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Solutions to some solvable modular operator equations

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Abstract. We find explicit solution of the operator equation $TXT^* - SXS^* = A$ in the general setting of the adjointable operators between Hilbert C^*-modules, using some block operator matrices. Furthermore, we obtain solutions to the solvable operator equation $TXR - SYQ = A$ over Hilbert C^*-modules, when both $\operatorname{ran}(T) + \operatorname{ran}(S)$ and $\operatorname{ran}(R^*) + \operatorname{ran}(Q^*)$ are closed.

1. Introduction and Preliminary

Xu and Sheng [11] showed that a bounded adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. Djordjević in [3] obtain explicit solution of the operator equation $A^*X + X^*A = B$ for Hilbert space operators, such that this solution is expressed in terms of the Moore- Penrose inverse of the operator A. In this paper, using block operator matrices and the Moore-Penrose inverse properties, we provide a new approach to the study of the equation $TXT^* - SXS^* = A$ for adjointable Hilbert module operators with closed ranges.

The operator equation $TXR - SYQ = A$ was studied by [1, 10, 12] for finite matrices. In this paper we obtain solutions to the operator equation $TXR - SYQ = A$ when both $\operatorname{ran}(T) + \operatorname{ran}(S)$ and $\operatorname{ran}(R^*) + \operatorname{ran}(Q^*)$ are closed in general setting of adjointable operators between in Hilbert C^*-modules. This solution is also expressed in terms of the Moore- Penrose inverse of the operator A.

Throughout this paper, $\mathcal{A}$ is a C^*-algebra. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules. A mapping $T : \mathcal{X} \to \mathcal{Y}$ is $\mathcal{A}$-linear, provided that for all $x, y \in \mathcal{X}$, all $\lambda, \mu \in \mathbb{C}$ and all $a \in A$ the following hold:

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty, \qquad T(xa) = T(x)a.$$

$\mathcal{A}$-linear mappings will be called *operators*. An operator $T : \mathcal{X} \to \mathcal{Y}$ is adjointable, if there exists an operator $T^* : \mathcal{Y} \to \mathcal{X}$ such that for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$ the following holds:

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

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If T is adjointable, then T^* is unique, and both T and T^* are bounded.

Let $\mathcal{L}(\mathcal{X},\mathcal{Y})$ be the set of the adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. For any $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, the range, the null space of T are denoted by $\mathrm{ran}(T)$ and $\ker(T)$ respectively. In case $\mathcal{X} = \mathcal{Y}$, $\mathcal{L}(\mathcal{X},\mathcal{X})$ which we abbreviate to $\mathcal{L}(\mathcal{X})$, is a C^*-algebra. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity.

Theorem 1.1. *(See Theorem 3.2 of [7]) Suppose that $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ has closed range. Then*

- $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$, with complement* $\mathrm{ran}(T^*)$.
- $\mathrm{ran}(T)$ *is orthogonally complemented in $\mathcal{Y}$, with complement* $\ker(T^*)$.
- *The map $T^* \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ has closed range.*

The Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ satisfying the following conditions:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

It is well-known that $T^\dagger$ exists if and only if $\mathrm{ran}(T)$ is closed, and in this case $(T^\dagger)^* = (T^*)^\dagger$ (see [11]).

Let $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ has a closed range. Then $TT^\dagger$ is the orthogonal projection from $\mathcal{Y}$ onto $\mathrm{ran}(T)$ and $T^\dagger T$ is the orthogonal projection from $\mathcal{Y}$ onto $\mathrm{ran}(T^*)$. Here the term "projection" means a self adjoint idempotent operator.

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp, \quad \mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \tag{1.1}$$

where $T_1 \in \mathcal{L}(\mathcal{M},\mathcal{N})$, $T_2 \in \mathcal{L}(\mathcal{M}^\perp,\mathcal{N})$, $T_3 \in \mathcal{L}(\mathcal{M},\mathcal{N}^\perp)$ and $T_4 \in \mathcal{L}(\mathcal{M}^\perp,\mathcal{N}^\perp)$. Note that $P_{\mathcal{M}}$ denotes the projection corresponding to $\mathcal{M}$.

