0}^{\infty} |s_n| \sum_{k=0}^{\infty} \left| \frac{a_{nk}}{r_k} \right| \cdot |x_k r_k| \le \left(\sum_{n=0}^{\infty} |s_n| \cdot \sup_k \left| \frac{a_{nk}}{r_k} \right| \right) \cdot \|x\|_{\mathbf{r}} \\ &= \left(\sum_{n=0}^{\infty} |s_n| \cdot |s_n(A)| \right) \cdot \|x\|_{\mathbf{r}} = \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \right) \cdot \|x\|_{\mathbf{r}} < \infty, \end{aligned}$$

that is, $Ax \in \ell_1(\mathbf{s})$, and so $A \in (\ell_1(\mathbf{r}), \ell_1(\mathbf{s}))$. This completes the proof of the sufficiency of the condition in (11). □

Now we give a direct proof of [1, Corollary 1] without applying [1, Theorem 2].

Theorem 3.6. ([1, Corollary 1]) *Every infinite matrix* $A = (a_{nk})_{n,k=0}^{\infty}$ *is in* $(\ell_1(\mathbf{r}), \ell_1(\mathbf{s}))$ *for some pair of sequence* $\mathbf{r}, \mathbf{s} \in \mathcal{U}$.

Proof. Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix. We write

$$b_k = \max_{0 \le l,j \le k} |a_{lj}| \text{ and put } r_k = 1 + b_k \text{ for } k = 0, 1, \dots.$$

Then we obviously have $\mathbf{r} \in \mathcal{U}$ and it follows that $|a_{nk}| \le \max\{b_k.b_n\} \le (1 + b_k)(1 + b_n)$ for all n and k, hence

$$\left|\frac{a_{nk}}{r_k}\right| \le \frac{(1 + b_n)(1 + b_k)}{(1 + b_k)} = 1 + b_n < \infty \text{ for each } n \in \mathbb{N}_0.$$

By Theorem 3.5, there is $\mathbf{s} \in \mathcal{U}$ such that $A \in (\ell_1(\mathbf{r}), \ell_1(\mathbf{s}))$. □

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