In fact $T_1 = P_{\mathcal{N}}TP_{\mathcal{M}}, \quad T_2 = P_{\mathcal{N}}T(1-P_{\mathcal{M}}), T_3 = (1-P_{\mathcal{N}})TP_{\mathcal{M}}, \quad T_4 = (1-P_{\mathcal{N}})T(1-P_{\mathcal{M}})$.

The proof of the following Lemma can be found [6, Corollary 1.2.] or [5, Lemma 1.1.].

Lemma 1.2. *Suppose that $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules $\mathcal{X} = \mathrm{ran}(T^*) \oplus \ker(T)$ and $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(T^*)$:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix},$$

where T_1 is invertible. Moreover

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix}.$$

Lemma 1.3. *Let $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$. Then T^*T is positive in $\mathcal{L}(\mathcal{X})$, and TT^* is positive in $\mathcal{L}(\mathcal{Y})$.*

Proof. Since $\mathcal{L}(\mathcal{X})$ is a C^*-algebra, the proof is finished if $\mathcal{X} = \mathcal{Y}$. In a general case, recall that T is positive in $\mathcal{L}(\mathcal{X})$ if and only if $\langle Tx, x\rangle$ is positive in $\mathcal{L}(\mathcal{X})$ for all $x \in \mathcal{X}$ (see [8], Proposition 2.1.3). Now, for all $x \in \mathcal{X}$ we have that $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle$ is positive in $\mathcal{L}(\mathcal{X})$, implying that T^*T is positive in $\mathcal{L}(\mathcal{X})$. Analogously, TT^* is positive in $\mathcal{L}(\mathcal{Y})$. □

Lemma 1.4. (See Lemma 1.2. of [9]) Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Let $\mathcal{X}_1$, $\mathcal{X}_2$ be closed submodules of $\mathcal{X}$, and let $\mathcal{Y}_1$, $\mathcal{Y}_2$ be closed submodules of $\mathcal{Y}$ such that $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of submodules $\mathcal{X} = \mathrm{ran}(T^*) \oplus \ker(T)$ and $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(T^*)$:

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix}$$

Then $D = T_1 T_1^* + T_2 T_2^* \in \mathcal{L}(\mathrm{ran}(T))$ is positive and invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{bmatrix}. \tag{1.2}$$

We also have:

$$T = \begin{bmatrix} T_1 & 0 \\ T_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix}, \tag{1.3}$$

where $F = T_1^* T_1 + T_3^* T_3 \in \mathcal{L}(\mathrm{ran(T^*)})$ is positive and invertible. Moreover,

$$T^\dagger = \begin{bmatrix} F^{-1} T_1^* & F^{-1} T_2^* \\ 0 & 0 \end{bmatrix}. \tag{1.4}$$

Notice that positivity of D and F follow from Lemma 1.3.

2. The solutions to some operator equations

In this section, we will study the operator equations $TXT^* - SXS^* = A$ and $TXQ - SYR = A$ where X and Y are the unknown operators.

The proof of the following lemma is the same as in the matrix case.

Lemma 2.1. *Suppose that $\mathcal{X}, \mathcal{Y}$ are Hilbert $\mathcal{A}$-modules, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ have closed ranges, and let $A \in \mathcal{L}(\mathcal{Y})$. Then the equation*

$$TXS = A, \tag{2.1}$$

has a solution $X \in \mathcal{L}(\mathcal{X})$ if and only if

$$TT^\dagger AS^\dagger S = A. \tag{2.2}$$

In which case, any solution X to Eq. (2.1) is of the form

$$X = T^\dagger AS^\dagger. \tag{2.3}$$

Lemma 2.2. *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, let $Q \in \mathcal{L}(\mathcal{X})$ and $P \in \mathcal{L}(\mathcal{Y})$ be orthogonal projections, and let TQ and PT have closed ranges. Then*

1. $(TQ)^\dagger = Q(TQ)^\dagger$,
2. $(PT)^\dagger = (PT)^\dagger P$.

Proof. (i) Since $\mathrm{ran}(TQ)$ is closed, the operator $(TQ)^\dagger$ exists, therefore $\mathrm{ran}((TQ)^\dagger) = \mathrm{ran}((TQ)^*) = \mathrm{ran}(QT^*) \subseteq \mathrm{ran}(Q)$. Hence $Q(TQ)^\dagger = (TQ)^\dagger$. The proof for (ii) is similar. □

Theorem 2.3. *Suppose that $\mathcal{X}, \mathcal{Y}$ are Hilbert $\mathcal{A}$-modules, $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges, $A \in \mathcal{L}(\mathcal{Y})$. If the operator equation*

$$TXT^* - SXS^* = A \ , \ X \in \mathcal{L}(\mathcal{X}), \tag{2.4}$$

is solvable, then

$$X = -S^\dagger A(S^\dagger)^* + S^\dagger T((1 - SS^\dagger)T)^\dagger A(T^*(1 - SS^\dagger))^\dagger T^*(S^\dagger)^*$$

is a solution of Eq. (2.4).

Proof. Since S, T have closed ranges, we have $\mathcal{X} = \text{ran}(T^*) \oplus \ker(T)$ and $\mathcal{Y} = \text{ran}(S) \oplus \ker(S^*)$. Hence by matrix form (1.1) with these orthogonally complemented submodules, we get

$$T = \begin{bmatrix} T_1 & 0 \\ T_3 & 0 \end{bmatrix} : \begin{bmatrix} \text{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \text{ran}(S) \\ \ker(S^*) \end{bmatrix} \tag{2.5}$$

and

$$S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \text{ran}(S) \\ \ker(S^*) \end{bmatrix} \tag{2.6}$$

and $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \text{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \text{ran}(T^*) \\ \ker(T) \end{bmatrix}$ and $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} \text{ran}(S) \\ \ker(S^*) \end{bmatrix} \to \begin{bmatrix} \text{ran}(S) \\ \ker(S^*) \end{bmatrix}$. The Eq. (2.4) which can be written in an equivalent form

$$\begin{aligned} &\begin{bmatrix} T_1 & 0 \\ T_3 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} T_1^* & T_3^* \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} S_1^* & 0 \\ S_2^* & 0 \end{bmatrix} \\ = &\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \end{aligned}$$

that is

$$\begin{bmatrix} T_1X_1T_1^* - S_1X_1S_1^* - S_1X_2S_2^* - S_2X_3S_1^* - S_2X_4S_2^* & T_1X_1T_3^* \\ T_3X_1T_1^* & T_3X_1T_3^* \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

Therefore

$$T_1X_1T_1^* - S_1X_1S_1^* - S_1X_2S_2^* - S_2X_3S_1^* - S_2X_4S_2^* = A_1, \tag{2.7}$$
$$T_1X_1T_3^* = A_2, \tag{2.8}$$
$$T_3X_1T_1^* = A_3, \tag{2.9}$$
$$T_3X_1T_3^* = A_4. \tag{2.10}$$

Using the matrix form (1.1) we get that $T_3 = (1 - P_{\text{ran(S)}})TP_{\text{ran}(T^*)} = (1 - P_{\text{ran(S)}})TT^\dagger T = (1 - P_{\text{ran(S)}})T$. Since ran(T) is closed, we get that ran(T_3) is closed [4]. Since Eq. (2.4) is solvable, we conclude that Eq. (2.10) is solvable. Then by Lemma 2.1, $X_1 = T_3^\dagger A_4(T_3^*)^\dagger$ is a solution to Eq. (2.10). Therefore, by Eq. (2.7), we have

$$S_1X_1S_1^* + S_1X_2S_2^* + S_2X_3S_1^* + S_2X_4S_2^* = -A_1 + T_1T_3^\dagger A_4(T_3^*)^\dagger T_1^*, \tag{2.11}$$

which can be written in an equivalent form

$$\begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} S_1^* & 0 \\ S_2^* & 0 \end{bmatrix} = \begin{bmatrix} -A_1 + T_1T_3^\dagger A_4(T_3^*)^\dagger T_1^* & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.12}$$

Eq. (2.4) is solvable, therefore Eq. (2.12) is solvable. By Lemma 2.1 a solution to Eq. (2.12) is

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} -A_1 + T_1T_3^\dagger A_4(T_3^*)^\dagger T_1^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1^* & 0 \\ S_2^* & 0 \end{bmatrix}^\dagger. \tag{2.13}$$

On the other hand, we have

$$\begin{aligned}
&SS^\dagger ASS^\dagger = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},\\
&(1-SS^\dagger)A(1-SS^\dagger) = \begin{bmatrix} 0 & 0 \\ 0 & A_4 \end{bmatrix},\\
&(1-SS^\dagger)T = \begin{bmatrix} 0 & 0 \\ T_3 & 0 \end{bmatrix},\\
&SS^\dagger T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

T_3 has closed range, hence $((1-SS^\dagger)T)^\dagger = \begin{bmatrix} 0 & T_3^\dagger \\ 0 & 0 \end{bmatrix}$ is the Moore-Penrose of $(1-SS^\dagger)T = \begin{bmatrix} 0 & 0 \\ T_3 & 0 \end{bmatrix}$. Therefore

$$\begin{aligned}
&\begin{bmatrix} -A_1 + T_1T_3^\dagger A_4(T_3^*)^\dagger T_1^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -A_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & T_3^\dagger \\ 0 & 0 \end{bmatrix}\times\\
&\begin{bmatrix} 0 & 0 \\ 0 & A_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ (T_3^\dagger)^* & 0 \end{bmatrix}\begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix}\\
=\ & -SS^\dagger ASS^\dagger + SS^\dagger T((1-SS^\dagger)T)^\dagger(1-SS^\dagger)A(1-SS^\dagger) \qquad (2.14)\\
&\times(((1-SS^\dagger)T)^\dagger)^*(SS^\dagger T)^*\\
=\ & -SS^\dagger ASS^\dagger + SS^\dagger T((1-SS^\dagger)T)^\dagger A(T^*(1-SS^\dagger))^\dagger(T^*SS^\dagger). \qquad (2.15)
\end{aligned}$$

The last equality is obtained from (1) and (2) of Lemma 2.2.

Now, by equations (2.13) and (2.15) and this fact that $SS^\dagger(S^\dagger)^* = (S^\dagger SS^\dagger)^* = (S^\dagger)^*$, it follows that

$$\begin{aligned}
X &= -S^\dagger SS^\dagger ASS^\dagger(S^\dagger)^* + S^\dagger SS^\dagger T((1-SS^\dagger)T)^\dagger A(T^*(1-SS^\dagger))^\dagger T^*SS^\dagger(S^\dagger)^*\\
&= -S^\dagger A(S^\dagger)^* + S^\dagger T((1-SS^\dagger)T)^\dagger A(T^*(1-SS^\dagger))^\dagger T^*(S^\dagger)^*.
\end{aligned}$$

□

Remark 2.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules. We use the notation $\mathcal{X}\oplus\mathcal{Y}$ to denote the direct sum of $\mathcal{X}$ and $\mathcal{Y}$, which is also a Hilbert $\mathcal{A}$-module whose $\mathcal{A}$-valued inner product is given by*

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle,$$

for $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$, $i = 1, 2$. To simplify the notation, we use $x\oplus y$ to denote $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{X}\oplus\mathcal{Y}$.

Proposition 2.5. *Suppose that $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}$ are Hilbert $\mathcal{A}$-modules, $T,S \in \mathcal{L}(\mathcal{Z},\mathcal{W})$, $R,Q \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, $A \in \mathcal{L}(\mathcal{X},\mathcal{W})$ such that S, Q, $T(1-S^\dagger S)$ and $R(1-Q^\dagger Q)$ have closed ranges. Suppose the equation*

$$TXR - SYQ = A\ ,\ \ X,Y \in \mathcal{L}(\mathcal{Y},\mathcal{Z}) \qquad (2.16)$$

is solvable. Then

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R & 0 \\ Q & 0 \end{bmatrix}^\dagger.$$

Proof. Taking $H = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} : \mathcal{Z}\oplus\mathcal{Z} \to \mathcal{W}\oplus\mathcal{W}$ has closed range, let $\{z_n \oplus x_n\}$ be sequence chosen in $\mathcal{W}\oplus\mathcal{W}$, such that $T(z_n) + S(x_n) \to y$ for some $y \in \mathcal{Z}$. Then

$$(1 - SS^\dagger)T(z_n) = (1 - SS^\dagger)(T(z_n) + S(x_n)) \to (1 - SS^\dagger)(y).$$

Since ran((1 − SS†)T) is assumed to be closed. Hence, $(1 - SS^\dagger)(y) = (1 - SS^\dagger)T(z_1)$ for some $z_1 \in \mathcal{Z}$. It follows that $y - T(x_1) \in \ker(1 - SS^\dagger) =$ ran(S), hence $y = T(z_1) + S(x)$ for some $x \in \mathcal{Z}$. Therefore H has closed range, hence $H^\dagger$ exists. Also, we take $K = \begin{bmatrix} R & 0 \\ Q & 0 \end{bmatrix} : \mathcal{X}\oplus\mathcal{X} \to \mathcal{Y}\oplus\mathcal{Y}$. Similar argument shows that K^* has closed range, hence by Theorem 3.2 of [7] implies that K has closed range, so $K^\dagger$ exists. Finally, let $Z = \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} : \mathcal{Y}\oplus\mathcal{Y} \to \mathcal{Z}\oplus\mathcal{Z}$ and $B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{X}\oplus\mathcal{X} \to \mathcal{W}\oplus\mathcal{W}$, hence Eq. (2.18) get into

$$HZK = B. \tag{2.17}$$

Lemma 2.1 implies that

$$Z = H^\dagger A K^\dagger.$$

□

Theorem 2.6. *Suppose that $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}$ are Hilbert $\mathcal{A}$-modules, $T, S \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$, $R, Q \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $A \in \mathcal{L}(\mathcal{X}, \mathcal{W})$ such that* ran(T) + ran(S) *and* ran(R*) + ran(Q*) *are closed and operator equation*

$$TXR - SYQ = A \ , \quad X, Y \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}) \tag{2.18}$$

is solvable, then

$$X = T^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger R^*, \tag{2.19}$$
$$Y = -S^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger Q^*. \tag{2.20}$$

Proof. Since ran(T) + ran(S) and ran(R*) + ran(Q*) are closed then by Lemma 4 of [2] and Corollary 5 of [2] respectively, imply that $\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} : \mathcal{Z}\oplus\mathcal{Z} \to \mathcal{W}\oplus\mathcal{W}$ and $\begin{bmatrix} R & 0 \\ Q & 0 \end{bmatrix} : \mathcal{X}\oplus\mathcal{X} \to \mathcal{Y}\oplus\mathcal{Y}$. Since Eq. (2.18) is equivalent to

$$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix}\begin{bmatrix} R & 0 \\ Q & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.21}$$

So Eq. (2.21) is solvable. Again by applying Lemma 4 of [2] and Corollary 5 of [2] and Lemma 2.1 we have

$$\begin{aligned}
\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} &= \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R & 0 \\ Q & 0 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} T^*(TT^* + SS^*)^\dagger & 0 \\ S^*(TT^* + SS^*)^\dagger & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (R^*R + Q^*Q)^\dagger R^* & (R^*R + Q^*Q)^\dagger Q^* \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} T^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger R^* & T^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger Q^* \\ S^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger R^* & S^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger Q^* \end{bmatrix}.
\end{aligned}$$

Since Eq. (2.18) is solvable, then $S^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger R^* = T^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger Q^* = 0$ and

$$X = T^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger R^*,$$
$$Y = -S^*(TT^* + SS^*)^\dagger A(R^*R + Q^*Q)^\dagger Q^*.$$

□

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